International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(1)**(2023), 41–48
Available Online: http://ijmaa.in

Cesàro Difference Sequence Spaces and its Dual

Shadab Ahmad Khan[1,*]

[1]*Department of Mathematics, SGN Government PG College, Muhammadabad, Gohna, Mau, Uttar Pradesh, India*

Abstract

The difference sequence spaces $c_0(\Delta), c(\Delta)$ and $\ell_\infty(\Delta)$ were introduced by Kizmaz [4]. Et [8] introduced the Cesàro difference sequence spaces $X_p(\Delta^m)\,(1 \leq p < \infty), X_\infty(\Delta^m)$ and determine their generalized Köthe-Toeplitz duals and some of the related matrix transformations. In this paper, we compute η-duals of $C_1(\Delta), C_1(\Delta^2)$ and $X_\infty(\Delta^2)$, the matrix classes $(C_1(\Delta), \ell_\infty), (C_1(\Delta), c; p), (C_1(\Delta), C_0), (C_1(\Delta^2), \ell_\infty), (C_1(\Delta^2), c)$, and $(C_1(\Delta^2), c_0)$ are also characterized.

Keywords: Schauder basis; matrix map; BK space.
2020 Mathematics Subject Classification: 40C05, 40A05, 46A45.

1. Introduction

Let ω denote the linear space of all complex sequences over $\mathbb{C}$ (the field of complex numbers). ℓ_∞, c and c_0 denote the space of all bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. A complete metric linear space is called a Frèchet space. Let X be a linear subspace of ω such that X is a Frèchet space with continuous coordinate projections. Then we say that X is a FK space. If the metric of a FK space is given by a complete norm, then we say that X is a BK space. We say that a FK space X has AK, or has the AK property, if (e_k), the sequence of unit vectors, is a Schauder basis for X. A sequence space X is called

(i) normal (or solid) if $y = (y_k) \in X$ whenever $|y_k| \leq |x_k|, k \geq 1$, for some $x = (x_k) \in X$,

(ii) monotone if it contains the canonical preimages of all its stepspaces,

(iii) sequence algebra if $xy = (x_k y_k) \in X$ whenever $x = (x_k), y = (y_k) \in X$,

(iv) convergence free when, if $x = (x_k)$ is in X and if $y_k = 0$ whenever $x_k = 0$, then $y = (y_k)$ is in X.

*Corresponding author (sakhanan15@gmail.com)

Let X be a sequence space and define

$$X^{\alpha} = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty, \forall x \in X \right\}$$
$$X^{\eta} = \left\{ a = (a_k) : \sum_k |a_k x_k|^r < \infty, \forall x \in X \right\}, \quad \text{where } r \geq 1.$$

Taking $r = 1$ in above definition we get α - dual of X. Then X^{α}, and X^{η} are called the $\alpha-$, and η-duals of X, respectively. A sequence space $x = (x_k)$ of complex numbers is said to be $(C,1)$ summable (or Cesàro summable of order 1) to $l \in \mathbb{C}$ if $\lim_{k\to\infty} \sigma_k = l$, where $\sigma_k = \frac{1}{k}\sum_{i=1}^{k} x_i$. By C_1 we shall denote the linear space of all $(C,1)$ summable sequences of complex numbers over $\mathbb{C}$, i.e.,

$$C_1 = \left\{ x = (x_k) \in \omega : \left(\frac{1}{k} \sum_{i=1}^{k} x_i \right) \in c \right\}$$

It is easy to see that C_1 is a BK space normed by

$$\|x\| = \sup_k \frac{1}{k} \left| \sum_{i=1}^{k} x_i \right|, x = (x_k) \in C_1$$

During the last 35 years, a large amount of work has been carried out by many mathematicians regarding various generalizations of difference sequence spaces of Kizmaz [4]. The notion of difference sequence space was introduced by Kizmaz [4] in 1981 as follows: $X(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in X\}$ for $X = \ell_\infty, c, c_0$; where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$ (the set of natural numbers). Quite recently, Cesàro summable difference sequence space $C_1(\Delta)$ has been introduced by Bhardwaj and Gupta [14, 15] as follows: $C_1(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in C_1\}$ i.e., $C_1(\Delta) = \left\{ x = (x_k) \in \omega : \left(\frac{1}{k} \sum_{i=1}^{k} \Delta x_i \right) \in c \right\}$.
The Cesàro sequence space

