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Norms on Infinite Dimensional Space

Research Article

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Abstract: It is well known that on a finite dimensional linear space, any two norms are always equivalent. But, is the converse also true? That is if all norms on a linear space X are equivalent, then is X necessarily finite dimensional? In this article, we try to find the answer to this question.

MSC: AMS Classification

Keywords: Norm, Infinite dimensional space, Equivalent norms.

1. Introduction

It is known to all of us that in a finite dimensional norm linear space any two norms are always equivalent [5]. Now the question is whether the inverse of this statement is true. To answer this question we have to first study about infinite dimensional space. So in the first part of this article we have discussed various definitions and examples of both finite and infinite dimensional spaces. From the examples discussed here we can conclude that in case of infinite dimensional normed linear space there may exist some norms which are not equivalent.

2. Preliminaries

In this section we have listed some important definitions and examples.

Definition 2.1. *Let $V(F)$ be a vector space (F is either $\mathbb{R}$ or $\mathbb{C}$). A norm denoted by $\|.\|$ is a function from X to $\mathbb{R}$ which satisfies the following conditions:*

(1). $\|x\| \geq 0, \forall x \in X$

(2). $\|x\| = 0 \iff x = 0, \forall x \in X$

(3). $\|kx\| = |k|.\|x\|, \forall\ x \in X$ *and* $k \in \mathbb{F}$

(4). $\|x + y\| \leq \|x\| + \|y\|, \forall x \in X.$

A vector space X together with $\|.\|$ function i.e. $(X, \|.\|)$ is called a Normed space.

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Definition 2.2. *A Normed space $(X, \|.\|)$ is called finite dimensional if X is finite dimensional. Otherwise if X is infinite dimensional then $(X, \|.\|)$ is called infinite dimensional normed space.*

Example 2.3. *Define a function $\|.\| : \mathbb{R} \to \mathbb{R}$ by $\|x\| = |x|$ where $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$ $\forall x \in \mathbb{R}$. Then $(\mathbb{R}, \|.\|)$ is a nomed linear space.*

Example 2.4. *Let $X = \mathbb{R}^n = \{x = (x_1, x_2, x_3, ..., x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}$ .Then we can define $\|.\|_k = (\sum\limits_{i=1}^{n} |x_i|^k)^{\frac{1}{k}}$ by choosing any $1 \leq k \leq \infty$. Then $(\mathbb{R}^n, \|.\|_k)$ is a normed linear space. We can have various norms on $\mathbb{R}^n$ by varying the value of k. In particular for $k = 2$ the norm $\|.\|_2 = (\sum\limits_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$ is called Euclidean norm. However, for $0 < k < 1$, $\|.\|_k$ does not define a norm on $\mathbb{R}^n$. We can see this by an example. If we take $k = \frac{1}{2}$ and $n = 2$. Let $x = (1, 0), y = (0, 1)$ in $\mathbb{R}^2$. Then we get $x + y = (1, 1)$. Now $\|x\|_{\frac{1}{2}} = 1$ and $\|y\|_{\frac{1}{2}} = 1$ But $\|x + y\|_{\frac{1}{2}} = 4$. Therefore, $\|x + y\|_{\frac{1}{2}} > \|x\|_{\frac{1}{2}} + \|y\|_{\frac{1}{2}}$, which violates the Triangular inequality.*

These were some examples of finite dimensional normed linear spaces. Now we will discuss some examples of infinite dimensional normed linear spaces.

Example 2.5. *Let $X = \{p = p(t)$: p is a polynomial of any degree$\}$. Therefore X be the space of polynomials of all degrees. Now we can define a norm $\|.\|$ on X by $\|p\| = sup_{t\in[0,1]}|p(t)|$. Then X is an infinite dimensional normed linear space.*

Example 2.6. *Consider $X = C\,[0, 1] \;= \{x : [0, 1] \to \mathbb{R} | x is\ continous\}$. Define a norm $\|.\|_1 = \int\limits_0^1 |x(t)|dt$. Then X is an infinite dimensional normed linear space.*

