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On a Problem Of Maximization of Some Indicators in the Discrete Time Models of Economic Dynamics

Research Article

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Abstract: We consider n single-product models, each of which is given by the production function and the safety coefficient. The total workforce is also given which is distributed between these models in such a way that to maximize the total consumption, total production and total national wealth. As the production functions of the models is considered the Cobb-Douglas function with constant elasticity substitution (CES). The conditions are given under which the total consumption, total production and total national wealth reaches maximum.

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1. Introduction

Let n single–product models be given, each of which is given by the production function F_i and safety coefficient v_i. The following denotations are used: K_i-basic foundations, W_i-consumption, $\eta_i = \frac{K_i}{l_i}-$capital-labor ratio, $0 \leq l_i \leq L$, $\sum\limits_{i=1}^{n} l_i = L$, ω_i-specific consumption (rate of wages paid) of the $i-$th model, respectively, $i = \overline{1,n}$. $f_i(n) = F_i(\eta_i, 1)$. [1, 4–7] The total number of the labor-force L is given which is distributed between these models in such a way that to maximize the total consumption, total production and total national wealth:

$$\sum_{i=1}^{n} W_i(l_i) \to \max, \tag{1}$$

$$\sum_{i=1}^{n} F_i(k_i l_i) \to \max, \tag{2}$$

$$\sum v_i K_i + F_i(K_i, l_i) \to \max \tag{3}$$

Under the condition $0 \leq l_i \leq L$, $\sum l_i = L$. Here $W_i(l_i)-$ consumption foundation, F_i-production, $v_i K_i + F_i(K_i, l_i)-$national wealth in the $i-$th, $i = \overline{1,n}$ model under the assumption that the specific consumption ω is chosen by the following formulas [1]

$$\omega = \frac{f(\eta) - \eta f'(\eta)}{v + f'(\eta)},$$

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where $\eta = \eta(L)$ is a root of the equation

$$\eta = \frac{M}{L} - \frac{f(\eta) - \eta f'(\eta)}{v + f'(\eta)}.$$

The situation arises: if in one of the models of the production function "much better" than the other, the entire workforce should be directed to this model [2]. However, it does not always happen.

2. Main Result

Let the vector $\bar{l} = (\bar{l}_1, \ldots, \bar{l}_n)$ be a solution of the problem

$$\sum_{i=1}^{n} \tilde{f}_i(l_i) \to \max,$$

with the condition $l_i \geq 0$, $\sum_{i=1}^{n} l_i = L$. Here $\tilde{f}_i(l_i)$ is twice continuously differentiable function, $i = \overline{1,n}$. Denote

$$I_1 = \left\{ i | \bar{l}_i = 0 \right\},$$
$$I_2 = \left\{ i | \bar{l}_i > 0 \right\}.$$

Then [3] there exists a number $\lambda > 0$, such that

$$\begin{aligned} \tilde{f}_i'(\bar{l}_i) &= \lambda, \quad i \in I_2, \\ \tilde{f}_i(\tilde{l}_i) &\leq \lambda, \quad i \in I_1. \end{aligned} \tag{4}$$

Theorem 2.1. *Let F_i be Cobb-Douglas function, $i = \overline{1,n}$.. [1, 3–5] Then in the problem (1)-(3) the vector $\bar{l}$ belongs to the inside of the cone R_+^n ($\bar{l} >> 0$).*

Proof. Using the formula

$$W'(L) = M \left(\frac{\delta}{u} \right)' = \frac{\delta' u - u' \delta}{u^2} \eta'(L) = \frac{u' \delta - u \delta'}{\beta^2 - uv}, \tag{5}$$

It is not difficult to check that in the case when $F$is Cobb-Douglas production function, then the production function is infinite at zero [9, 10]. Really, as follows from (5)

$$\begin{aligned} W'(L) &= \frac{(v + Ar\eta^{r-1})A\eta^r(1-r) - (1-r)Ar\eta^{r-1}\eta^r(v\eta^{1-r} + A)}{(v + Ar\eta^{r-1})^2 - \eta^r(v\eta^{1-r} + A)Ar(r-1)\eta^{r-2}} \\ &= \frac{A(1-r)\eta^2 \left[v + Ar\eta^{r-1} - vr\eta^{r-1}\eta^{1-r} - Ar\eta r - 1 \right]}{(v + Ar\eta^{r-1})^2 + Ar(1-r)(v\eta^{r-1} + Ar\eta^{2r-2})} \\ &= \frac{vA(1-r)\eta^r}{\left(v + Ar\frac{1}{\eta^{1-r}} \right)^2 + Ar(1-r)\left(v\frac{1}{\eta^{1-r}} + \frac{A}{\eta^{2-2r}} \right)}. \end{aligned}$$

Therefore considering $0 < r < 1$, we get

$$W'(0) = \lim_{\eta \to \infty} W'(\eta) = \infty.$$

Besides, $F_2'(0) = +\infty$. Therefore the second relation in (4) does not take place. It gives $I_1 = \emptyset$. □

Note 2.2. *For the any production function with $F'(0) = +\infty$ the problems (2), (3) have a solution $\bar{l} >> 0$.*

