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Unique Fixed Points and Mappings in Hyperconvex Metric Spaces

Research Article

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Abstract: In this present paper we study the basic structure of hyperconvex spaces and will prove the uniqueness of fixed point theorems for uniformly $k-$lipschitzian mappings in hyperconvex metric spaces.

Keywords: Hyperconvex space, uniformly $k-$lipschitzian mapping, fixed point, nonexpansive mappings.

1. Introduction

In 1956 Aronszajn and Panitchpakdi [2] proved that a hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically embedded. Isbell [14] showed that every metric space is isometric to a subspace of a unique hyperconvex space called the injective envelope. Dress [8] rediscovered the notion "injective envelope" as the tight span in the context of optimal networks and phylogenetic analysis. Sine [23] and Soardi [24] proved independently that nonexpansive mappings which satisfy $d(Fz, Fu) \leq d(z,u)$, $z, u \in M$ defined on a bounded hyperconvex space has fixed points. Since then, a number of fixed-point results in hyperconvex spaces were obtained of both topological and metric character. Baillon [3] proved that any intersection of hyperconvex spaces with a certain finite intersection property is a nonempty hyperconvex space also proved that the set of unique fixed points of a commuting family of nonexpansive mappings acting on a bounded hyperconvex space is a nonexpansive retract of M.

In this present paper focus is made on the properties and uniqueness of fixed-point sets of uniformly $k-$Lipschitzian mappings on hyperconvex spaces. Uniformly Lipschitzian mappings [11], are natural generalization of nonexpansive mappings. We define a mapping $F : M \to M$ is uniformly $k-$Lipschitzian if $d(F^m z, F^m u) \leq Kd(z,u)$ for each $z, u \in M$ and $n \in \mathbb{N}$, e.g. Lipschitzian periodic mappings. Lifschitz [20] proved that if C is a convex, closed and bounded subset of a Hilbert space and $k < 2$, then every uniformly $k-$Lipschitzian mapping $F : S \to S$ has a fixed point.

Hilbert and hyperconvex spaces are two extremes and yet there are some similarities between them from the geometrical point of view. Lang [21] proved that every uniformly $k-$Lipschitzian mapping with $k < \sqrt{2}$ in a bounded hyperconvex

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space with the property (P) has a fixed point. This result was later generalized in [5]. Deeper discussion of hyperconvex spaces can be seen [8, 9, 10].

The simplest examples of hyperconvex spaces are the set of real numbers $\mathbb{R}$, or a finite-dimensional real banach spaces endowed with the maximum norm. while the Hilbert space l^2 fails to be hyperconvex, the spaces L^∞ and l^∞ are hyperconvex. We show that a general linking construction yielding hyperconvex spaces.

Moreover, in this spaces paths between points are restricted, they must pass through certain unique points. The Theorem 3.1. if $\{F_p : p \in H\}$ is a mappings of uniformly $k-$Lipschitzian mappings on a hyperconvex space M with $K < \sqrt{2}$ and the orbits are bounded, then unique Fixed point of H is a Holder continuous retract of M.

In Theorem 3.2 and 3.3, We will study convex subsets of convex metric spaces and will show that the collection of all convex subsets of a hyperconvex metric space is uniformly normal and will prove under suitable conditions two fixed point theorems for uniformly $k-$lipschitzian mappings in hyperconvex metric spaces, also we will prove the uniqueness of the fixed points in these theorems. It should be mentioned that the maps, even though they are not intrinsically continuous, they are asymptotically continuous.

2. Basic Properties and Definitions

since a hyper-convex (M, d) is a metric space in which there is only one path between two points z and u, this would imply that if w is a point between z and u, by which we mean if $d(z, w) + d(w, u) = d(z, u)$ then we know that w is actually on the path between z and u, this will motivate the next concept of a metric interval.

