International Journal of Mathematics And its Applications
Volume 5, Issue 3–C (2017), 261–275.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Cone $\mathcal{C}$-class Function on New Contractive Conditions of Integral Type on Complete Cone S-metric Spaces

Research Article

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Abstract: In this paper, we generalised the concept of a new contractive conditions of integral type on complete cone S-metric spaces via cone C-class function.

MSC: Primary 47H10; Secondary 54H25.

Keywords: Integral-type contractive conditions, Fixed point ,Cone S-metric, C-class function.

1. Introduction and Mathematical Preliminaries

In 2012 [8] Sedghi. S et. al introduced the concept of generalization of fixed point theorems in S-metric spaces. Rahman M.U and Sarwar M are discussed in fixed point results of Altman integral type mappings in S-metric spaces in [9]. In recently, Nihal Yilmaz Ozgur, Nihal Tas [7] are discuss new contractive conditions of integral type on complete S-metric spaces. In 2007, Huang and Zhang [17] introduced the concept of cone metric spaces and fixed point theorems of contraction mappings; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point. In 1984, M.S. Khan, M. Swalech and S. Sessa [15] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function. In 2002, Branciari in [18] introduced a general contractive condition of integral type. Farshid Khojasteh et.al, [16] discuss some fixed point theorems of integral type contraction in cone metric spaces.

In this paper we discuss generalised result on cone C-class function on new contractive conditions of integral type on complete cone S-metric spaces. In [17], let E be a Banach space. A subset P of E is called a cone if and only if:

(1). P is closed, nonempty and $P \neq 0$.

(2). $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b.

(3). $P \cap (-P) = \{0\}$.

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Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $int\ P \neq 0$ and $\leq$ is partial ordering with respect to P.

Example 1.1. *Let $K > 1$ be given. Consider the real vector space with*

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(C1) $d(x, y) \geq 0$, *and* $d(x, y) = 0$ *if and only if* $x = y\ \forall x, y \in X$,

(C2) $d(x, y) = d(y, x)$, $\forall x, y \in X$,

(C3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$,

Then (X, d) is called a cone metric space simply CMS.

Lemma 1.3 ([20])**.** *Every regular cone is normal.*

Example 1.4. *Let $E = R^2$*

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a Cone metric space.

Definition 1.5. *Let $X \neq \emptyset$ be any set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $u, v, z, a \in X$.*

(S1) $S(u, v, z) \geq 0$.

(S2) $S(u, v, z) = 0$ *if and only if* $u = v = z$.

(S3) $S(u, v, z) \leq S(u, u, a) + S(v, v, a) + S(z, z, a)$.

Then the function S is called an S-metric on X and the pair (X, S) is called an S-metric space simply SMS.

Example 1.6 ([6])**.** *Let X be a non empty set, d is ordinary metric space on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.*

Definition 1.7 ([21]). *Suppose that E is a real Banach space, then P is a cone in E with $intP \neq \emptyset$, and $\leq$ is partial ordering with respect to P. Let X be a nonempty set, a function $d : X \times X \times X \to E$ is called a cone S metric on X if it satisfies the following conditions with*

(CS1) $S(u, v, z) \geq 0$.

(CS2) $S(u, v, z) = 0$ *if and only if* $u = v = z$.

(CS3) $S(u, v, z) \leq S(u, u, a) + S(v, v, a) + S(z, z, a)$.

Then the function S is called an cone S-metric on X and the pair (X, S) is called an cone S-metric space simply CSMS.

Example 1.8. *Let $E = R^2$, $P = \{(x, y) : x, y \geq 0\}$, $X = R$ and $d : X \times X \times X \to E$ such that then $S(x, y, z) = (d(x, z) + d(y, z), \alpha(d(x, z) + d(y, z)))$, $(\alpha > 0)$ is an cone S- metric on X.*

Example 1.9. *Let (X, d) be a cone metric space. Define $S : X \times X \times X \to E$ by $S(x, y, z) = d(x, z) + d(y, z) + d(z, x)$ for every $x, y, z \in X$*

Example 1.10. *Let $E = R^3$, $P = \{(x, y, z) : x, y, z \geq 0\}$, $X = R$ and $d : X \times X \times X \to E$ such that*

$$S(u, u, u) = (0, 0, 0) = S(v, v, v)$$

$$S(u, v, v) = (0, 1, 1) = S(v, u, v) = S(u, u, v)$$

$$S(v, u, u) = (0, 1, 0) = S(u, v, u) = S(u, v, u)$$

Here (x, S) is cone S metric space but not a G-cone metric space since $S(u, u, v) \neq S(u, v, v)$.

Lemma 1.11. *Let (X, S) be an cone S-metric space . Then we have $S(u, u, v) = S(v, v, u)$.*

Definition 1.12. *Let (X, S) be an cone S-metric space.*

(1). A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$. That is, there exists $n_0 \in N$ such that for all $n \geq n_0, S(u_n, u_n, u) \ll c$ for each $c \in E$, $0 \ll c$. We denote this by $\lim_{n\to\infty} u_n = u$ or $\lim_{n\to\infty} S(u_n, u_n, u) = 0$.

(2). A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$. That is, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(u_n, u_n, u_m) \ll c$ for each $c \in E$, $0 \ll c$.

(3). The cone S-metric space (X, S) is called complete if every Cauchy sequence is convergent.

