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Some Identities Relating Euler Type Integrals And Certain Polynomials

Research Article

N. U. Khan[1], M. M. Eltikali[2]*, T. M. Elfrgani[2] and A. Paul[2]

1 Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh, India.

2 Department of Mathematics, Sam Higginbottom Institute of Agriculture Technology and Sciences, Allahabad, India.

Abstract: In this paper, we establish new results involving Euler type integral by making use of some known integrals. On an account of the general nature of the functions involved, we obtained certain new integrals follows as a pplications of the main result involving Laguerre, Jacobi, Hermite, Gegenbauer and Chebyshev polynomials.

Keywords: Euler integral, exponantial function and polynomials.

1. Introduction

The integrals can be used to derive summation formulae, representations, generating relations and other properties for the new and known families of special functions. The Beta function [5] $\beta(p,q)$ is a function of two complex variables p and q, defined by the Eulerian integral of the first kind

$$\beta(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad (\Re(p),\ \Re(q)>0). \tag{1}$$

A general class of polynomials with essentially arbitrary coefficients is defined and represented in the following form [11]:

$$S_n^m[x]=\sum_{k=0}^{[n/m]}(-n)_{mk}A_{n,k}\frac{x^k}{k!}, \qquad n=0,\ 1,\ 2,\ \cdots \tag{2}$$

where m is an arbitary positive integer, the coefficients $A_{n,k}(n,k\geq 0)$ are arbitrary constant, real or complex and $(\lambda)_n$ denote the pochhammer symbol. By suitably specializing the coeffients $A_{n,k}$ the polynomials $S_n^m[x]$ can be reduce to the classical orthogonal polynomials including for example the Laguerre polynomials [12]:

$$L_n^{(\alpha)}(y)=\sum_{k=0}^{\infty}\binom{n+\alpha}{n-k}\frac{(-y)^k}{k!}=\binom{n+\alpha}{n}\,{}_1F_1\left[\begin{matrix}-n; & \\ & y\\ \alpha+1; & \end{matrix}\right], \tag{3}$$

* E-mail: mahmoodeltikali@gmail.com

in which case $m=1,\ A_{n,k}=\begin{pmatrix} n+\alpha \\ n \end{pmatrix} \frac{1}{(\alpha+1)_k}$

$$S_n^1(y) \longrightarrow L_n^{(\alpha)}(y)$$

The Jacobi polynomials [12]:

$$P_n^{(\alpha,\beta)}(y)=\sum_{k=0}^{\infty}\begin{pmatrix} n+\alpha \\ n-k \end{pmatrix}\begin{pmatrix} n+\beta \\ k \end{pmatrix}\frac{(y-1)^k}{2}\left(\frac{y+1}{2}\right)^{n-k}. \tag{4}$$

$$=\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,\alpha+\beta+n+1; \\ \alpha+1; \end{matrix}\ \frac{1-y}{2}\right], \tag{5}$$

in which case $m=1,\quad A_{n,k}=\begin{pmatrix} n+\alpha \\ n \end{pmatrix} \frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k}$

$$S_n^1(y) \longrightarrow P_n^{(\alpha,\beta)}(1-2y)$$

The polynomial $S_n^m[x]$ defined in (2) can be further reduced to the Hermite polynomials $H_n[x]$, (see [12]) Gegenbauer polynomials $C_n^{\nu}(x)$, Legendre polynomials $P_n[x]$, and Chebyshev polynomials $T_n[x]$ and $U_n[x]$ of the first and second kind (see [6]). A general class of polynomials $S_n^m[x]$ also includes the hypergeometric polynomials as the Bessel polynomials [4], Gould-Hopper polynomials [2], Brafman polynomials [1], extended Jacobi polynomials [10] and their generalization studied in the literature [9]. We consider the known integrals defined as [3] as:

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}dx=\frac{1}{2}\frac{\beta(p+\frac{1}{2},\frac{1}{2})}{a(4ab+c)^{p+\frac{1}{2}}}, \tag{6}$$

where $a>0,\ b>0,\ c>-4ab,\ p>\frac{-1}{2}$ and $\beta(p,q)$ is Beta function given by (1).

2. Main Results

Theorem 2.1.

$$\int_0^{\infty}\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}dx=\frac{1}{2}\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\frac{\beta(p-r+\frac{1}{2},\frac{1}{2})}{a(4ab+c)^{p-r+\frac{1}{2}}} \tag{7}$$

where $a>0,\ b>0,\ c>-4ab,\ p>\frac{-1}{2}$ *and* $\beta(p-r+\frac{1}{2},\frac{1}{2})$ *is Beta function.*

Proof. Using series expansion of exponential function, the l.h.s of equation (7) becomes

$$\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\int_0^{\infty}\left((ax+\frac{b}{x})^2+c\right)^r\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}dx=\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{r-p-1}dx$$

Now, by using the integral (6) and after simplification, we get the required result. □

Theorem 2.2. *The following integral involving Srivastava polynomials holds true:*

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1}S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{\sigma}\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a\,(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\sum_{k=0}^{[n/m]}(-n)_{mk}\ A_{n,k}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})\ y^k}{(4ab+c)^{-k\sigma-r}\ k!} \tag{8}$$

Proof. Using (7), the l.h.s of equation (8) becomes

$$\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta+r-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{\sigma}\right]dx \tag{9}$$

Now using (2) in (9), after simplification we get

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{\sigma}\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\sum_{k=0}^{[n/m]}(-n)_{mk}A_{n,k}\frac{y^k}{k!}\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{k\sigma+r+\eta-1}dx \tag{10}$$

Now, using (6) in (10), after simplification we get the result (8). □

3. Applications

(1).