Definition 2.1. *Let (M, d) be a metric space and let $z, u \in M$. An arc from z to u is the image of a topological embedding $\alpha : [p, q] \to M$ of a closed interval $[p, q]$ of $\mathbb{R}$ such that $\alpha(p) = z$ and $\alpha(q) = u$. A geodesic segment from z to u is the image of an isometric embedding $\alpha : [p, q] \to M$ such that $\alpha(p) = z$ and $\alpha(q) = u$. The geodesic segment will be called metric segment and denoted by $[z, u]$ throughout this work.*

Definition 2.2. *A metric space (M, d) be a hyperconvex metric space if*

$$\bigcap_{\alpha\in\Gamma} B(z_\alpha, t_\alpha) \neq \emptyset$$

for any collection of closed balls $\{B(z_\alpha, t_\alpha)\}\, \alpha \in \Gamma$ such that $d(z_\alpha, z_\beta) \leq t_\alpha + t_\beta,\ \alpha, \beta \in \Gamma$. It is not difficult to see that hyperconvex metric spaces are complete. We will use this fact several times. In hyperconvex metric spaces, $t(z) = \frac{\delta(z)}{2}$, whenever the orbits are bounded. The sets $S(z), SS(z)$ are nonempty since diam $S(z) \leq 2t(z)$. Furthermore,

$$S(z) = \bigcap_{\alpha\in G} B(F_p z, t(z)).$$

Definition 2.3. *the centre of the orbit of $z \in M$ is the set $S(z) = \{u \in M : r(u, z) = r(z)\}$ and the centre of $S(z)$ is defined by*

$$SS(z) = \bigcap_{u\in s(z)} B(u, r(z)) \cap S(z)$$

Definition 2.4. *Let M be a hyperconvex space. A mapping $F : S \to S$ of a subset S of M is said to be lipschitizn if there exists a non-negative number R such that $d(Fz, Fu) \leq Rd(z, u)$ for all z and $u \in S$. The smallest such R is called $k-$lipschitz constant and will be denoted by (K). Same mapping is called uniformly $k-$lipschitzian if $\sup_{m\geq 1} K(F^m) < \infty$ respectively $\sup_{m\geq m_0} K(F^m) < \infty$ for some $m_0 \geq 1$. Note that uniformly $k-$lipshitzian mappings need not be continuous. If $K(F) \leq 1$, then F is called nonexpansive and eventually nonexpansive if $\sup_{m\geq m_0} K(F^m) \leq 1$ for some $m_0 \geq 1$. It is well known fact that if map is uniformly $k-$lipshitzian, then one may find an equivalent distance for which the map is non-expansive. Indeed, let $F : S \to S$ be uniformly $k-$lipshitzian. Setting $\tau(z, u) = \sup\{Kd(F^m z, F^m u) : m = 0, 1, 2, 3 \ldots, \}$ for $z, u \in S$.*

Theorem 2.5. *let M be a bounded metric space and let $\{H_\beta\}\beta \in \Gamma$ be a decreasing family of nonempty hyperconvex subsets of M. Then $\bigcap_{\beta\in\Gamma} H_\beta \neq \emptyset$ and is hyperconvex.*

3. Main Results

Definition 3.1. *Let (M, d) be a hyperconvex metric space with $k < \sqrt{2}$ and the orbits $O(z)$ are bounded and if $\{F_p : p\varepsilon H\}$ is a mapping of uniformly $k-$lipshitzian mappings, then the set of fixed point H is nonempty.*

Proof. Assume that $k \in (1, \sqrt{2})$ without loss of generality. For fixed point $z_1 \in M$ choose $z_2 \in CC(z_1)$. Such that for all $p, q \in H$

$$\begin{aligned} d(F_p z_2, F_q z_1) &\leq \sup_{p,q\in H} \{F_p z_2, F_p z_1) + d(F_p z_1, F_q z_1) + d(F_q z_1, F_q z_2)\} \\ &\leq K[d(z_2, F_q z_1) + d(F_p z_2, z_1)] \\ &\leq r(z_1) + (k-1) r(z_1) = kr(z_1) \end{aligned}$$

From hyper-convexity for every $p \in H \ \exists\ T_p \in M$ such that

$$T_p \in \bigcap_{q\varepsilon H} B(F_q z_1, r(z_1)) \bigcap B(F_p z_2, (K-1) r(z_1)) \Rightarrow T_p \in C(z_1)$$

then $d(T_p, F_p z_2) \leq (k-1) r(z_1)$ which in turn implies $d(F_p z_2, F_q z_2) \leq Kd(z_2, F_q z_2) \leq K^2 r(z_1)$. For every $p, q \in H$. Hence

$$\varphi(z_2) \leq \frac{K^2}{2} \varphi(z_1).$$

Next select $z_3 CC(z_2)$ and estimate $\varphi(z_3)$ in a similar way. Continuing in the same way, we have a sequence $\{z_n\}$ such that

$$\varphi(z_{n+1}) \leq \frac{K^2}{2} \varphi(z_n) \text{ and } \varphi(z_{n+1}, z_n) = \frac{\varphi(z_n)}{2}.$$