$$\text{ces}_p = \left\{ x = (x_k) \in \omega : \|x\|_p = \left(\sum_n \frac{1}{n} \sum_{k=1}^{n} |x_k|\, p \right)^{\frac{1}{p}} < \infty \right\}, 1 \leq p < \infty$$

and

$$\text{ces}_\infty = \left\{ x = (x_k) \in \omega : \|x\|_\infty = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced and studied by Shiue [6] in 1970 and it was observed that $\ell_p \subset \text{ces}_p (1 < p < \infty\ )$ is strict, although it does not hold for $p = 1$. Ng and Lee [11] in 1978 defined and studied the Cesàro sequence spaces X_p and X_∞ of nonabsolute type as follows:

$$X_p = \left\{ x = (x_k) \in \omega : \|x\|_p = \left(\sum_n \frac{1}{n} \sum_{k=1}^{n} |x_k|\, p \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \leq p < \infty$$
$$X_\infty = \left\{ x = (x_k) \in \omega : \|x\|_\infty = \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\}$$

The inclusion $\text{ces}_p \subset X_p, (1 \leq p < \infty)$ is strict. Orhan [1, 2] defined and studied the Cesàro difference spaces $X_p(\Delta)$ and $X_\infty(\Delta)$ (in fact, Orhan used C_p instead of $X_p(\Delta)$ and C_∞ instead of $X_\infty(\Delta)$), by replacing $x = (x_k)$ with $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ in the spaces X_p and X_∞ of Ng and Lee [11] as follows:

$$X_p(\Delta) = \left\{ x = (x_k) \in \omega : \|x\|_p = \left(\sum_n \frac{1}{n} \left| \sum_{k=1}^{n} \Delta x_k \right| p \right)^{\frac{1}{p}} < \infty \right\}, 1 \leq p < \infty$$

and

$$X_\infty(\Delta) = \left\{ x = (x_k) \in \omega : \|x\|_\infty = \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| < \infty \right\}$$

and it was shown that for $1 \leq p < \infty$, the inclusions $X_p \subset X_p(\Delta)$ and $X_\infty \subset X_\infty(\Delta)$ are strict. Using this notion of generalized difference sequence space, Et [8], defined the Cesàro difference sequence space $X_p\left(\Delta^m\right)$ and $X_\infty\left(\Delta^m\right)$ (in fact, Et used $C_p\left(\Delta^m\right)$ instead of $X_p\left(\Delta^m\right)$ and $C_\infty\left(\Delta^m\right)$ instead of $X_\infty\left(\Delta^m\right)$) as follows:

$$X_p\left(\Delta^m\right) = \left\{ x = (x_k) \in \omega : \|x\|_p = \left(\sum_n \frac{1}{n} \left| \sum_{k=1}^{n} \Delta^m x_k \right| p \right)^{\frac{1}{p}} < \infty \right\}, 1 \leq p < \infty$$

and

$$X_\infty\left(\Delta^m\right) = \left\{ x = (x_k) \in \omega : \|x\|_\infty = \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right| < \infty \right\}$$

If we take $m = 1, X_p\left(\Delta^m\right)$ and $X_\infty\left(\Delta^m\right)$ reduce to the spaces C_p and C_∞ of Orhan [1,2], respectively. The space $X_\infty\left(\Delta^m\right)$ for $m = 2$ was independently introduced and studied by Mursaleen et al. [9]. Bhardwaj, Gupta and Karan [16] introduced the difference sequence space $C_1\left(\Delta^2\right)$ as follows:

$$C_1\left(\Delta^2\right) = \left\{ x = (x_k) \in \omega : \left(\frac{1}{k} \sum_{i=1}^{k} \Delta^2 x_i \right) \in c \right\}$$

The difference sequence space $X_\infty\left(\Delta^2\right) = \left\{ x = (x_k) \in \omega : \left(\frac{1}{k} \sum_{i=1}^{k} \Delta^2 x_i \right) \in \ell_\infty \right\}$ strictly includes the sequence space $C_1\left(\Delta^2\right)$.

In this paper, we show that $C_1(\Delta)$ strictly includes the spaces $c_0(\Delta)$ and $c(\Delta)$ but overlaps with $\ell_\infty(\Delta)$ and the non-absolute type sequence spaces $X_\infty\left(\Delta^2\right)$ and $C_1\left(\Delta^2\right)$ are BK spaces, none of which is perfect. Finally the η-duals of $C_1(\Delta), C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$ are computed, the matrix classes $\left(C_1(\Delta), \ell_\infty\right), \left(C_1(\Delta), c; p\right), \left(C_1(\Delta), c_0\right), \left(C_1\left(\Delta^2\right), \ell_\infty\right), \left(C_1\left(\Delta^2\right), c\right)$ and $\left(C_1\left(\Delta^2\right), c_0\right)$ are also characterized.