In the following lemma we see the relationship between a cone metric and an cone S-metric.

Lemma 1.13. *Let (X, d) be a cone metric space. Then the following properties are satisfied:*

(1). $S(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$ is an cone S-metric on X.

(2). $u_n \to u$ in $\{X, d\}$ if and only if $u_n \to u$ in (X, S_d) :

(3). $\{u_n\}$ is Cauchy in $\{X, d\}$ if and only if $\{u_n\}$ is Cauchy in (X, S_d) :

(4). $\{X, d\}$ is complete if and only if (X, S_d) is complete.

Definition 1.14 ([1]). *A mapping $F : P^2 \to P$ is called C-class function if it is continuous and satisfies following axioms:*

(1). $F(s, t) \leq s$;

(2). $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

Note for some F we have that $F(0,0) = 0$. We denote C-class functions as $\mathcal{C}$.

Example 1.15 ([1]). *The following functions $F : P^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:*

(1). $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;

(2). $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;

(3). $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(4). $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(5). $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6). $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0, \infty)$, $F(s,t) = s \Rightarrow t = 0$;

(7). $F(s,t) = s \log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(8). $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;

(9). $F(s,t) = s\beta(s)$, $\beta : [0, \infty) \to (0, 1)$, and is continuous, $F(s,t) = s \Rightarrow s = 0$;

(10). $F(s,t) = s - \frac{t}{k+t}, F(s,t) = s \Rightarrow t = 0$;

(11). $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12). $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$, here $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;

(13). $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$.

(14). $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$, $F(s,t) = s \Rightarrow s = 0$.

(15). $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, here $\phi : [0, \infty) \to [0, \infty)$ is a upper semi continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,

(16). $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0, \infty)$, $F(s,t) = s \Rightarrow s = 0$;

Definition 1.16 ([3]). *A function $\psi : P \to P$ is called an altering distance function if the following properties are satisfied:*

(1). ψ is non-decreasing and continuous,

(2). $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.17 ([1]). *An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : P \to P$ such that $\varphi(t) > 0$, $t \ll 0$ and $\varphi(0) \geq 0$.*

We denote this set with Φ_u

Definition 1.18. *Suppose that P is a normal cone in E. $a, b \in E$ and $a < b$. we define*

$$[a,b] = \{x \in E : x = tb + (1-t)a, for\ some t \in [0,1]\}$$
$$[a,b) = \{x \in E : x = tb + (1-t)a, for\ some t \in [0,1)\} \tag{1}$$

Definition 1.19. *The set $\{a = x_0, x_1.x_2 \cdots , x_n = b\}$ is called a partition for $[a, b]$ if and only if the sets $\{x_{t-1}, x_t\}_{t=1}^{n}$ are pairwise disjoint and $[a, b] = \{\bigcup_{t=1}^{n}[x_{i-1}, x_t) \cup \{b\}\}$*

Definition 1.20. *For each partition Q of $[a, b]$ and each increasing function $\zeta : [a, b] \to P$, we define cone lower summation and cone upper summation as*

$$\begin{aligned} L_n^{con}(\zeta, Q) &= \sum_{t=0}^{n-1} \zeta(x_t)\|x_t - x_{t+1}\| \\ U_n^{con}(\zeta, Q) &= \sum_{t=0}^{n-1} \zeta(x_{t+1})\|x_t - x_{t+1}\| \end{aligned} \tag{2}$$

Respectively.

Definition 1.21. *Suppose that P is a normal cone in E. $\zeta : [a, b] \to P$ is called an integrable function on $[a, b]$ with respect to cone P or to simplicity, Cone integrable function, if and only if for all partition Q of $[a, b]$, $\lim_{n\to\infty} L_n^{con}(\zeta, Q) = S^{con} = \lim_{n\to\infty} U_n^{con}(\zeta, Q)$, where S^{con} must be unique. We show the common value S^{con} by $\int_a^b \zeta(x)d_P(x)$ to simplicity $\int_a^b \zeta d_p$*

Definition 1.22. *The function $\zeta : P \to E$ is called subadditive cone integrable function if and only if for all $a, b \in P$,*

$$\int_0^{a+b} \zeta d_p \le \int_0^a \zeta d_p + \int_0^b \zeta d_p$$

Example 1.23. *Let $E = X = R, d(x, y) = |x - y|, P = (0, \infty)$, and $\zeta(t) = \frac{1}{(t+1)}$ for all $t > 0$. Then for all $a, b \in P$,*

$$\int_0^{a+b} \frac{dt}{(t+1)} = \ln(a + b + 1), \int_0^a \frac{dt}{(t+1)} = \ln(a + 1), \int_0^b \frac{dt}{(t+1)} = \ln(b + 1)$$

Since $ab \ge 0$, then $a + b + 1 \le a + b + 1 + ab = (a + 1)(b + 1)$. Therefore

$$\ln(a + b + 1) \le \ln(a + 1) \le \ln(b + 1)$$

This shows that ζ is an example of subadditive cone integrable function.