Definition 2.7 ([1])**.** *Let two norms $\|.\|_1$ and $\|.\|_2$ defined on same vector space X are said to be comparable if either $\|x\|_1 \leq c_1\|x\|_2$ or $\|x\|_2 \leq c_2\|x\|_1$ $\forall x \in X$ and for some $c_1, c_2 \in \mathbb{R}^+$ is satisfied. If first one satisfied then $\|x\|_2$ is said to be stronger than $\|x\|_1$ and $\|x\|_1$ is weaker than $\|x\|_2$. Similarly if second one is satisfied then $\|x\|_1$ is said to be stronger than $\|x\|_2$ and $\|x\|_2$ is weaker than $\|x\|_1$.*

Definition 2.8. *Two norms defined over a same linear space is said to be equivalent if one is weaker or stronger than the other and conversly.*

3. Main Section

Theorem 3.1 ([4])**.** *Let $\|.\|_1$ and $\|.\|_2$ be two norms on a vector space X. Then $\|.\|_1$ and $\|.\|_2$ are equivalent iff $\exists$ c_1 and $c_2 > 0$ such that $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1$ for all $x \in X$.*

Theorem 3.2 ([6])**.** *On a finite dimensional vector space X, any norm $\|.\|_1$ is equivalent to any other norm $\|.\|_2$ defined on this.*

Now we have some examples of infinite dimensional space and norms on them which are not equivalent.

Example 3.3. *Let $X = \{p = p(t)$: p is a polynomial of any degree$\}$. Therefore X be the space of polynomials of all degrees. X is an infinite dimensional norm linear space. Define two norms $\|.\|_1$ and $\|.\|_2$ defined on X as follows*

$$\|p\|_1 = \sup_{t\in[0,1]} |p(t)|$$
$$\|p\|_2 = \sum_{i=0}^{n} |a_i|, \text{ where } p(t) = \sum_{i=0}^{n} a_i t^i$$

Then we can prove that the norms $\|.\|_1$ and $\|.\|_2$ are not equivalent.

Proof. For $p(t) = a_0 + a_1 t + ... + a_n t^n$

$$\begin{aligned} |p(t)| &= |a_0 + a_1 t + ... + a_n t^n| \\ &\leq |a_0| + |a_1||t| + |a_2||t|^2 + ... + |a_n||t|^n \\ \sup_{t\in[0,1]} |p(t)| &\leq \sum_{i=0}^{n} |a_i| \\ \Rightarrow \|p\|_1 &\leq \|p\|_2 \end{aligned}$$

Let if possible $\exists \ \alpha > 0$ such that,

$$\|p\|_2 \leq \alpha \|p\|_1 \ \ \forall \ \ p(t) \in X$$

Let

$$\begin{aligned} p_n(t) &= 1 - t + t^2 - t^3 + ... + t^{2n-2} - t^{2n-1} \ \text{ for all } \ n = 1, 2, 3, ... \\ &= (1-t)(1 + t^2 + t^4 + ... + t^{2n-2}). \\ \therefore \ \ p_1 &= 1 - t \\ p_2 &= 1 - t + t^2 - t^3 \\ p_3 &= 1 - t + t^2 - t^3 + t^4 - t^5. \ \text{Then} \\ \|p_n\|_1 &= \sup_{t\in[0,1]} |p_n(t)| = 1 \\ \|p_n\|_2 &= 2n. \ \text{Since,} \\ \|p_n\|_2 &\leq \alpha \|p_n\|_1 \ \ \forall \ \ n \\ \Rightarrow 2n &\leq \alpha \ \ \forall \ \ n \\ \Rightarrow n &\leq \frac{\alpha}{2} \ \ \forall \ \ n; \text{a contradiction.} \end{aligned}$$

Therefore, $\|p\|_2 \leq \alpha \|p\|_1$ is not possible. So, $\|.\|_1$ and $\|.\|_2$ are not equivalent. □

Example 3.4. *Consider $X = C\,[0,1] = \{x : [0,1] \to \mathbb{R} | x \ \textit{is continous}\}$. Define two norms $\|.\|_\infty$ and $\|.\|_1$ as follows:*

$$\begin{aligned} \|.\|_\infty &= \max_{t\in[0,1]} |x(t)| \\ \|.\|_1 &= \int_0^1 |x(t)| dt \end{aligned}$$

These two norms are not equivalent.