2. Topological Properties of $C_1(\Delta), C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$

Theorem 2.1. $\ell_\infty \subset C_1(\Delta)$, *the inclusion being strict.*

Proof. Let $x = (x_k) \in \ell_\infty$. Then there exists $M > 0$ such that $|x_1 - x_{k+1}| \leq M$ for all $k \geq 1$, and so $\frac{1}{k} \sum_{i=1}^{k} \Delta x_i \to 0$ as $k \to \infty$. For strict inclusion, observe that $(k) \in C_1(\Delta)$ but $(k) \notin \ell_\infty$. □

Theorem 2.2. *$C_1 \subset C_1(\Delta)$, the inclusion being strict.*

Proof. For $x = (x_k) \in C_1$, we have $\lim_{k\to\infty} \frac{1}{k}x_k = 0$, and so $\frac{1}{k}\sum_{i=1}^{k} \Delta x_i \to 0$ as $k \to \infty$. □

Inclusion is strict in view of the example cited in Theorem 2.1.

Theorem 2.3. *$c(\Delta) \subset C_1(\Delta)$, the inclusion being strict.*

Proof. Inclusion is obvious since $c \subset C_1$. To see that the inclusion is strict, consider the sequence $x = (x_k) = (1, 2, 1, 2, 1, 2, \dots)$. □

Theorem 2.4. *$C_1(\Delta)$ is a BK space normed by*

$$\|x\|_\Delta = |x_1| + \sup_k \frac{1}{k}\left|\sum_{i=1}^{k} \Delta x_i\right|, x = (x_k) \in C_1(\Delta).$$

Theorem 2.5. *$C_1(\Delta)$ and $X_\infty\left(\Delta^2\right)$ are not separable but $C_1\left(\Delta^2\right)$ is separable.*

Corollary 2.6. *$C_1(\Delta)$ and $X_\infty\left(\Delta^2\right)$ does not have a schauder basis.*

Theorem 2.7. *$C_1(\Delta)$ is not normal (solid) and hence neither perfect nor convergence free.*

Proof. Taking $x = (x_k) = (k-1)$ and $y = (y_k) = \left((-1)^k(k-1)\right)$, we see that $x \in C_1(\Delta)$ but $y \notin C_1(\Delta)$ although $|y_k| \le |x_k|, k \ge 1$ and so $C_1(\Delta)$ is not normal. It is well known [12] that every perfect space, and also every convergence free space, is normal and consequently $C_1(\Delta)$ is neither perfect nor convergence free. □

Theorem 2.8. *$C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$ are neither monotone nor sequence algebra.*

Theorem 2.9. *$C_1(\Delta) \subset C_1\left(\Delta^2\right)$, the inclusion being strict.*

Proof. Inclusion is trivial as $C_1 \subset C_1(\Delta)$. To see that the inclusion is strict, consider the sequence $x = (x_k) = \left(k^2\right)$. Then $(\Delta x_k) = (-3, -5, -7, \dots) \notin C_1$ but $\left(\Delta^2 x_k\right) = (2, 2, 2, \dots) \in C_1$. □

Theorem 2.10. *[16] $C_1\left(\Delta^2\right) \subset X_\infty\left(\Delta^2\right)$, the inclusion being strict.*

Theorem 2.11. *$C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$ are BK spaces normed by $\|x\|_{\Delta^2} = |x_1| + |x_2| + \sup_k \frac{1}{k}\left|\sum_{i=1}^{k} \Delta^2 x_i\right|$.*

Theorem 2.12.

(a) $C_1\left(\Delta^2\right)$ is a closed subspace of $X_\infty\left(\Delta^2\right)$.

(b) $C_1\left(\Delta^2\right)$ is a nowhere dense subset of $X_\infty\left(\Delta^2\right)$.

Theorem 2.13. *[16] $C_1\left(\Delta^2\right)$ does not have the AK property.*

Theorem 2.14. *[16] The difference sequence spaces $C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$ are not normal (solid) and hence neither perfect nor convergence free.*

3. η - duals of $C_1(\Delta), C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$

In this section, we compute the η-duals of $C_1(\Delta), C_1\left(\Delta^2\right)$ and $X_\infty\left(\Delta^2\right)$ and show that these difference sequence spaces are not perfect. Convenience, we have used the notation $C_\infty\left(\Delta^2\right)$ instead of $X_\infty\left(\Delta^2\right)$.