Theorem 1.24 ([7])**.** *Let (X, S) be a complete S-metric space, $h\epsilon(0, 1)$, the function $\zeta : [0, \infty) \to [0, \infty)$ be defined as for each $\epsilon > 0, \int_0^\epsilon \zeta d_p > 0$ and $T : X \to X$ be a self-mapping of X such that*

$$\int_0^{S(Tu,Tu,Tv)} \zeta(t)dt \le h \int_0^{S(u,u,v)} \zeta(t)dt$$

for all $u, v \in X$. Then T has a unique fixed point $w \in X$ and we have $\lim_{n\to\infty} T^n u = w$, for each $u \in X$.

Theorem 1.25 ([7])**.** *Let (X, S) be a complete S-metric space, the function $\zeta : [0, \infty) \to [0, \infty)$ be defined as for each $\epsilon > 0, \int_0^\epsilon \zeta(t)dt > 0$ and $T : X \to X$ be a self-mapping of X such that*

$$\int_0^{S(Tu,Tu,Tv)} \zeta(t)dt \le h_1 \int_0^{S(u,u,v)} \zeta(t)dt + h_2 \int_0^{S(Tu,Tu,v)} \zeta(t)dt + h_3 \int_0^{S(Tv,Tv,u)} \zeta(t)dt + h_4 \int_0^{\max\{S(Tu,Tu,v),S(Tv,Tv,u)\}} \zeta(t)dt$$

for all $u, v \in X$ with non negative real numbers $h_i (i \in \{1, 2, 3, 4\})$ satisfying $\max\{h_1 + 3h_3 + 2h_4, h_1 + h_2 + h_3\} < 1$, Then T has a unique fixed point $w \in X$ and we have $\lim_{n\to\infty} T^n u = w$, for each $u \in X$.

2. Main Result

Theorem 2.1. *Let (X,S) be a complete cone S-metric space and P is a normal cone, $\psi : P \to P$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$, the function $\zeta : P \to P$ be defined as for each $\epsilon > 0, \int_0^{\epsilon} \zeta d_p > 0$ and $T : X \to X$ be a self-mapping of X such that*

$$\psi(\int_0^{S(Tu,Tu,Tv)} \zeta d_p) \leq F(\psi(\int_0^{S(u,u,v)} \zeta d_p), \varphi(\int_0^{S(u,u,v)} \zeta d_p)). \tag{3}$$

for all $u,v \in X$, Then T has a unique fixed point $w \in X$ and we have $\lim_{n\to\infty} T^n u = w$, for each $u \in X$.

Proof. Let $u_0 \in X$ and the sequence $\{u_n\}$ be defined as $T^n u_0 = u_n$. Suppose that $u_n \neq u_{n+1}$ for all n. Using the inequality (3), we obtain

$$\begin{aligned}\psi(\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p) &\leq F(\psi(\int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p), \varphi(\int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p)) \\ &\leq \psi(\int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p).\end{aligned} \tag{4}$$

so

$$\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p \leq \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p \tag{5}$$

Since $\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p > 0$, there exists $r \geq 0$ such that $\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p = r$. If $r > 0$, then take limit for $n \to \infty$, we get $\psi(r) \leq F(\psi((r), \varphi(r))$. So $\psi(r) = 0$ or $\varphi(r) = 0$. Thus $r = 0$, which is a contradiction. Thus, we conclude that $r = 0$, that is,

$$\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p = 0,$$

since for each $\epsilon > 0, \int_0^{\epsilon} \zeta d_p > 0$, implies $\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0$. Now we show that the sequence $\{u_n\}$ is a Cauchy sequence. Assume that $\{u_n\}$ is not Cauchy. Then there exists an$\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ such that $m_k < n_k < m_{k+1}$ with

$$S(u_{m_k}, u_{m_k}, u_{n_k}) \geq \epsilon \text{ and} \tag{6}$$

$$S(u_{m_k}, u_{m_k}, u_{n_{k-1}}) < \epsilon \tag{7}$$

Hence using Lemma (1.11), we have

$$S(u_{m_{k-1}}, u_{m_{k-1}}, u_{n_{k-1}}) \leq 2S(u_{m_{k-1}}, u_{m_{k-1}}, u_{m_k}) + S(u_{n_{k-1}}, u_{n_{k-1}}, u_{m_k}) < 2S(u_{m_{k-1}}, u_{m_{k-1}}, u_{m_k}) + \epsilon$$

and

$$\lim_{k\to\infty} \int_0^{S(u_{m_{k-1}}, u_{m_{k-1}}, u_{m_{k-1}})} \zeta d_p \leq \int_0^{\epsilon} \zeta d_p \tag{8}$$

Using the inequalities (3), (6) and (8) we obtain

$$\begin{aligned}\psi(\int_0^{\epsilon} \zeta d_p) &\leq \psi(\int_0^{S(u_{m_k},u_{m_k},u_{n_k})} \zeta d_p) \\ &\leq F(\psi(\int_0^{S(u_{m_k-1},u_{m_k-1},u_{n_k-1})} \zeta d_p), \varphi(\int_0^{S(u_{m_k-1},u_{m_k-1},u_{n_k-1})} \zeta d_p)) \leq F(\psi(\int_0^{\epsilon} \zeta d_p), \varphi(\int_0^{\epsilon} \zeta d_p))\end{aligned}$$

