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A New Extended Mittag-Leffler Function

Research Article

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Abstract: The main objective of this paper is defined a new extended Mittag-Leffler function using the modified extended classical Beta function due to Pucheta (see [8]). We will study some basic properties and evaluate Mellin transform.

Keywords: Modified extended beta function, Fractional Calculus, Mellin Transform, Extended Mittag-Leffler function.

1. Introduction

As it is known in 1997 M. Chaurhry introduced the extended classical Beta function as a generalization of the Euler Beta function and it is defined as (see [5]).

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}\,dt \tag{1}$$

where

$$R_e(p) \geq 0, \quad \min\{R_e(x), R_e(y)\} > 0$$

For more details see ([2–4]). Note that if $p = 0$ (1) it is reduced to classical Beta function. The Mittag-Leffler one parameters function is defined by the following series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \tag{2}$$

Where $\Gamma(,)$ denotes the classical Gamma function. The two parameters Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{3}$$

and tree parameters Mittag-Leffler function is defined as

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{4}$$

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For more details see ([1]). Afterwards, M. Ozarslan in 2014 (see [7]) use extended Beta function to defined an extended Mittag-Leffler function as

$$E_{\xi,\beta}^{(\delta,c)}(z) = \sum_{n=0}^{\infty} \frac{B_p(\delta+n, c-\delta)}{B(\delta, c-\delta)} \frac{(c)_n}{\Gamma(\xi n+\beta)} \frac{z^n}{n!} \tag{5}$$

Where

$$p \geq 0, \quad R_e(c) > R_e(\delta) > 0$$

for more details (see [7]). Note that if $p = 0$ and $c = 1$, (5) it is reduced to the tree parameters Mittag-Leffler function. Recently P. Pucheta introduce a generalization of the classical gamma function, given by the following expression (see [8]).

$$\Gamma^{\alpha}(x) = \int_0^{\infty} t^{x-1} E_{\alpha}(-t) dt \tag{6}$$

where

$$R_e(x) > 0 \ \ and \ \ E_{\alpha}(-t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{\Gamma(\alpha n+1)} \ \text{Mittag-Leffler function}$$

and the new modified extended classical Beta function is defined as

$$B_b^{\alpha}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha}(-bt(1-t)) dt \tag{7}$$

where

$$R_e(x) > 0, R_e(y) > 0 \ \ and \ \ R_e(p) \geq 0$$

Note that if $p = 0$ and $\alpha = 1$, (7) is reduces to classical Beta function.

Definition 1.1 (Mellin Transform [1]). *The Mellin transform of a function Φ is defined by the following integral*

$$M\{\Phi(z)\}(s) = \int_0^{\infty} z^{s-1} \Phi(z) dz \tag{8}$$

Definition 1.2 (Wright Fuction [1]). *The more general function ${}_q\Psi_p(z)$ is defined for $z \in \mathbb{C}$, complex $a_i, b_j \in \mathbb{C}$ and real $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, p; j = 1, , q)$ by the series*

$${}_q\Psi_p(z) = {}_q\Psi_p \left[\begin{matrix} (a_i, \alpha_1)_{1,p} \\ \\ (b_j, \beta_j)_{1,q} \end{matrix} \ , z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i n + a_i)}{\prod_{j=1}^{q} \Gamma(\beta_j n + b_j)} \frac{z^n}{n!} \tag{9}$$

2. Main Result

2.1. A New Extended Mittag-Leffler Function

In this section we introduce a new extended Mittag-Leffler function. Consider some of their properties and the transform Mellin is evaluate.

Definition 2.1. *Let $p \geq 0$, $\xi, \beta, c, \delta \in \mathbb{C}$ such as $R_e(c) > R_e(\delta) > 0$ and $R_e(\xi) > 0$, $R_e(\beta) > 0$. The new extended Mittag-Leffler function is defined as follows series:*

$$E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z) = \sum_{n=0}^{\infty} \frac{B_p^{\alpha}(\delta+n, c-\delta)}{B(\delta, c-\delta)} \frac{(c)_n}{\Gamma(\xi n+\beta)} \frac{z^n}{n!} \tag{10}$$

where $(c)_n$ *is Pochhammer symbol is defined as:*

$$(c)_n = \begin{cases} 1 & \text{if } n=0 \\ c(c+1\dots(c+n-1)) & \text{if } n\in\mathbb{N} \end{cases}$$

and $B_p^{\alpha}(,)$ *is the new extended modified Beta function.*

It may be observed that if $p=0,\ \alpha=c=1$, we obtain $E_{\xi,\beta}^{(\delta,1,1,0)}(z)=E_{\xi,\beta}^{\delta}(z)$

Theorem 2.2 (Integral Representation)**.** *Let* $p\geq 0,\ \xi,\beta,c,\delta\in\mathbb{C}$ *such as* $R_e(c)>R_e(\delta)>0$ *and* $R_e(\xi)>0,\ R_e(\beta)>0$. *Then*

$$E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)=\frac{1}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))E_{\xi,\beta}^{c}(tz)dt \tag{11}$$