It follows that $\{z_n\}$ is a Cauchy sequence converging to a fixed point $z_0 \in H$. Since $\varphi(z_0) \leq 2Kd(z_0, z_n) + \varphi(z_n) \to 0$. □

Theorem 3.2. *Let (M, d) be a hyperconvex metric space and N be a closed, bounded, nonempty, convex subset of M with diam $N > 0$. Let $L : N \to N$ be eventually uniformly $k-$lipschitzian such that*

$$\tau(F) = \lim_{s\to\infty} \sup K(F^s) < \frac{3}{2}$$

Then F has a fixed point.

Proof. Let $r > 0$ such that $\tau(F) < r < \frac{3}{2}$. By the definition of $\tau(F)$, there exists $s_0 \geq 1$ such that $K(F^s) \leq r$ for $s \geq s_0$. Next let $z \in R$ and set

$$d(z) = \lim_{s \to \infty} \sup d(z, F^s(z))$$

And

$$t(z) = \inf \left\{ \varrho > 0 : \exists\ s \geq 1 \ \textit{such that}\ R \cap \left(\bigcap_{\imath \geq s} B(F^\imath z, \varrho) \right) \neq \emptyset \right\}$$

Since the diameter of the set R is finite, which implies $t(z) \leq diam(R)$ is finite. Next, for each $\varepsilon > 0$, we define

$$S_\varepsilon(z) = \bigcup_{m \geq 0} \left(\bigcap_{\imath \geq m} B(F^\imath, t(z) + \varepsilon) \right).$$

Then for each $\varepsilon > 0$ the set $S_\varepsilon(z)$ is nonempty ($S_\varepsilon(z) \cap R \neq \emptyset$) and convex. Compactness of $S(M)$ implies that

$$S(z) = \bigcap_{\varepsilon > 0} \widetilde{S_\varepsilon} \cap R \neq \emptyset.$$

Let $w^* \in S(z)$, then w^* and $t(z)$ have the properties of unique fixed point:

(1). for any $\varepsilon > 0$, $\exists\ m_0 \geq 1$ such that for any $m \geq m_0$ we have $F^m(z) \in B(w^*, t(z) + \varepsilon)$.

(2). for any $w^* \in R$ and $0 < t < t(z)$, then set $\{\imath : d(F^\imath(z), w^*) > t\}$ is infinite.

Observe that if $t(z) = 0$ or if $d(w^*) = 0$, then $\lim_{m \to \infty} F^m(z) = w^*$. Let us prove in this case we have $Fw^* = w^*$ indeed, let $m_0 \geq 1$ such that F^m is $k-$lipschitzian for any $m \geq m_0$. In particular, F^m will be continuous for $(m \geq m_0)$. So for $M \geq 1$, we have $\lim_{m \to \infty} F^{m+M}(z) = F^M(w^*)$. But

$$\lim_{m \to \infty} F^{m+M}(z) = \lim_{m \to \infty} F^m(z) = w^*.$$

So for $M \geq m_0$, we have $F^M(w^*) = w^*$. This clearly implies $F^{M+1}(w^*) = w^*$. Combining the two, we get $F(w^*) = w^*$. Assume that $t(z) > 0$ and $d(w^*) > 0$. Let $\in > 0$, $\in \leq d(w^*)$ and select $\jmath$ big enough so that $d(w^*, F^\jmath w^*) \geq d(w^*) - \varepsilon$. By using property, there exists $m_0 \geq 1$ such that if $\imath \geq m_0$, then $d(w^*, F^\imath z) \leq t(z) + \varepsilon \leq R(t(z) + \varepsilon)$ if $\jmath \leq m_0$ then $d(w^*, F^\jmath z) \geq t(z) - \varepsilon \geq R(t(z) - \varepsilon)$ were m_0 is chosen so that $K(F^m) \leq R$ for $m \geq m_0$. Thus if $\imath - \jmath \geq m_0$, we have

$$d(F^\jmath(z), F^\imath(z)) \leq Rd\left(w^*, F^{\imath - \jmath}(z)\right) \leq R(t(z) + \varepsilon).$$

Considering the midpoint p of the interval $[w^*, F^\jmath(z)]$ and using the property of uniformly convexity, we have

$$d(p, F^\imath(z)) + \frac{1}{2} d(w^*, F^\jmath(z)) \leq R(t(z) + \varepsilon).$$