So $\psi(\int_0^{\epsilon}\zeta d_p)=0$ or $\varphi(\int_0^{\epsilon}\zeta d_p)=0$. Thus $\int_0^{\epsilon}\zeta d_p=0$, which is a contradiction with our assumption . So the sequence $\{u_n\}$ is Cauchy. Using the completeness hypothesis, there exists $w\in X$ such that $\lim_{n\to\infty}T^n u_0=w$. From the inequality (3) we find

$$\psi(\int_0^{S(Tw,Tw,u_{n+1})}\zeta d_p)\leq F(\psi(\int_0^{S(w,w,u_n)}\zeta d_p),\varphi(\int_0^{S(w,w,u_n)}\zeta d_p))$$

Therefore

$$\lim_{n\to\infty}\|\psi(\int_0^{S(Tw,Tw,x_{n+1})}\zeta d_p)\|\leq K\|\psi(\int_0^{S(Tw,Tw,w)}\zeta d_p)\| \text{ where } K>0$$

So $\psi(\int_0^{S(Tw,Tw,w)}\zeta d_p)=0$ or $\varphi(\int_0^{S(Tw,Tw,w)}\zeta d_p)=0$. Thus $\int_0^{S(Tw,Tw,w)}\zeta d_p=0$, which implies that $S(Tw,Tw,w)\ll 0$. Thus $Tw=w$. Now we show the uniqueness of the fixed point. Suppose that w_1 is another fixed point of T. Using the inequality (3) we have

$$\psi(\int_0^{S(w,w,w_1)}\zeta d_p)=\psi(\int_0^{S(w,w,w_1)}\zeta d_p)\leq F(\psi(\int_0^{S(w,w,w_1)}\zeta d_p),\varphi(\int_0^{S(w,w,w_1)}\zeta d_p))$$

So $\psi(\int_0^{S(w,w,w_1)}\zeta d_p)=0$ or $\varphi(\int_0^{S(w,w,w_1)}\zeta d_p)=0$. Thus $\int_0^{S(w,w,w_1)}\zeta d_p=0$. Using the $\int_0^{\epsilon}\zeta d_p>0$ we get $w=w_1$. Consequently, the fixed point w is unique. □

With choice $F(s,t)=hs$, $0<h<1$,$\psi(t)=t$, in theorem (2.1) we have

Corollary 2.2 ([7])**.** *Let (X,S) be a complete cone S-metric space and P is a normal cone, $h\epsilon(0,1)$, the function $\zeta:P\to P$ be defined as for each $\epsilon>0$, $\int_0^{\epsilon}\zeta d_p>0$ and $T:X\to X$ be a self-mapping of X such that*

$$\int_0^{S(Tu,Tu,Tv)}\zeta d_p\leq h\int_0^{S(u,u,v)}\zeta d_p \tag{9}$$

for all $u,v\in X$. Then T has a unique fixed point $w\in X$ and we have $\lim_{n\to\infty}T^n u=w$, for each $u\in X$.

Example 2.3. *Let $X=R$, $k=10$ be a fixed real number and function $S:X\times X\times X\to[0,\infty)$ be defined as*

$$S(u,v,z)=\frac{z}{k+1}(|v-z|+|v+z-2u|)$$

for all $u,v,z\in R$. It can be ready seen that the function S is an cone S-metric. Now we show that cone S metric can not be generated by cone metric ρ. On the contrary, we assume that there exists a metric ρ such that

$$S(u,v,z)=\rho(u,z)+\rho(v,z) \tag{10}$$

for all $u,v,z\in R$.

$$\rho(u,z)=\frac{10}{11}|u-z| \tag{11}$$

Similarly, we have $S(v,v,z)=2\rho(v,z)=\frac{20}{11}(|v-z|+|v+z-2u|)$ and

$$\rho(v,z)=\frac{10}{11}|v-z| \tag{12}$$

Using the equalities above equation (10), (11) and (12), we obtain

$$\frac{10}{11}(|v-z|+|v+z-2u|)=\frac{10}{11}|u-z|+\frac{10}{11}|v-u|$$

which is a contradiction, S is not generated by any metric and (R,S) is a complete cone S-metric space. $T:R\to R$ and $Tu=\frac{u}{4}$ for all $u\in R$; $\zeta:P\to P$ where $P=(0,\infty)$ as $\zeta(t)=2t$. Let $F(s,t)=s-t$ for all $s,t\in[0,\infty)$. Also define $\varphi,\psi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t$ and $\varphi(t)=\frac{t}{2}$.

$$F(\psi(\int_0^{S(u,u,v)}\zeta d_p),\varphi(\int_0^{S(u,u,v)}\zeta d_p))=\psi(\int_0^{S(u,u,v)}\zeta d_p)-\varphi(\int_0^{S(u,u,v)}\zeta d_p) \tag{13}$$

From equation (13), we have

$$\begin{aligned}F(\psi(\int_0^{\epsilon}\zeta(t)d_p(t)),\varphi(\int_0^{\epsilon}\zeta(t)d_p(t)))&=\psi(\int_0^{\epsilon}\zeta(t)d_p(t))-\varphi(\int_0^{\epsilon}\zeta(t)d_p(t))\\&=\psi(\int_0^{\epsilon}2td_p(t))-\varphi(\int_0^{\epsilon}2td_p(t))\\&=\epsilon^2-\frac{\epsilon^2}{2}>0\end{aligned}$$

for all $\epsilon>0$, T satisfies the inequalities (3).

$$\frac{100}{4(121)}|u-v|^2\leq\frac{4\times100}{121}|u-v|^2\quad\forall u,v\in R$$

T has a unique fixed point $u=0$.