Theorem 3.1. $[C_1(\Delta)]^n = \left\{a = (a_k) : \sum_k k^r |a_k|^r < \infty\right\} = D_1.$

Proof. Let $a = (a_k) \in D_1$. For any $x = (x_k) \in C_1(\Delta)$, we have $\left(\frac{1}{k}\sum_{i=1}^{k} \Delta x_i\right) \in c$, i.e., $\frac{1}{k}(x_1 - x_{k+1}) \in c$ and so there exists some $M > 0$ such that $|x_k| \leq M(k-1) + x_1$ for $k \geq 1$ and hence $\sup_k k^{-1}|x_k| < \infty$, which implies that

$$\sum_k |a_k x_k|^r = \sum_k \left(k^r |a_k|^r\right)\left(k^{-r}|x_k|^r\right) < \infty$$

Thus, $a = (a_k) \in [C_1(\Delta)]^\eta$.
Conversely, let $a = (a_k) \in [C_1(\Delta)]^\eta$. Then $\sum_k |a_k x_k|^r < \infty$ for all $x = (x_k) \in C_1(\Delta)$. Taking $x_k = k$ for all $k \geq 1$, we have $x = (x_k) \in C_1(\Delta)$ whence $\sum_k k^r |a_k|^r < \infty$. □

Remark 3.2. *It is well known [13] that $[c_0(\Delta)]^\eta = [c(\Delta)]^\eta = [\ell_\infty(\Delta)]^\eta = D_1$, so we conclude that $[c_0(\Delta)]^\eta = [c(\Delta)]^\eta = [\ell_\infty(\Delta)]^\eta = [C_1(\Delta)]^\eta$, i.e. the η-duals of $c_0(\Delta), c(\Delta), \ell_\infty(\Delta)$ and $C_1(\Delta)$ coincide.*

Theorem 3.3. $[C_1(\Delta)]^{\eta\eta} = \left\{a = (a_k) : \sup_k k^{-r}|a_k|^r < \infty\right\} = D_2.$

Proof. Taking $m = 1$ and $X = c$ in the Theorem 2.11 of [13], we have $[c(\Delta)]^{\eta\eta} = \left\{a = (a_k) : \sup_k k^{-r}|a_k|^r < \infty\right\}$ and the result follows in view of Remark 3.2. □

Corollary 3.4. $C_1(\Delta)$ *is not perfect.*

The proof follows at once when we observe that the sequence $\left((-1)^k(k-1)\right) \in [C_1(\Delta)]^{\eta\eta}$ but does not belong to $C_1(\Delta)$.

Theorem 3.5. $\left[C_\theta\left(\Delta^2\right)\right]^\eta = \left\{a = (a_k) : \sum_k k^{2r}|a_k|^r < \infty\right\} = D_1$, *where* $\theta \in \{1, \infty\}$.

Proof. Let $a = (a_k) \in D_1$. For $\theta \in \{1, \infty\}$

(i) $(x_k) \in C_\theta(\Delta)$ implies $x_k = O(k)$

(ii) $(x_k) \in C_\theta\left(\Delta^2\right)$ implies $x_k = O\left(k^2\right)$.

We have $\sup_k k^{-2r}|x_k|\, r < \infty$ for all $x = (x_k) \in C_\theta\left(\Delta^2\right)$, which implies that

$$\sum_k |a_k x_k|^r = \sum_k \left(k^{2r}|a_k|^r\right)\left(k^{-2r}|x_k|^r\right) < \infty.$$

Thus $a = (a_k) \in \left[C_\theta\left(\Delta^2\right)\right]^\eta$.
Conversely, let $a = (a_k) \in \left[C_\theta\left(\Delta^2\right)\right]^\eta$. Then $\sum_k |a_k x_k|\, r < \infty$ for all $x = (x_k) \in C_1\left(\Delta^2\right)$. Taking $x_k = k^2$ for all $k \geq 1$, we have $x = (x_k) \in C_\theta\left(\Delta^2\right)$ whence $\sum_k k^{2r}|a_k|^r < \infty$. □