Equivalently,

$$d(p, F^\imath(z)) \leq R(t(z) + \varepsilon) - \frac{1}{2} d(w^*, F^\jmath(z)) \leq R(t(z) + \varepsilon) - \frac{1}{2}(d(w^*) - \varepsilon)$$

Therefore, we have

$$t(z) \leq R(t(z) + \varepsilon) - \frac{1}{2}(d(w^*) - \varepsilon)$$

From the definition of $t(z)$. Since ε was arbitrary we get

$$t(z) \leq Rt(z) - \frac{1}{2} d(w^*)$$

Or equivalently

$$d\left(w^{*}\right)\leq 2\left(R-1\right)t\left(z\right)$$
$$d(w^{*})\leq bt(z)\leq bd\left(z\right)$$

Where $b=2(R-1)<1$ and

$$d\left(w^{*},z\right)\leq d\left(z\right)+t\left(z\right)\leq 2d\left(z\right).$$

To complete the proof, fix $z\in R$ and a sequence $\{z_m\}$, construct by induction with $z_0=z$, such that $d(z_{m+1})\leq bd\left(z_m\right)$ and $d\left(z_{m+1},z_m\right)\leq 2d\left(z_m\right)$. For $m=1,\ldots,$ if we have $d\left(z_m\right)=0$ for some m, then a fixed point of F. Otherwise, we have

$$d(z_{m+1},z_m)\leq 2d(z_m)\leq 2b^m d\left(z\right)$$

Which implies that the sequence $\{z_m\}$ is Cauchy, therefore $\lim\limits_{m\to\infty} z_m=w^{*}\in R$ exists. Also

$$d\left(w^{*},F^{\imath}\left(w^{*}\right)\right)\leq d\left(w^{*},z_m\right)+d\left(z_m,F^{\imath}\left(z_m\right)\right)+d\left(F^{\imath}\left(z_m\right),F^{\imath}\left(w^{*}\right)\right)$$

For any $\imath\geq 1$. If we chose $\imath$ large enough to assume that $K\left(F^{\imath}\right)\leq R$, then

$$d\left(w^{*},F^{\imath}\left(w^{*}\right)\right)\leq\left(R+1\right)d\left(w^{*},z_m\right)+d\left(z_m,F^{\imath}\left(z_m\right)\right).$$

This implies

$$d\left(w^{*}\right)\leq\left(R+1\right)d\left(w^{*},z_m\right)+d\left(z_m\right)$$

Hence $d\left(w^{*}\right)=0$ which implies $Fw^{*}=w^{*}$. Let w^{**} be the another fixed point of F thus we have

$$d\left(w^{*},w^{**}\right)\leq\left(R+1\right)d\left(\left(w^{*},z_m\right)+d\left(z_m\right)\right)+d\left(\left(w^{**},z_m\right)+d\left(z_m\right)\right)$$

Thus implies $d\left(w^{*},w^{**}\right)=0\Rightarrow w^{*}=w^{**}\in F(w^{*})$. Thus have a unique fixed point. □

Theorem 3.3. *let (M,d) be a hyperconvex metric space and N be a nonempty closed, convex subset of M with $\dim(N)>0$. Let $F:N\to N$ be eventually uniformly $K-$lipschitzian mapping such that*

$$\tau\left(F\right)=\lim_{m\to\infty}\sup K(F^m)<2$$

Then F has a fixed point provided that F has bounded orbits.

Proof. Let $r>0$ such that $\tau(F)<r<2$. By definition of $\tau(F)$, therexists $m_0\geq 1$ such that $K(F^m)\leq R$ for any $m\geq m_0$. Let $u\in N$ and set $T(u)=\inf\{d>0:\ \exists\ z\in S$ such that for any $m\geq 1\ d(F^m\left(z\right),u)\leq d\}$. Since the orbit of u is bounded, we get $T(u)<\infty$. Assume that $T(u)=0$. Then for all $\epsilon>0$, there exists $z_\epsilon\in S$ such that $d\left(F^m\left(z_\epsilon\right),u\right)<\epsilon$ for any $m\geq 1$. If we choose $i\geq m_0$, then we get