Theorem 2.4. *Let (X,S) be a complete cone S-metric space and P is a normal cone, $\psi:P\to P$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$, the function $\zeta:P\to P$ be defined as for each $\epsilon\gg0,\int_0^{\epsilon}\zeta d_p\gg0$ and $T:X\to X$ be a self-mapping of X such that*

$$\begin{aligned}\psi(\int_0^{S(Tu,Tu,Tv)}\zeta d_p)\leq F(\psi(h_1\int_0^{S(u,u,v)}\zeta d_p+h_2\int_0^{S(Tu,Tu,v)}\zeta d_p+h_3\int_0^{S(Tv,Tv,u)}\zeta d_p\\+h_4\int_0^{\max\{S(Tu,Tu,v),S(Tv,Tv,u)\}}\zeta d_p),\varphi(h_1\int_0^{S(u,u,v)}\zeta d_p+h_2\\\int_0^{S(Tu,Tu,v)}\zeta d_p+h_3\int_0^{S(Tv,Tv,u)}\zeta d_p+h_4\int_0^{\max\{S(Tu,Tu,v),S(Tv,Tv,u)\}}\zeta d_p))\end{aligned} \tag{14}$$

for all $u,v\in X$ with non negative real numbers $h_i(i\in\{1,2,3,4\})$ satisfying $\max\{h_1+3h_3+2h_4,h_1+h_2+h_3\}=1$. *Then T has a unique fixed point $w\in X$ and we have $\lim_{n\to\infty}T^nu=w$, for each $u\in X$.*

Proof. Let $u_0\in X$ and the sequence $\{u_n\}$ be defined as $\lim_{n\to\infty}T^nu_0=u_n$ Suppose that $u_n\neq u_{n+1}$ for all n. Using the inequality (14), the condition (S2) and Lemma (1.11)we get

$$\psi(\int_0^{S(u_n,u_n,u_{n+1})}\zeta d_p)=\psi(\int_0^{S(Tu_{n-1},Tu_{n-1},Tu_n)}\zeta d_p)$$

$$\leq F(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_2 \int_0^{S(u_n,u_n,u_n)} \zeta d_p + h_3 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta d_p + h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \zeta d_p),$$

$$\varphi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_2 \int_0^{S(u_n,u_n,u_n)} \zeta d_p + h_3 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta d_p + h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \zeta d_p))$$

$$= F(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_3 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta d_p + h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \zeta d_p), \varphi(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_3 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta d_p + h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \zeta d_p))$$

$$\leq F(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_3 \int_0^{2S(u_{n+1},u_{n+1},u_n)} \zeta d_p + h_3 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_4 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta d_p$$

$$+ h_4 \int_0^{S(u_{n+1},u_{n+1},u_n)} \zeta d_p), \varphi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_3 \int_0^{2S(u_{n+1},u_{n+1},u_n)} \zeta d_p + h_3 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + h_4 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta d_p + h_4 \int_0^{S(u_{n+1},u_{n+1},u_n)} \zeta d_p))$$

$$= F(\psi((h_1 + h_3 + h_4) \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + (2h_3 + h_4) \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p), \varphi(h_1 + h_3 + h_4) \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + (2h_3 + h_4) \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p))$$

$$\Rightarrow \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p \leq \frac{h_1 + h_3 + h_4}{1 - 2h_3 - h_4} \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p = \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p \tag{15}$$

Since $\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p > 0$, so there exists $r \geq 0$ such that $\lim_{n\to+\infty} \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p = r$. If $r > 0$, then take limit for $n \to \infty$, we get $\psi(r) \leq F(\psi(r), \varphi(r))$. So $\psi(r) = 0$ or $\varphi(r) = 0$. Thus $r = 0$, which is a contradiction. Thus, we conclude that $r = 0$, that is, $\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p = 0$, since for each $\epsilon > 0, \int_0^{\epsilon} \zeta d_p > 0$, implies $\lim_{n\to\infty} S(u_n, u_n, u_{n+1}) = 0$. By the similar arguments used in the proof of Theorem (2.1), we see that the sequence $\{u_n\}$ is Cauchy. Then there exists $w \in X$ such that $\lim_{n\to\infty} T^n u_0 = w$, since (X, S) is a complete cone S-metric space. From the inequality (14) we find

$$\psi(\int_0^{S(u_n,u_n,Tw)} \zeta d_p) = \psi(\int_0^{S(Tu_{n-1},Tu_{n-1},Tw)} \zeta d_p)$$
$$\leq F(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},Tw)} \zeta d_p + h_2 \int_0^{S(u_n,u_n,w)} \zeta d_p$$
$$+ h_3 \int_0^{S(Tw,Tw,u_{n-1})} \zeta d_p + h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(Tw,Tw,w)\}} \zeta d_p),$$
$$\varphi(h_1 \int_0^{S(u_{n-1},u_{n-1},Tw)} \zeta d_p + h_2 \int_0^{S(u_n,u_n,w)} \zeta d_p + h_3 \int_0^{S(Tw,Tw,u_{n-1})} \zeta d_p$$
$$+ h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(Tw,Tw,w)\}} \zeta d_p))$$