Proof. From the definition (10), using (7) and for the uniform convergence of the series, we obtain:

$$\begin{aligned} E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z) &= \sum_{n=0}^{\infty}\frac{B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\int_0^1 t^{\delta+n-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))dt\right)\frac{(c)_n}{B(\delta,c-\delta)\Gamma(\xi n+\beta)}\frac{z^n}{n!} \\ &= \frac{1}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))dt\sum_{n=0}^{\infty}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{(tz)^n}{n!} \\ &= \frac{1}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))E_{\xi,\beta}^{(c)}(tz)dt \end{aligned}$$

□

Remark 2.3. *Making a change of variable* $t=\frac{u}{1+u}$ *in the previous expression (11), we obtain:*

$$E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z) = \frac{1}{B(\delta,c-\delta)}\int_0^{\infty}\frac{u^{\delta-1}}{(1+u)^c}E_{\alpha}\left(-p\frac{u^2}{(1+u)}\right)\times E_{\xi,\beta}^{c}\left(\frac{uz}{(1+u)}\right)dt \tag{12}$$

Remark 2.4. *Taking* $t=sin^2\theta$ *in the previous expression (11), we obtain:*

$$E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z) = \frac{1}{B(\delta,c-\delta)}2\int_0^{\frac{\pi}{2}} sin^{2\delta-1}\theta cos^{2c-2\delta-1}E_{\alpha}(sin^2\theta cos^2\theta)\times E_{\xi,\beta}^{c}\left(zsin^2\theta\right)d\theta \tag{13}$$

Theorem 2.5 (Recurrence formula)**.** *Let* $p\geq 0,\ \xi,\beta,c,\delta\in\mathbb{C}$ *such as* $R_e(c)>R_e(\delta)>0$ *and* $R_e(\xi)>0,\ R_e(\beta)>0$. *Then*

$$E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)=\beta E_{\xi,\beta+1}^{(\delta,c,\alpha,p)}(z)+\xi z\frac{d}{dz}E_{\xi,\beta+1}^{(\delta,c,\alpha,p)}(z) \tag{14}$$

Proof. Starting for the right member of (14), we have

$$\begin{aligned} \beta E_{\xi,\beta+1}^{(\delta,c,\alpha,p)}(z)+\xi z\frac{d}{dz}E_{\xi,\beta+1}^{(\delta,c,\alpha,p)}(z) &= \sum_{n=0}^{\infty}\frac{B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta+1)}\frac{z^n}{n!} \\ &+ \xi z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta+1)}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{\beta B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta+1)}\frac{z^n}{n!} \\ &+ \sum_{n=0}^{\infty}\frac{\xi n B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta+1)}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(\xi n+\beta)B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{(\xi n+\beta)\Gamma(\xi n+\beta)}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{z^n}{n!} \\ &= E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z) \end{aligned}$$

Here we use property of the Gamma function $\Gamma(x+1)=x\Gamma(x)$. □

Theorem 2.6 (Derivative Formula). *Let $p \geq 0$, $\xi, \beta, c, \delta \in \mathbb{C}$ such as $R_e(c) > R_e(\delta) > 0$ and $R_e(\xi) > 0$, $R_e(\beta) > 0$, $k \in \mathbb{N}$. Then*

$$\frac{d^k}{dz^k}\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\} = (c)_k E_{\xi,\beta+\xi k}^{(\delta+k,c+k,\alpha,p)}(z) \tag{15}$$

Proof. From definition (10) and taking into account the property of the Pochhammer symbol $(c)_{n+j} = (c)_j(c+j)_n$, we obtain

$$\begin{aligned}
\frac{d^k}{dz^k}\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\} &= \frac{d^k}{dz^k}\left(\sum_{n=0}^{\infty}\frac{B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{z^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\frac{B_p^{\alpha}(\delta,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{d^k}{dz^k}\frac{z^n}{n!}\\
&= \sum_{n=k}^{\infty}\frac{B_p^{\alpha}(\delta+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{z^{n-k}}{(n-k)!}\\
&= \sum_{n=0}^{\infty}\frac{B_p^{\alpha}((\delta+k)+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{n+k}}{\Gamma(\xi(n+k)+\beta)}\frac{z^n}{n!}\\
&= (c)_k\sum_{n=0}^{\infty}\frac{B_p^{\alpha}((\delta+k)+n,c-\delta)}{B(\delta,c-\delta)}\frac{(c+k)_n}{\Gamma(\xi n+(\beta+\xi k))}\frac{z^n}{n!}\\
&= (c)_k E_{\xi,\beta}^{(\delta+k,c+k,\alpha,p)}(z)
\end{aligned}$$

□

Theorem 2.7. *Let $p \geq 0$, $\xi, \beta, c, \delta \in \mathbb{C}$ such as $R_e(c) > R_e(\delta) > 0$ and $R_e(\xi) > 0$, $R_e(\beta) > 0$, $n \in \mathbb{N}$. Then*

$$\frac{d^n}{dz^n}\left\{z^{\beta-1}E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z^{\xi})\right\} = z^{\beta-n-1}E_{\xi,\beta-n}^{(\delta,c,\alpha,p)}(z^{\xi}) \tag{16}$$