Proof. If possible consider $\|.\|_\infty$ and $\|.\|_1$ on $C\,[0,1]$ are equivalent. Then completeness of $\|.\|_\infty$ must imply completeness of $\|.\|_1$.

<u>*Claim :*</u> Let $\{x_n\}$ be a Cauchy sequence in $C\,[0,1]$. Then for any ε >0, there exists an N such that for all $m, n > N$ we have,

$$\|x_n - x_m\|_\infty = \max_{t\in[0,1]} |x_n(t) - x_m(t)| < \varepsilon \tag{1}$$

Hence for any fixed $t = t_0 \in [0,1]$

$$|x_n(t_0) - x_m(t_0)| < \varepsilon \text{for all } m, n > N$$

This shows $\{x_1(t_0), x_2(t_0), x_3(t_0), ...\}$ is a Cauchy sequence of real numbers for each $t_0 \in [0,1]$. Since $\mathbb{R}$ is complete, the sequence converges, say $x_n(t_0) \to x(t_0)$ as $n \to \infty$. This defines a function x on $[0,1]$. We show that $x \in C\,[0,1]$ and $x_n \to x$. Again, with $m \to \infty$ in (1) we have,

$$\max_{t\in[0,1]} |x_n(t) - x(t)| < \varepsilon \qquad (m > N)$$

Hence for every $t \in [0,1]$

$$|x_n(t) - x(t)| < \varepsilon \qquad (m > N)$$

This shows $\{x_n(t)\}$ converges to $x(t)$ on $[0,1]$. Since x_ns are continuous on $[0,1]$ and the convergence is uniform, the limit function x is continuous on $[0,1]$. Therefore $x \in C\,[0,1]$ and $x_n \to x$. Therefore $\|.\|_\infty$ is complete.

$\|.\|_1$ is not complete: We can give an example for it. Define a sequence $\{x_n\}$ in $C\,[0,1]$ as follows:

$$\begin{aligned} x_n(t) &= 0 \text{ if } t \in [0,1] \\ &= n\left(x - \frac{1}{2}\right) \text{ if } t \in \left[\frac{1}{2}, a_n\right] \\ &= 1 \text{ if } t \in [a_n, 1] \end{aligned}$$

The sequence is Cauchy because for every $\varepsilon > 0$

$$\|x_n - x_m\|_1 < \varepsilon \text{ when } m, m > \frac{1}{\varepsilon}$$

Here, $\|x_n - x_m\|_1$ is the area of the triangle shown in the figure. But this Cauchy sequence does not converge.

$$\begin{aligned} \|x_n - x_m\|_1 &= \int_0^1 |x_n(t) - x_m(t)| dt \\ &= \int_0^{\frac{1}{2}} |x_n(t)| dt + \int_{\frac{1}{2}}^{a_n} |x_n(t) - x(t)| dt + \int_{a_n}^1 |1 - x(t)| dt \end{aligned}$$

Therefore, $\|x_n - x_m\|_1 \to 0$ as $n \to \infty$ implies that each integral approaches zero and since x is continuous we have,

$$\begin{aligned} x(t) = 0 \quad \text{if} \quad t \in \left[0, \frac{1}{2}\right). \\ x(t) = 1 \quad \text{if} \quad t \in \left(\frac{1}{2}, 1\right]. \end{aligned}$$

But it is impossible for continuous function. So, $\{x_n\}$ does not converge. Therefore $c\,[0,1]$ is not complete with$\|.\|_1$ norm. Thus, $\|.\|_\infty$ and $\|.\|_1$ are not equivalent. □

Therefore from the above examples we have the observation that in infinite dimensional norm linear space there exists norms which are not equivalent. Now the question is whether this is a characteristics of infinite dimensional norm linear space. To answer this question we have the following theorem.