$\lim_{n\to\infty} \|\psi((h_3 + h_4) \int_0^{S(Tw,Tw,u_n)} \zeta d_p)\| = K\|\psi((h_3 + h_4) \int_0^{S(Tw,Tw,w)} \zeta d_p)\|$ where $K > 0$. So $\psi((h_3 + h_4) \int_0^{S(Tw,Tw,w)} \zeta d_p) = 0$ or $\varphi((h_3 + h_4) \int_0^{S(Tw,Tw,w)} \zeta d_p) = 0$. Thus $\int_0^{S(Tw,Tw,w)} \zeta d_p = 0$, which implies that $S(Tw, Tw, w) \ll 0$. Thus $Tw = w$. Now we show the uniqueness of the fixed point. Let w_1 be another fixed point of T. Using the inequality (14) and Lemma (1.11), we get

$$\psi(\int_0^{S(w,w,w_1)} \zeta d_p) = \psi(\int_0^{S(Tw,Tw,w_1)} \zeta d_p$$
$$\leq F(\psi(h_1 \int_0^{S(w,w,w_1)} \zeta(t)dt + h_2 \int_0^{S(w,w,w_1)} \zeta(t)dt + h_3 \int_0^{S(w_1,w_1,w)} \zeta d_p$$
$$+ h_4 \int_0^{\max\{S(w,w,w),S(w_1,w_1,w_1)\}} \zeta(t)dt), \varphi(h_1 \int_0^{S(w,w,w_1)} \zeta d_p$$
$$+ h_2 \int_0^{S(w,w,w_1)} \zeta(t)dt + h_3 \int_0^{S(w_1,w_1,w)} \zeta d_p$$
$$+ h_4 \int_0^{\max\{S(w,w,w),S(w_1,w_1,w_1)\}} \zeta d_p))$$

which implies

$$\psi(\int_0^{S(w,w,w_1)} \zeta d_p) \leq F(\psi((h_1 + h_2 + h_3) \int_0^{S(w,w,w_1)} \zeta d_p), \varphi((h_1 + h_2 + h_3) \int_0^{S(w,w,w_1)} \zeta d_p))$$
$$\leq \psi((h_1 + h_2 + h_3) \int_0^{S(w,w,w_1)} \zeta d_p) \leq \psi(\int_0^{S(w,w,w_1)} \zeta d_p)$$

So $\psi((h_1 + h_2 + h_3) \int_0^{S(w,w,w_1)} \zeta d_p) = 0$ or $\varphi((h_1 + h_2 + h_3) \int_0^{S(w,w,w_1)} \zeta(t)dt) = 0$. Then we obtain

$$\int_0^{S(w,w,w_1)} \zeta d_p = 0$$

that is, $w = w_1$ since $h_1 + h_2 + h_3 < 1$. Consequently, T has a unique fixed point $w \in X$. □

With choice $F(s,t) = hs$, $0<h<1$,$\psi(t) = t$,(replace h_i with hh_i) in Theorem (2.4) we have

Corollary 2.5 ([7]). *Let (X,S) be a complete cone S-metric space and P is a normal cone, $h\epsilon(0,1)$, the function $\zeta : P \to P$ be defined as for each $\epsilon \gg 0, \int_0^{\epsilon} \zeta d_p \gg 0$ and $T : X \to X$ be a self-mapping of X such that*

$$\int_0^{S(Tu,Tu,Tv)} \zeta d_p \leq h_1 \int_0^{S(u,u,v)} \zeta d_p + h_2 \int_0^{S(Tu,Tu,v)} \zeta d_p + h_3 \int_0^{S(Tv,Tv,u)} \zeta d_p + h_4 \int_0^{\max\{S(Tu,Tu,v),S(Tv,Tv,u)\}} \zeta d_p$$

for all $u,v \in X$ with non negative real numbers $h_i(i \in \{1,2,3,4\})$ satisfying $\max\{h_1+3h_3+2h_4, h_1+h_2+h_3\} < 1$. Then T has a unique fixed point $w \in X$ and we have $\lim_{n\to\infty} T^n u = w$, for each $u \in X$.

Example 2.6. *Let $X = R$ be the complete cone S-metric space with cone cone S-metric space defined in example (2.3). Let us define the self mapping $T : R \to R$ as*

$$Tu = \begin{cases} 2u+39 & u \in (0,3) \\ 90 & otherwise \end{cases}$$

forall $u \in R$ and define a function $\zeta : P \to P$ where $P = (0,\infty)$ as $\zeta(t) = 2t$

$$\int_0^{\epsilon} \zeta(t) d_p(t) = \int_0^{\epsilon} 2t d_p(t) = \epsilon^2 > 0 \quad \epsilon > 0.$$