Proof. Let $n \in \mathbb{N}$ such that $n = 1$. Thus, using integral representation (11) and (4), we obtain:

$$\begin{aligned}
\frac{d}{dz}\left\{z^{\beta-1}E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z^{\xi})\right\} &= \frac{1}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))\\
&\times \frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{(c)_n}{\Gamma(\xi n+\beta)}\frac{t^n z^{\xi n+\beta-1}}{n!}\right)dt\\
&= \frac{1}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))\\
&\times\ z^{\beta-2}\sum_{n=0}^{\infty}\frac{(c)_n}{\Gamma(\xi n+\beta-1)}\frac{(tz^{\xi})^n}{n!}dt\\
&= \frac{z^{(\beta-1)-1}}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))\\
&\times\ E_{\xi,\beta-1}^{c}(tz^{\xi})dt\\
&= z^{(\beta-1)-1}E_{\xi,\beta-1}^{(\delta,c,\alpha,p)}(z^{\xi})
\end{aligned}$$

□

Continuing with this some procedure n times, we obtain

$$\frac{d^n}{dz^n}\left\{z^{\beta-1}E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z^{\xi})\right\} = z^{\beta-n-1}E_{\xi,\beta-n}^{(\delta,c,\alpha,p)}(z^{\xi})$$

Theorem 2.8 (Mellin Transform). *Let $p \geq 0$, $\xi, \beta, c, \delta \in \mathbb{C}$ such as $R_e(c) > R_e(\delta) > 0$ and $R_e(\xi) > 0$, $R_e(\beta) > 0$. Then, the Mellin transform of the new extended Mittag-Leffler function is given by*

$$M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s) = \frac{\Gamma^{\alpha}(s)\Gamma(c-\delta-s+2)}{\Gamma(\delta)\Gamma(c-\delta)}\,{}_2\Psi_2\left[\begin{array}{c}(c,1),(\delta-s+2,1)\\ \\ (\xi,\beta),(c-2(s-2),1)\end{array},z\right] \tag{17}$$

Where ${}_2\Psi_2$ is the Wright generalized hypergeometric function.

Proof. From definition of Mellin transform, we obtain:

$$M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s)=\int_0^{\infty}p^{s-1}E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)dp \tag{18}$$

Using integral representaion (11), we obtain:

$$M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s)=\frac{1}{B(\delta,c-\delta)}\int_0^{\infty}p^{s-1}\left[\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\alpha}(-pt(1-t))E_{\xi,\beta}^{c}(tz)dt\right]dp \tag{19}$$

Now if we take $u=pt(1-t)$, exchange the orden of integration in (19) and using $\Gamma^{\alpha}(s)$, we obtain

$$M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s)=\frac{\Gamma^{\alpha}(s)}{B(\delta,c-\delta)}\int_0^1 t^{(\delta-s+2)-1}(1-t)^{(c-\delta-s+2)-1}E_{\xi,\beta}^{c}(tz)dt \tag{20}$$

From definition (4) and from uniform convergence of the series, we can exchange the orden of summation and integration in (20), we obtain

$$M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s)=\frac{\Gamma^{\alpha}(s)}{B(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{(c)_n z^n}{\Gamma(\xi n+\beta)n!}\int_0^1 t^{(\delta-s+2+n)-1}(1-t)^{(c-\delta-s+2)-1}dt \tag{21}$$

Thus

$$M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s)=\frac{\Gamma^{\alpha}(s)}{B(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{B(\delta-s+n+2,c-\delta-s+2)(c)_n z^n}{\Gamma(\xi n+\beta)n!} \tag{22}$$

Considering that $(c)_n=\frac{\Gamma(c+n)}{\Gamma(c)}$ and $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and inserting in (22), we get the result

$$\begin{aligned}M\left\{E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)\right\}(s)&=\frac{\Gamma^{\alpha}(s)\Gamma(c-\delta-s+2)}{\Gamma(\delta)\Gamma(c-\delta)}\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(c+n)\Gamma(\delta-s+2+n)}{\Gamma(\xi n+\beta)\Gamma(c-2(s-2)+n)}\\&=\frac{\Gamma^{\alpha}(s)\Gamma(c-\delta-s+2)}{\Gamma(\delta)\Gamma(c-\delta)}{}_2\Psi_2\left[\begin{matrix}(c,1),(\delta-s+2,1)\\ \\(\xi,\beta),(c-2(s-2),1)\end{matrix},z\right]\end{aligned}$$

□

Remark 2.9. *Putting $s=1$ in (18) we have the follows integral representation:*

$$\int_0^{\infty}E_{\xi,\beta}^{(\delta,c,\alpha,p)}(z)dp=\frac{\Gamma(c-\delta+1)}{\Gamma(\delta)\Gamma(c-\delta)}{}_2\Psi_2\left[\begin{matrix}(c,1),(\delta+1,1)\\ \\(\xi,\beta),(c+2,1)\end{matrix},z\right] \tag{23}$$

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