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{\sigma}\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r\ \beta(-\sigma k-r-\eta+\frac{1}{2},\ \frac{1}{2})}{r!\ (4ab+c)^{-r}}\ L_n^{(\alpha)}(y) \tag{11}$$

In order to prove the main integral (11), we consider the left hand side of (11) by I_1. Taking $m=1$ and $A_{n,k}=\begin{pmatrix} n+\alpha \\ n\end{pmatrix}\frac{1}{(\alpha+1)_k\ (4ab+c)^{k\sigma}}$ in (8) and using (2) and (6), we get

$$I_1=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\sum_{k=0}^{\infty}(-n)_k\begin{pmatrix} n+\alpha \\ n\end{pmatrix}\frac{1}{(\alpha+1)_k}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})y^k}{(4ab+c)^{-r}\ k!}. \tag{12}$$

Now, using (3) in (12), after simplification, we get the result (11).

(2).

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{\sigma}\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r\ \beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}\ r!}\ p_n^{(\alpha+\beta)}(1-2y). \tag{13}$$

In order to prove the main integral (13), we consider the left hand side of (13) by I_2. Taking $m=1$, $A_{n,k}=\begin{pmatrix} n+\alpha \\ n\end{pmatrix}\frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k\ (4ab+c)^{k\sigma}}$ and $y=\frac{1-z}{2}$ in (8) and using (2) and (6), we get

$$I_2=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\sum_{k=0}^{\infty}\frac{(-n)_k}{k!}\begin{pmatrix} n+\alpha \\ n\end{pmatrix}\frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}}\left(\frac{1-z}{2}\right)^k. \tag{14}$$

Now, using (5) in (14) after simplification, we get the result (13).

(3).

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^\sigma\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r\ \beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}\ r!}\ H_n(y). \tag{15}$$

In order to prove the main integral (15), we consider the left hand side of (15) by I_3. Taking $m=2$, $A_{n,k}=\frac{(2z)^n}{(4ab+c)^{k\sigma}}$ and $y=\frac{-1}{4z^2}$ in (8) and using (2) and (6), we get

$$I_3=\frac{(2z)^n}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r}{r!}\sum_{k=0}^{[\frac{n}{2}]}(-n)_{2k}\left(\frac{-1}{4z^2}\right)^k\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}\ k!} \tag{16}$$
$$=\frac{(2z)^n}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r}{r!}\sum_{k=0}^{[\frac{n}{2}]}(2)^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k\left(\frac{-1}{4z^2}\right)^k\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}\ k!} \tag{17}$$

Now, after simplification we get the result (15).

(4).

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^\sigma\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r\ \beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}\ r!}\ C_n^{(\nu)}(y). \tag{18}$$

In order to prove the main integral (18), we consider the left hand side of (18) by I_4. Taking $m=2$, $A_{n,k}=\begin{pmatrix} n+2\nu-1 \\ n \end{pmatrix}\frac{(n+2\nu)_k}{(4ab+c)^{k\sigma}\ (v+\frac{1}{2})_k}$ and $y=\frac{1-z}{2}$ in (8) and using (2) and (6), we get

$$I_4=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r}{r!}\begin{pmatrix} n+2\nu-1 \\ n \end{pmatrix}\frac{(-n)_k(n+2\nu)_k}{(v+\frac{1}{2})_k\ k!}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}}\left(\frac{1-z}{2}\right)^k \tag{19}$$

Now, after simplification we get the result (18).

(5).

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^\sigma\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r}{r!}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}}\ T_n(y). \tag{20}$$

In order to prove the main integral (20), we consider the left hand side of (20) by I_5. Taking $m=2$, $A_{n,k}=\frac{(n)_k}{(4ab+c)^{k\sigma}\ (\frac{1}{2})_k}$ and $y=\frac{1-z}{2}$ in (8) and using (2) and (6), we get

$$I_5=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^\infty\frac{(-q)^r}{r!}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-n)_k\,(n)_k}{(\frac{1}{2})_k\ k!}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}}\left(\frac{1-z}{2}\right)^k \tag{21}$$

Now, after simplification we get the result (20).

(6).

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{\eta-1} S_n^m\left[y\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^\sigma\right]\exp\left\{-q\left[\left(ax+\frac{b}{x}\right)^2+c\right]\right\}dx$$
$$=\frac{1}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}}U_n(y). \quad (22)$$

In order to prove the main integral (22), we consider the left hand side of (22) by I_6. Taking $m=2$, $A_{n,k}=\frac{(n+1)(n+1)_k}{(4ab+c)^{k\sigma}\ (\frac{3}{2})_k}$ and $y=\frac{1-z}{2}$ in (8) and using (2) and (6), we get

$$I_6 = \frac{(n+1)}{2a(4ab+c)^{-(\eta-\frac{1}{2})}}\sum_{r=0}^{\infty}\frac{(-q)^r}{r!}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-n)_k\,(n+1)_k}{(\frac{3}{2})_k\ k!}\frac{\beta(-\sigma k-r-\eta+\frac{1}{2},\frac{1}{2})}{(4ab+c)^{-r}}\left(\frac{1-z}{2}\right)^k \quad (23)$$

Now, after simplification we get the result (22).

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