T satisfy the inequality (14) in theorem (2.4) for $h_1 = h_2 = h_3 = 0$, $h_4 = \frac{1}{2}$ and the inequality (16) in theorem (2.7) for $h_1 = h_3 = h_5 = 0$, $h_2 = \frac{1}{3}$. Hence T has a unique fixed point 90. But T does not satisfy the inequality (16) in theorem (2.7) But T does not satisfy the inequality (3) in theorem (2.1). Indeed, if we take $u = 0$ and $v = 1$, then we obtain

$$\psi(\int_0^{10} 2t d_p(t)) = 100 \leq F(\psi(h \int_0^{3} 2t d_p(t)), \varphi(h \int_0^{3} 2t d_p(t))) \leq \psi(h \int_0^{3} 2t d_p(t)) \leq 9h$$

which is a contradiction since $h \in (0,1)$

Theorem 2.7. *Let (X,S) be a complete cone S-metric space and P is a normal cone, $\psi : P \to P$ is an altering distance function,$\varphi \in \Phi_u$ and $F \in \mathcal{C}$, the function $\zeta : P \to P$ be defined as for each $\epsilon \gg 0, \int_0^{\epsilon} \zeta d_p \gg 0$ and $T : X \to X$ be a self-mapping of X such that*

$$\begin{aligned} \psi(\int_0^{S(Tu,Tu,Tv)} \zeta d_p) \leq F(\psi(h_1 &\int_0^{S(u,u,v)} \zeta d_p + h_2 \int_0^{S(Tu,Tu,u)} \zeta d_p + h_3 \\ &\int_0^{S(Tu,Tu,v)} \zeta d_p + h_4 \int_0^{S(Tv,Tv,u)} \zeta d_p + h_5 \int_0^{S(Tv,Tv,v)} \zeta d_p \\ &+ h_6 \int_0^{\max\{S(u,u,v),S(Tu,Tu,u),S(Tu,Tu,v),S(Tv,Tv,u),S(Tv,Tv,v)\}} \zeta d_p), \\ &\varphi(h_1 \int_0^{S(u,u,v)} \zeta d_p + h_2 \int_0^{S(Tu,Tu,u)} \zeta d_p + h_3 \int_0^{S(Tu,Tu,v)} \zeta d_p \\ &+ h_4 \int_0^{S(Tv,Tv,u)} \zeta d_p + h_5 \int_0^{S(Tv,Tv,v)} \zeta d_p + h_6 \\ &\int_0^{\max\{S(u,u,v),S(Tu,Tu,u),S(Tu,Tu,v),S(Tv,Tv,u),S(Tv,Tv,v)\}} \zeta d_p)) \end{aligned} \tag{16}$$

for all $u, v \in X$ with non negative real numbers $h_i (i \in \{1,2,3,4,5,6\})$ satisfying $h_1+h_2+3h_4+h_5+3h_6, h_1+h_3+h_4+h_6 = 1$, then T has a unique fixed point $w \in X$ and we have $\lim_{n\to\infty} T^n u = w$, for each $u \in X$.

Proof. Let $u_0 \in X$ and the sequence $\{u_n\}$ be defined as $\lim_{n\to\infty} T^n u_0 = u_n$. Suppose that $u_n \neq u_{n+1}$ for all n. Using the inequality (14), the condition (S2) and Lemma (1.11), we get

$$\psi(\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p) = \psi(\int_0^{S(Tu_{n-1},Tu_{n-1},Tu_n)} \zeta d_p)$$

$$\leq F(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta(t)dt + h_2 \int_0^{S(u_n,u_n,u_{n-1})} \zeta d_p + h_3$$

$$\int_0^{S(u_n,u_n,u_n)} \zeta(t)dt + h_4 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta(t)dt + h_5 \int_0^{S(u_{n+1},u_{n+1},u_n)} \zeta d_p + h_6$$

$$\int_0^{\max\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n-1}),S(u_n,u_n,u_n),S(u_{n+1},u_{n+1},u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \zeta d_p),$$

$$\varphi(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta(t)dt + h_2 \int_0^{S(u_n,u_n,u_{n-1})} \zeta(t)dt + h_3 \int_0^{S(u_n,u_n,u_n)} \zeta d_p$$

$$+ h_4 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \zeta(t)dt + h_5 \int_0^{S(u_{n+1},u_{n+1},u_n)} \zeta d_p + h_6$$

$$\int_0^{\max\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n-1}),S(u_n,u_n,u_n),S(u_{n+1},u_{n+1},u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \zeta d_p)))$$

$$\leq F(\psi((h_1+h_2+h_4+h_6) \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p + (2h_4+h_5+2h_6)$$

$$\int_0^{S(u_{n+1},u_{n+1},u_n)} \zeta d_p), \varphi((h_1+h_2+h_4+h_6) \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta(t)dt$$

$$+ (2h_4+h_5+2h_6) \int_0^{S(u_{n+1},u_{n+1},u_n)} \zeta d_p))$$

which implies

$$\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p \leq (\frac{h_1+h_2+h_4+h_6}{1-2h_4-h_5-2h_6}) \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p = h \int_0^{S(u_{n-1},u_{n-1},u_n)} \zeta d_p \tag{17}$$

Since $\int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p > 0$, so there exists $r \geq 0$ such that $\lim_{n\to+\infty} \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p = r$. If $r > 0$, then take limit for $n \to \infty$, we get $\psi(r) \leq F(\psi(r), \varphi(r))$. So $\psi(r) = 0$ or $\varphi(r) = 0$. Thus $r = 0$, which is a contradiction. Thus, we conclude that $r = 0$, that is,

$$\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \zeta d_p = 0,$$

since for each $\epsilon > 0, \int_0^{\epsilon} \zeta d_p > 0$, implies $\lim_{n\to\infty} S(u_n, u_n, u_{n+1}) = 0$. By the similar arguments used in the proof of Theorem (2.1), we see that the sequence $\{u_n\}$ is Cauchy. Then there exists $w \in X$ such that $\lim_{n\to\infty} T^n u_0 = w$, since (X, S) is a complete cone S-metric space. From the inequality (16) we find