Remark 3.6. *It is well known [13] that* $\left[c_0\left(\Delta^2\right)\right]^\eta = \left[c\left(\Delta^2\right)\right]^\eta = \left[\ell_\infty\left(\Delta^2\right)\right]^\eta = D_1$*, so we conclude that* $\left[c_0\left(\Delta^2\right)\right]^\eta = \left[c\left(\Delta^2\right)\right]^\eta = \left[\ell_\infty\left(\Delta^2\right)\right]^\eta = \left[C_1\left(\Delta^2\right)\right]^\eta = \left[C_\infty\left(\Delta^2\right)\right]^\eta$ *i.e., the* η*-dual of* $c_0\left(\Delta^2\right), c\left(\Delta^2\right), \ell_\infty\left(\Delta^2\right), C_1\left(\Delta^2\right)$ *and* $C_\infty\left(\Delta^2\right)$ *coincide.*

Theorem 3.7. $\left[C_1\left(\Delta^2\right)\right]^{\eta\eta} = \left\{a = (a_k) : \sup_k k^{-2r}\left|a_k\right|^r < \infty\right\} = D_2.$

Proof. Taking $m = 2$ in the Theorem 2.11 of [13], we have $\left[C_\infty\left(\Delta^2\right)\right]^{\eta\eta} = \left\{a = (a_k) : \sup_k k^{-2r}\left|a_k\right|^r < \infty\right\}$ and the result follows in view of Remark 3.6. □

Corollary 3.8. $C_1\left(\Delta^2\right)$ *and* $C_\infty\left(\Delta^2\right)$ *are not perfect space.*

4. Matrix Maps

Finally, we characterize certain matrix classes. For any complex infinite matrix $A = (a_{nk})$ we shall write $A_n = (a_{nk})_{k\in\mathbb{N}}$ for the sequence in the n^{th} row of A. If X, Y are any two sets of sequences, we denote by (X, Y) the class of all those infinite matrices $A = (a_{nk})$ such that the series $A_n(x) = \sum_k a_{nk}x_k$ converges for all $x = (x_k) \in X \quad (n = 1, 2, \ldots)$ and the sequence $Ax = (a_{nk})_{k\in\mathbb{N}}$ is in Y for all $x \in X$.

Theorem 4.1. *[5] Let X and Y be BK spaces and suppose that* $A = (a_{nk})$ *is an infinite matrix such that* $\left(\sum_k a_{nk}x_k\right)_{n\in\mathbb{N}} \in Y$ *for each* $x \in X$*, i.e.,* $A \in (X, Y)$*, then* $A : X \to Y$ *is a bounded linear operator.*

Theorem 4.2. $A \in (C_1(\Delta), \ell_\infty)$ *if and only if* $\sup_n \sum_{k=2}^{\infty} (k-1)\left|a_{nk}\right| < \infty.$

Remark 4.3. *If* $x = (x_k) \in C_1(\Delta)$*, then there exists some* $l \in \mathbb{C}$ *such that* $\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \Delta x_i = l$*. We shall call* l *the* $C_1(\Delta)$ *limit of the sequence* (x_k) *and by* $(C_1(\Delta), c; P)$ *we shall denote that subset of* $(C_1(\Delta), c)$ *for which* $C_1(\Delta)$ *limits are preserved.*

Theorem 4.4. *[14]* $A \in (C_1(\Delta), c; P)$ *if and only if*

(i) $\sup_n \sum_{k=2}^{\infty} (k-1)\left|a_{nk}\right| < \infty,$

(ii) $\lim_{n\to\infty} \sum_k (k-1)a_{nk} = -1,$

(iii) $\lim_{n\to\infty} a_{nk} = 0$ *for each* k*,*

(iv) $\lim_{n\to\infty} \sum_n a_{nk} = 0.$

Theorem 4.5. $A \in (C_1(\Delta), c_0)$ *if and only if*

(i) $\sup_n \sum_{k=2}^{\infty} (k-1)\left|a_{nk}\right| < \infty,$

(ii) $\lim_{n\to\infty} \sum_k (k-1)a_{nk} = 0,$

(iii) $\lim_{n\to\infty} a_{nk} = 0$ *for each* k*,*

(iv) $\lim_{n\to\infty} \sum_n a_{nk} = 0.$

Theorem 4.6. *$A = (a_{nk}) \in \left(C_1\left(\Delta^2\right), \ell_\infty\right)$ if and only if*

(i) $\sup_n |\sum_k a_{nk}| < \infty$,

(ii) $\sum_k k^2 a_{nk}$ *converges for each* $n \in \mathbb{N}$ *and*

(iii) $(R_{nk}) \in C_1(\Delta), \ell_\infty)$ *where* $R_{nk} = \sum_{v=k+1}^{\infty} a_{nv}$.