Theorem 3.5. *In every infinite dimensional space we can always define two norms which are not equivalent.*

Proof. For simplicity, we can consider a separable infinite dimensional vector space X. Consider a norm $\|.\|$ on X such that $\|e_i\| = 1$ for all i, where $\{e_1, e_2, ...\}$ be a basis of X. Let $x \in X$. Then x has a unique representation

$$x = \sum_{i=1}^{\infty} \alpha_i e_i$$

where all but finitely many α_i s are zero. Define $\phi : X \to \mathbb{F}$ such that

$$\phi(x) = \sum_{k=1}^{\infty} k\alpha_k, \quad \text{where} \quad x = \sum_k \alpha_k e_k$$

ϕ is linear: Let $x, y \in X$, $\gamma \in \mathbb{F}$

$$\begin{aligned} x = \sum_k \alpha_k e_k \quad &\text{and} \quad y = \sum_k \beta_k e_k \\ \therefore \quad x + \gamma y &= \sum_k (\alpha_k + \gamma\beta_k) e_k \\ \therefore \quad \phi(x + \gamma y) &= \sum_k k(\alpha_k + \gamma\beta_k) \\ &= \sum_k k\alpha_k + \gamma \sum_k k\beta_k \\ &= \phi(x) + \gamma\phi(y) \end{aligned}$$

ϕ is discontinous: Let, $y_k = \dfrac{e_k}{\sqrt{k}}$. Then $\|y_k\| = \dfrac{1}{\sqrt{k}} \to 0$ as $k \to \infty$. Therefore $y_k \to 0$ in X. But $\phi(y_k) = \dfrac{1}{\sqrt{k}}\phi(e_k) = \dfrac{k}{\sqrt{k}} = \sqrt{k} \to \infty$ as $k \to \infty$. Therefore, $\phi(y_k) \nrightarrow 0$ as $k \to \infty$ in $\mathbb{F}$. Thus ϕ is discontinuous. Define, $\|.\|_\phi = \|x\| + |\phi(x)|$. Now we show that $\|.\|_\phi$ is norm.

(1). Since $\|x\| \geq 0$ and $|\phi(x)| > 0$ therefore $\|.\|_\phi \geq 0$ for all $x \in X$.

(2). Again if $x = 0$ then $\|x\| = 0$ and $|\phi(x)| = 0$. Therefore, $\|.\|_\phi = 0$. Conversly, if $\|.\|_\phi = 0$ implies $\|x\| + |\phi(x)| = 0$ implies $\|x\| = 0$ and $|\phi(x)| = 0$ implies $x = 0$.

(3). $\|\alpha x\|_\phi = \|\alpha x\| + |\phi(\alpha x)| = |\alpha|\|x\| + |\alpha||\phi(x)| = |\alpha|\|x\|_\phi$

(4). $\|x + y\|_\phi = \|x + y\| + |\phi(x + y)| = \|x + y\| + |\phi(x) + \phi(y)| \leq \|x\| + \|y\| + |\phi(x)| + |\phi(y)| = \|x\|_\phi + \|y\|_\phi$.

Also $\|x\| \leq \|x\| + |\phi(x)| = \|x\|_\phi$. If possible, $\exists\ \alpha > 0$ such that $\|x\|_\phi \leq \alpha\|x\|$ for all $x \in X$. Then we have, $\|y_k\| \to 0$ implies $\|y_k\|_\phi \to 0$ implies $\|y_k\| + |\phi(y_k)| \to 0$ implies $|\phi(y_k)| \to 0$ in X, which is a contradiction. So no α exists such that $\|x\|_\phi \leq \alpha\|x\|$ for all $x \in X$. Hence $\|.\|_\phi$ and $\|.\|$ are not equivalent. So, in case of infinite dimensional norm linear space, we always have non equivalent norms. □

4. Conclusion

Thus we arrive at the conclusion that if X is a vector space such that any two norms on X are always equivalent, then X must be finite dimensional because in a infinite dimensional linear space there always exist non equivalent norms. Therefore if all the norms on a linear space are equivalent then the space must be finite dimensional.

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