Theorem 2.3. *Consider the models* (F_1, v_1) *and* (F_2, v_2)*, where* F_i *is a function with constant elasticity of substitution [1, 4, 5, 8]*

$$F_i(K, L) = (A_i K^{-\rho_i} + \beta_i L^{-\rho_i})^{\frac{1}{\rho_i}}, \quad i = \overline{1, 2},$$

and $\rho_i > 0$*. Then the total consumption* $\sum_{i=1}^{2} W_i(l_i)$ *reaches its maximum on the interval* $[0, L]$ *in the point* $l_j = L$*, if and only if when* $L \le \bar{L}_j$ *and* $W_j'(L) \ge \frac{1}{v_i} B_i^{-\frac{1}{\rho_i}}$*, and total production reaches maximum in the point* L *if and only if when* $F_j'(L) \ge B_i^{-\frac{1}{\rho_i}}$*,* $i \ne j$*. Here* $\bar{L}_j$ *is the only maximum point for the function* W_j *on the positive semi-axis.*

Proof. It follows from (4) that the set I in the problems (1), (2) is not empty, if there exists i such that

$$W_i'(0) \le W_j'(\bar{l}_j) = \lambda_W,$$
$$F_i'(0) \le F_j'(\bar{l}_j) \ = \lambda_F.$$

$j \in I_2$, $\lambda_W > 0$, $\lambda_F > 0$, respectively. It should be noted that $\bar{l}_j$ must be less than $\bar{L}_j$, since $W_j'(l) \le 0$ for all $l \le \bar{L}_j$. Let the set I_2 consists of one number j. Then $W_j(L) \ge W_i'(0)$; $F_j'(L) \ge F_i'(0)$ for all $i \ne j$. To finish the proof we show that

$$W_i'(0) = \frac{1}{v_i} B_i^{-\frac{1}{\rho_i}}$$
$$F_i'(0) = B_i^{-\frac{1}{\rho_i}}.$$

Since

$$W' = \frac{B\eta^{\rho+1} y^{\frac{1}{\rho}} \left(vB\eta^{\rho} - vA\rho - A\rho\, y^{-\frac{1}{\rho}} \right)}{vy^{\frac{1}{\rho}+2v} A y^{\frac{1}{\rho}+1} + A^2 + A(\rho+1)\left(vy^{\frac{1}{\rho}+1} \right) B\eta^{\rho}},$$

then dividing to dividing the numerator and denominator of the last expression by $\eta^{2\rho+2}$, we get

$$W' = \frac{vB^2 \left(\frac{A}{\eta^{\rho}} + B \right)^{\frac{1}{\rho}} - \left(\frac{A}{\eta^{\rho(\rho+1)}} + \frac{B}{\eta^{\rho}} \right)^{\frac{1}{\rho}} - \frac{A\rho}{\eta^{\rho+1}}}{v^2 \left(\frac{A}{\eta^{\rho}} + B \right)^{\frac{2\rho+2}{\rho}} + \rho v A \left(\frac{A}{\eta^{2\rho}} + \frac{B}{\eta^{\rho}} \right)^{\frac{1}{\rho}+1} + \frac{A^2}{\eta^{2\rho+2}} + vA(\rho+1)\left(\frac{A}{\eta^{(\rho+2)\rho}} + \frac{B}{\eta^{\rho^2+\rho}} \right)^{\frac{1}{\rho}}}.$$

From this

$$\lim_{L \to +0} W'(L) = \lim_{\eta \to +\infty} W'(\eta) = \frac{vB^2 B^{\frac{1}{\rho}}}{v^2 B^{2+\frac{2}{\rho}}} = \frac{1}{vB^{\frac{1}{\rho}}}.$$

As

$$F'(L) = \left(\frac{A}{K^{\rho}} + \frac{B}{L^{\rho}} \right)^{-\frac{1}{\rho}-1} BL^{-\rho-1} = B\left(A\frac{L^s}{K^{\rho}} + B \right)^{-\frac{1}{\rho}+\rho},$$

then

$$F_i'(0) = B_i^{1-\frac{1}{\rho_i}-1} = B_i^{-\frac{1}{\rho_i}}.$$

□

Theorem 2.4. *For the arbitrary production function* F *in the problem (3) is valid* $\bar{l}_i < 1$*.*

Proof. The maximum in (3) is reached at $\bar{l}_i < 1$ if and only if, when

$$v_j K_j + F_j(K_j, L) \ge \sum_{i=1}^{n} v_i K_i + F_i(K_i, l_i), \ \ 0 \le l_i \le L, \quad \sum_{i=1}^{n} l_i = L.$$

From this

$$F_j(K_j, L) - F_j(K_j, l_j) \ge \sum_{i \ne j} v_i K_i + F_i(K_i, l_i), \quad 0 \le l_i \le L, \ \sum_{i=1}^{n} l_i = L.$$

Introduce the function

$$F(l_j) = F_j(K_j, L) - F_j(K_0, l_j).$$

Let $l_j \to L$. Then $F(l_j) \to 0$ and $l_i \to 0, \quad \forall i \neq j$. Therefore

$$\sum_{i=1}^{n} (v_i K_i + F_i(K_i, l_i)) \to \sum_{i=1}^{n} v_i K^i > 0.$$

Thus there exists the points l_j which provide

$$F(l_j) \leq \sum_{i \neq j} (v_i K_i + F_i(K_i, l_i)).$$

□

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