Proof. Let $(a_{nk}) \in \left(C_1\left(\Delta^2\right), \ell_\infty\right)$. Then the series $\sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$ and $(\sum_k a_{nk}x_k\Big) \in \ell_\infty$ for all $x = (x_k) \in C_1\left(\Delta^2\right)$. Condition (i) and (ii) follow easily since the sequences $\left(k^2\right) = \left(1^2, 2^2, 3^2, \ldots\right)$ and $(1, 1, 1, \ldots)$ belong to $C_1\left(\Delta^2\right)$. For all $x = (x_k) \in C_1\left(\Delta^2\right)$, Abel's summation by parts yields $\sum_{k=1}^{m} a_{nk}x_k = -\sum_{j=1}^{m-1} \Delta x_j R_{nj} + R_{nm}\sum_{j=1}^{m-1} \Delta x_j + x_1 \sum_{j=1}^{m} a_{nj}$, where $R_{nj} = \sum_{k=j+1}^{\infty} a_{nk}$ and $m, n \in \mathbb{N}$. Proceeding as in Theorem 3.9, we have $\left|R_{nm}\sum_{j=1}^{m-1} \Delta x_j\right| \to 0$ as $m \to \infty$ and so $\sum_k a_{nk}x_k = -\sum_j \Delta x_j R_{nj} + x_1 \sum_j a_{nj}$ for all $x = (x_k) \in C_1\left(\Delta^2\right)$ and $n \in \mathbb{N}$. As $\sup_n \left|\sum_k a_{nk}\right| < \infty$ and $A = (a_{nk}) \in \left(C_1\left(\Delta^2\right), \ell_\infty\right)$, so $\left(\sum_j R_{nj}\Delta x_j\right) \in \ell_\infty$. Thus $\left(R_{nj}\right) \in (C_1(\Delta), \ell_\infty)$.

Conversely, using (ii), (iii), in equation ?, we have $\sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$ and $x = (x_k) \in C_1\left(\Delta^2\right)$. Proceeding as above, we get $\sum_k a_{nk}x_k = -\sum_j \Delta x_j R_{nj} + x_1 \sum_j a_{nj}$ for all $x = (x_k) \in C_1\left(\Delta^2\right)$ and $n \in \mathbb{N}$ and result follows. □

References

[1] C. Orhan, *Cesàro difference sequence spaces and related matrix transformations*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 32(8)(1983), 55-63.

[2] C. Orhan, *Matrix transformations on Cesàro difference sequence spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A_1 Math., 33(1984), 1-8.

[3] F. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukr. Math. J., 55(1)(2003), 136-147.

[4] H. Kizmaz, *On certain sequence spaces*, Can. Math. Bull., 24(2)(1981), 169-176.

[5] I. J. Maddox, *Elements of Functional Analysis*, 2^{nd} Edn., Cambridge University Press, Cambridge, (1988).

[6] J. S. Shiue, *On the Cesàro sequence spaces*, Tamking J. Math., 1(1970), 19-25.

[7] M. Et and R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., 21(4)(1995), 377-386.

[8] M. Et, *On some generalized Cesàro difference sequence spaces*, İstanbul Üniv. Fen Fak. Mat. Derg., 55/56(1996/97), 221-229.

[9] Mursaleen, M. A. Khatib and Qamaruddin, *On difference Cesàro sequence spaces of non-absolute type*, Bull. Cal. Math. Soc., 89(1997), 337-342.

[10] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Dekker, New York, (1981).

[11] P. N. Ng and P. Y. Lee, *Cesàro sequence spaces of non-absolute type*, Comment Math., 20(1978), 429-433.

[12] R. G. Cooke, *Infinite Matrices and Sequence Spaces*, Macmillan & Co, London, (1950).

[13] S. A. Khan and A. A. Ansari, *η-duals of generalized difference sequence spaces*, American International Journal of Research in Science, Technology, Engineering & Mathematics, 14(2)(2016), 122-125.

[14] V. K. Bhardwaj and S. Gupta, *Cesàro summable difference sequence space*, J. Inequal. Appl., 2013(2013), 315.

[15] V. K. Bhardwaj and S. Gupta, *Correction: Cesàro summable difference sequence space*, J. Inequal. Appl., 2014(2014), 11.

[16] V. K. Bhardwaj, S. Gupta and R. Karan, *Köthe-Toeplitz duals and matrix transformations of Cesàro difference sequence spaces of second order*, J. Math. Analysis, 5(2)(2014), 1-11.