$$d\left(F^{m+i}\left(z_\epsilon\right),F^{i}\left(u\right)\right)<R\epsilon$$

Which implies $d\left(u,F^{i}\left(u\right)\right)<\epsilon\left(1+R\right)$, for any $i\geq m_0$. Since ϵ was arbitrary, we get $F^{i}\left(u\right)=u$, for any $i\geq m_0$. So $F^{m0}\left(u\right)=F^{m0+1}\left(u\right)=u$ implies $F\left(u\right)=u$. Now let $T(u)>0$. Since $r<2$. Let $R<q<2$ and $p>1$ such that $\forall\ z,u\in S$, $\forall\ t>0$ with $d\left(z,u\right)>t$, there exists $w^{*}\in[z,u]$ such that

$$\{d\left(z,l\right)\leq qt\Rightarrow d\left(l,w^{*}\right)\leq t$$

Similarly $\{d(u,l) \leq pt \Rightarrow d(l,w^*) \leq t$. By Letting $\lambda < 0$ such that $\varphi = \min\{p\lambda, q\lambda/2\}$, we construct a sequence $\{u_m\} \in S$ by induction such that $T(u_{m+1}) \leq \lambda T(u_m)$. And $d(u_m, u_{m+1}) \leq (\lambda + \varphi) T(u_m)$. Let $u_1 \in S$ and assume $u_1 \ldots u_m$ are known. Again if $T(u_m) = 0$ we are done. Assuming that $T(u_m) > 0$, then there exists $j \geq m_0$ such that

$$\lambda T(u_m) \leq d\left(F^j(u_m), u_m\right)$$

And $z \in S$ with $d(F^n(z), u_m) \leq \varphi T(u_m) \forall\ n \geq 1$. Let $z^* = F^j(z)$. Then for $i \geq 1$ we have

$$F^i(z^*) = F^{i+j}(z) \in B(u_m, \varphi T(u_m)) \subset B(u_m, x\varphi T(u_m))$$

Which implies

$$d\left(F^i(z^*), F^j(u_m)\right) = d\left(F^{i+j}(z), F^j(u_m)\right) \leq Rd\left(F^i z, u_m\right) \leq R\varphi T(u_m) \leq q\lambda t(u_m),$$

Hence

$$F^i(z^*) \in B(u_m, p\lambda T(u_m)) \cap B\left(F^j(u_m), q\lambda T(u_m)\right) = D$$

Since $q < 2$, $\exists\ \ L \in [u_m, F^j(u_m)] \subset S$ such that $D \subset B(l, \lambda T(u_m))$ yielding $F^i(z^*) \in B(L, \lambda T(u_m))$ for all $i \geq 1$. Thus $T(l) \leq \lambda T(u_m)$. Set $u_{m+1} = l$, then $T(u_{m+1}) \leq \lambda T(u_m)$ and $d(u_{m+1}, u_m) \leq d\left(u_{m+1}, F^i(z^*)\right) + d\left(F^i(z^*), u_m\right) \leq \lambda T(u_m) + \varphi T(u_m) \leq (\lambda + \varphi) T(u_m)$.

Clearly shows that $\{u_m\}$ is a Cauchy sequence $\therefore \lim\limits_{m\to\infty} u_m = w^* \in S$ exists. Let $\in > 0$, so that there exists $m_1 \geq m_0$ such that $\forall\ \ m \geq m_1$, $d(w^*, u_m) < \epsilon$. And hence

$$d\left(F^i(z), w^*\right) \leq d(w^*, u_m) + T(u_m) + \epsilon.$$

Thus $T(w^*) \leq d(w^*, u_m) + T(u_m) + \epsilon$ yielding $T(w^*) = 0$. This will imply $F(w^*) = w^*$. Let w^{**} be the another fixed point such that

$$\begin{aligned} d(w^*, w^{**}) \leq d(Fw^*, Fw^{**}) &\leq d\left(F^i(z), w^*\right) + d\left(w^{**}, F^i(z)\right) \\ &\leq d(w^*, u_m) + d(w^{**}, u_m) + T(u_m) + \epsilon \end{aligned}$$

Thus $T(w) \leq d(w^*, w^{**}) + T(u_m) + \epsilon$ yielding $T(w) = 0 \Rightarrow d(w^*, w^{**}) = 0$. This implies $w^* = w^{**} \in F(W^*)$. Thus there exists a unique fixed point $w^* = w^{**} \in F(w^*)$. □

Corollary 3.4. *Every uniformly $k-$lipschitzian mappings in a hyperconvex space with $k < 2$ has a unique fixed point.*

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