$$\begin{aligned}
\psi(\int_0^{S(u_n,u_n,Tw)} \zeta d_p) &= \psi(\int_0^{S(Tu_{n-1},Tu_{n-1},Tw)} \zeta d_p \\
&\leq F(\psi(h_1 \int_0^{S(u_{n-1},u_{n-1},w)} \zeta(t)dt + h_2 \int_0^{S(u_n,u_n,u_{n-1})} \zeta d_p \\
&+ h_3 \int_0^{S(u_n,u_n,w)} \zeta d_p + h_4 \int_0^{S(Tw,Tw,u_{n-1})} \zeta(t)dt + h_5 \int_0^{S(Tw,Tw,w)} \zeta d_p \\
&+ h_6 \int_0^{\max\{S(u_{n-1},u_{n-1},w),S(u_n,u_n,u_{n-1}),S(u_n,u_n,w),S(Tw,Tw,u_{n-1}),S(Tw,Tw,w)\}} \zeta d_p),
\end{aligned}$$

$$\begin{aligned}
&\varphi(h_1 \int_0^{S(u_{n-1},u_{n-1},w)} \zeta d_p + h_2 \int_0^{S(u_n,u_n,u_{n-1})} \zeta d_p \\
&+ h_3 \int_0^{S(u_n,u_n,w)} \zeta d_p + h_4 \int_0^{S(Tw,Tw,u_{n-1})} \zeta d_p + h_5 \int_0^{S(Tw,Tw,w)} \zeta d_p \\
&+ h_6 \int_0^{\max\{S(u_{n-1},u_{n-1},w),S(u_n,u_n,u_{n-1}),S(u_n,u_n,w),S(Tw,Tw,u_{n-1}),S(Tw,Tw,w)\}} \zeta d_p))
\end{aligned}$$

$\lim_{n\to\infty} \|\psi((h_4 + h_5 + h_6) \int_0^{S(Tw,Tw,u_n)} \zeta d_p)\| \leq K\|\psi((h_4 + h_5 + h_6) \int_0^{S(Tw,Tw,w)} \zeta d_p)\|$, where $K > 0$. So $\psi((h_4 + h_5 + h_6) \int_0^{S(Tw,Tw,w)} \zeta d_p) = 0$ or $\varphi((h_4 + h_5 + h_6) \int_0^{S(Tw,Tw,w)} \zeta d_p) = 0$. Thus $\int_0^{S(Tw,Tw,w)} \zeta d_p = 0$, which implies that $S(Tw,Tw,w) \ll 0$. Thus $Tw = w$. Now we show the uniqueness of the fixed point. Let w_1 be another fixed point of T. Using the inequality (16) and Lemma (1.11), we get

$$\begin{aligned}
\psi(\int_0^{S(w,w,w_1)} \zeta d_p) &= \psi(\int_0^{S(Tw,Tw,Tw_1)} \zeta d_p) \\
&\leq F(\psi(h_1 \int_0^{S(w,w,w_1)} \zeta(t)dt + h_2 \int_0^{S(w,w,w)} \zeta d_p \\
&+ h_3 \int_0^{S(w,w,w_1)} \zeta d_p + h_4 \int_0^{S(w_1,w_1,w)} \zeta(t)dt + h_5 \int_0^{S(w_1,w_1,w_1)} \zeta d_p \\
&+ h_6 \int_0^{\max\{S(w,w,w_1),S(w,w,w),S(w,w,w_1),S(w_1,w_1,w),S(w_1,w_1,w_1)\}} \zeta d_p), \\
&\varphi(h_1 \int_0^{S(w,w,w_1)} \zeta(t)dt + h_2 \int_0^{S(w,w,w)} \zeta d_p \\
&+ h_3 \int_0^{S(w,w,w_1)} \zeta d_p + h_4 \int_0^{S(w_1,w_1,w)} \zeta(t)dt + h_5 \int_0^{S(w_1,w_1,w_1)} \zeta d_p \\
&+ h_6 \int_0^{\max\{S(w,w,w_1),S(w,w,w),S(w,w,w_1),S(w_1,w_1,w),S(w_1,w_1,w_1)\}} \zeta d_p))
\end{aligned}$$

which implies

$$\psi(\int_0^{S(w,w,w_1)} \zeta d_p) \leq F(\psi((h_1+h_3+h_4+h_6)\int_0^{S(w,w,w_1)} \zeta d_p), \varphi((h_1+h_3+h_4+h_6)\int_0^{S(w,w,w_1)} \zeta d_p))$$
$$\leq \psi((h_1+h_3+h_4+h_6)\int_0^{S(w,w,w_1)} \zeta d_p) \leq \psi(\int_0^{S(w,w,w_1)} \zeta d_p)$$

So $\psi((h_1+h_3+h_4+h_6)\int_0^{S(w,w,w_1)} \zeta(t)dt) = 0$ or $\varphi((h_1+h_3+h_4+h_6)\int_0^{S(w,w,w_1)} \zeta d_p) = 0$. Then we obtain

$$\int_0^{S(w,w,w_1)} \zeta d_p = 0$$

that is, $w = w_1$ since $h_1+h_3+h_4+h_6 < 1$. Consequently, T has a unique fixed point $w \in X$. □

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