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# Some Fixed Point Theorems for JSC Contraction in Complete Metric Space

**Research Article**

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**Abstract:** Recently, Hussain et al. [1] introduced the concept of JS-contraction. In this paper, we introduce a new method of proofs that allows to prove fixed point theorems for JSC- contraction in complete metric space.These theorems generalizes the results of [6].

**MSC:** 47H10, 54H25.

**Keywords:** Fixed point theorem, JS-contraction, JSC-contraction.



## 1. Introduction

The Banach contraction principle [2] is the first important result on fixed points for contractive-type mappings, which states that each Banach contraction $T : X \to X$ (that is., there exists $\lambda \in (0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for each $x, y \in X$) has a unique fixed point, provided that $(X, d)$ is a complete metric space. The main purpose of this paper is to show that the results concerned in metric spaces with JSC-contraction in [6] are consequences of Theorem 1.3. Before going to the main results. Let us recall the basic definitions and theorems. The concepts of Ciric contraction and JS-contraction have been introduced, respectively, by Ciric [4] and Hussain et al. [1] as follows.

**Definition 1.1** ([4]). *Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a ciric contraction if there exist non-negative numbers $q, r, s, t$ with $q + r + s + 2t < 1$ such that*

$$d(Tx, Ty) \leq qd(x, y) + rd(x, Tx) + sd(y, Ty) + t[d(x, Ty) + d(y, Tx)], \quad \forall \ x, y \in X. \tag{1}$$

**Definition 1.2** ([1]). *Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a JS-contraction if there exist $\psi \in \Psi$ and non-negative numbers $q, r, s, t$ with $q + r + s + 2t < 1$ such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y))^q \psi(d(x, Tx))^r \psi(d(y, Ty))^s \psi(d(x, Ty) + d(y, Tx))^t, \quad \forall \ x, y \in X. \tag{2}$$

*where $\Psi$ is the set of all functions $\psi : [0, +\infty) \to [1, +\infty)$ satisfying conditions:*

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*($\psi_1$) $\psi$ is non-decreasing and $\psi(t) = 1$ if and only if $t = 0$;*

*($\psi_2$) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;*

*($\psi_3$) there exist $r \in (0,1)$ and $l \in (0, +\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$;*

*($\psi_4$) $\psi(a+b) \le \psi(a)\psi(b)$ for all $a, b > 0$.*

For convenience, we denote by $\Psi_1$, the set of all non-decreasing functions $\psi$:$(0, +\infty) \to (1, +\infty)$ satisfying ($\psi_2$) and ($\psi_3$) and by $\Psi_2$, the set of all functions $\psi$:$[0, +\infty) \to [1, +\infty)$ satisfying ($\psi_1$), ($\psi_2$) and ($\psi_4$).

**Theorem 1.3** ([4]). *Let $(X, d)$ be a complete metric space, and $T : X \to X$ be a Ciric contraction. Then $T$ has a unique fixed point in $X$.*

**Theorem 1.4** ([5]). *Let $(X, d)$ be a complete metric space, and $T : X \to X$,Assume that there exist $\psi \in \Psi_1$ and $k \in (0,1)$ such that $\forall\ x, y \in X$, $d(Tx, Ty) \neq 0 \Longrightarrow \psi(d(Tx, Ty)) \le \psi(d(x, y))^k$. Then $T$ has a unique fixed point in $X$.*

**Definition 1.5** ([6]). *For $\psi \in \Psi_2$ and $t \in [0, +\infty)$, set $\eta(t) = ln(\psi(t))$. Then it is easy to check that $\eta : [0, +\infty) \to [0, +\infty)$ has the following properties:*

($\eta_1$) *$\eta$ is non-decreasing, and $\eta(t) = 0$ if and only if $t = 0$;*

($\eta_2$) *for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \eta(t_n) = 0$ if and only if $\lim_{n\to\infty}(t_n) = 0$;*

($\eta_3$) *$\eta(a+b) \le \eta(a) + \eta(b)$ for all $a, b > 0$.*

*Since ($\eta_1$) and ($\eta_2$) are clear, we only show ($\eta_3$), we have*

$$\eta(a+b) = ln(\psi(a+b)) \le ln(\psi(a)\psi(b)) = ln(\psi(a)) + ln(\psi(b)) = \eta(a) + \eta(b).$$

**Lemma 1.6** ([6]). *Let $(X, d)$ be a metric space, and $\psi \in \Psi_2$. Then $(X, D)$ is a metric space, where $D(x, y) = \eta(d(x, y)) = ln(\psi(d(x, y)))$.*

**Lemma 1.7** ([6]). *Let$(X, d)$ be a metric space, and $\psi \in \Psi_2$. Then $(X, D)$ is complete if and only if $(X, d)$ is complete, where $D(x, y) = \eta(d(x, y)) = ln(\psi(d(x, y)))$.*

**Lemma 1.8** ([6]). *Let $(X, d)$ be a metric space, and $T : X \to$ be a JS-contraction with $\psi \in \Psi_2$. Then $T$ is a Ciric contraction in $(X, D)$, where $D(x, y) = \eta(d(x, y)) = ln(\psi(d(x, y)))$.*

## 2. Main results

In this section, we introduce a new metric $D$ in a given metric space $(X, d)$ induced by the metric $d$, and then we prove that $(X, D)$ is complete if and only if $(X, d)$ is complete. Then we show that each JSC-contraction with $\psi \in \Psi_2$ in $(X, d)$ is certainly a Ciric contraction in $(X, D)$.

**Definition 2.1.** *Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a JSC-contraction if there exist $\psi \in \Psi$ and non-negative numbers $q, r, s, t$ with $q + r + s + 2t < 1$ such that*

$$\psi(d(Tx, Ty)) \le q\psi(d(x, y)) + r\psi(d(x, Tx)) + s\psi(d(y, Ty)) + t\psi(d(x, Ty) + d(y, Tx)),\ \ \forall\ x, y \in X. \tag{3}$$

*where $\Psi$ is the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying conditions:*

*($\psi_1$) $\psi$ is non-decreasing and $\psi(t) = 0$ if and only if $t = 0$;*

*($\psi_2$) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \psi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$;*

*($\psi_3$) there exist $r \in (0, 1)$ and $l \in (0, +\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)}{t^r} = l$;*

*($\psi_4$) $\psi(a + b) \leq \psi(a) + \psi(b)$ for all $a, b > 0$.*

For convenience, we denote by $\Psi_1$, the set of all non-decreasing functions $\psi$:$(0, +\infty) \to (0, +\infty)$ satisfying ($\psi_2$) and ($\psi_3$) and by $\Psi_2$, the set of all functions $\psi$:$[0, +\infty) \to [0, +\infty)$ satisfying ($\psi_1$), ($\psi_2$) and ($\psi_4$).

**Remark 2.2.**

*(1). If $f(t)=\sqrt{t}$, $t \geq 0$. then $f \in \Psi \cap \Psi_1 \cap \Psi_2$.*

*(2). If $g(t) = t$ for $t \geq 0$, then $g \in \Psi_2$, but $g \notin \Psi \cup \Psi_1$. Since $\frac{t}{t^r} = 0$ for each $r \in (0, 1)$, that is, ($\psi_3$) is not satisfied.*

*(3). Clearly $\Psi \subseteq \Psi_1$ and $\Psi \subseteq \Psi_2$.*

**Theorem 2.3.** *Let $(X, d)$ be a complete metric space, and $T : X \to X$. Assume that there exist $\psi \in \Psi_1$ and $k \in (0, 1)$ such that $\forall x, y \in X$,*

$$d(Tx, Ty) \neq 0 \Longrightarrow \psi(d(Tx, Ty)) \leq k\psi(d(x, y)). \tag{4}$$

*Then $T$ has a unique fixed point in $X$.*

The Banach contraction principle follows immediately from this theorem. Indeed, let $T : X \to X$ and $k \in (0, 1)$ be such that (4) holds. Then if we choose $\psi(t) = \sqrt{t} \in \Psi_1$ and $k = \sqrt{\lambda}$ in (4), then we get $\sqrt{d(Tx, Ty)} \leq \sqrt{\lambda}\sqrt{d(x, y)}$, that is, $d(Tx, Ty) \leq \lambda d(x, y)$, $\forall\ x, y \in X$, which means that $T$ is a Banach contraction.

**Definition 2.4.** *For $\psi \in \Psi_2$ and $t \in [0, +\infty)$, set $\eta(t) =\| \psi(t) \|$. Then it is easy to check that $\eta : [0, +\infty) \to [0, +\infty)$ has the following properties:*

*($\eta_1$) $\eta$ is non-decreasing, and $\eta(t) = 0$ if and only if $t = 0$;*

*($\eta_2$) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \eta(t_n) = 0$ if and only if $\lim_{n\to\infty} (t_n) = 0$;*

*($\eta_3$) $\eta(a + b) \leq \eta(a) + \eta(b)$ for all $a, b > 0$.*

*Since ($\eta_1$) and ($\eta_2$) are clear, we only show ($\eta_3$), we have*

$$\eta(a + b) =\| \psi(a + b) \|\leq\| \psi(a) + \psi(b) \|\leq\| \psi(a) \| + \| \psi(b) \|= \eta(a) + \eta(b).$$

**Lemma 2.5.** *Let$(X, d)$ be a metric space, and $\psi \in \Psi_2$. Then $(X, D)$ is a metric space, where $D(x, y) = \eta(d(x, y)) =\| \psi(d(x, y)) \|$.*

*Proof.* For each $x \in X$, we have $D(x, x) = \eta(d(x, x)) = 0$ by ($\eta_1$). For all $x, y \in X$, $D(x, y) = 0$,we have $\eta(d(x, y)) = 0$. Hence $d(x, y) = 0$ by ($\eta_1$). Hence $D(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$, we have $D(x, y) = \eta(d(x, y)) = \eta(d(y, x)) = D(y, x)$ for all $x, y \in X$. For all $x, y, z \in X$ with $z \neq x$ and $z \neq y$, by ($\eta_1$) and ($\eta_3$) we have,

$$\begin{aligned} D(x, y) &= \eta(d(x, y)) \\ &\leq \eta(d(x, z) + d(z, y)) \\ &\leq \eta(d(x, z)) + \eta(d(z, y)) \\ &= D(x, z) + D(z, y) \end{aligned}$$

we have $D(x,y) = D(x,z) = D(x,z) + D(y,z)$, for all $x \in X$ and $y = z \in X$ by $(\eta_1)$. Also we have $D(x,y) = D(z,y) = D(x,z) + D(z,y)$, for all $x = z \in X$ and $y \in X$ by $(\eta_1)$. For all $x = y = z \in X$, we have $D(x,y) = 0 = D(x,z) + D(y,z)$ by $(\eta_1)$. Hence, for all $x, y, z \in X$, we always have $D(x,y) \leq D(x,z) + D(z,y))$. This shows that $(X, D)$ is a metric space. □

**Lemma 2.6.** *Let$(X,d)$ be a metric space, and $\psi \in \Psi_2$. Then $(X,D)$ is complete if and only if $(X,d)$ is complete, where $D(x,y) = \eta(d(x,y)) =\| \psi(d(x,y)) \|$.*

*Proof.* Suppose that $(X,d)$ is complete. Let $\{x_n\}$ be a Cauchy sequence of $(X,D)$, that is $\lim\limits_{m,n\to\infty} D(x_n,x_m) = 0$. Then we have $\lim\limits_{m,n\to\infty} \eta(d(x_n,x_m)) = 0$, hence $\lim\limits_{m,n\to\infty} d(x_n,x_m) = 0$ by $(\eta_2)$. Moreover, by the completeness of $(X,d)$ there exists $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n,x) = 0$. So we have $\lim\limits_{n\to\infty} D(x_n,x) = \lim\limits_{n\to\infty} \eta(d(x_n,x)) = 0$ by $(\eta_2)$. Hence $(X,D)$ is complete. Similarly, we can show that if $(X,D)$ is complete, then $(X,d)$ is complete. □

**Lemma 2.7.** *Let $(X,d)$ be a metric space, and $T : X \to$ be a JSC-contraction with $\psi \in \Psi_2$. Then $T$ is a Ciric contraction in $(X,D)$, where $D(x,y) = \eta(d(x,y)) =\| \psi(d(x,y)) \|$.*

*Proof.* It follows from (3) that, for all $x,y \in X$,

$$\begin{aligned}
D(Tx,Ty) &= \eta(d(Tx,Ty)) =\| \psi(d(Tx,Ty)) \| \\
&\leq\| q\psi(d(x,y)) + r\psi(d(x,Tx)) + s\psi(d(y,Ty)) + t\psi(d(x,Ty) + d(y,Tx)) \| \\
&\leq\| q\psi(d(x,y)) \| + \| r\psi(d(x,Tx)) \| + \| s\psi(d(y,Ty)) \| + \| t(\psi(d(x,Ty) + d(y,Tx)) \| \\
&=| q \|\| \psi(d(x,y)) \| + | r \|\| \psi(d(x,Tx)) \| + | s \|\| \psi(d(y,Ty)) \| + | t \|\| \psi(d(x,Ty) + d(y,Tx)) \| \\
&= qD(x,y) + rD(x,Tx) + sD(y,Ty) + t[D(x,Ty) + D(y,Tx)]
\end{aligned}$$

Therefore (1) is satisfied with respect to metric $D$. Hence $T$ is a Ciric contraction in $(X,D)$. □

**Theorem 2.8.** *Let $(X,d)$ be a complete metric space, $T : X \to X$ be a JSC-contraction with $\psi \in \Psi_2$.Then $T$ has a unique fixed point in $X$.*

*Proof.* Since $(X,d)$ is a complete metric space, $(X,D)$ is also a complete metric space by Lemma 2.6, we know that $T$ is a Ciric contraction in $(X,D)$ by Lemma 2.7. Therefore $T$ has a unique fixed point in $X$ by Theorem 1.3. □

**Theorem 2.9.** *Theorem 2.8 implies Theorem 1.3.*

*Proof.* Let $\psi(t) = t$ for $t \geq 0$. Clearly, $t \in \Psi_2$ by Remark 2.2 (2). By (3) we have,

$$\psi(d(Tx,Ty)) \leq q\psi(d(x,y)) + r\psi(d(x,Tx)) + s\psi(d(y,Ty)) + t\psi(d(x,Ty) + d(y,Tx)).$$

$$d(Tx,Ty) \leq qd(x,y) + rd(x,Tx) + sd(y,Ty) + t[d(x,Ty) + d(y,Tx)],$$

for all $x,y \in X$. Which implies that a Ciric contraction $T : X \to X$ is certainly a JSC-contraction with $\psi(t) = t$. Thus Theorem 1.3 immediately follows from Theorem 2.8. □

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