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Solution of Non-linear Time-fractional Generalized Hirota-Satsuma Coupled Korteweg-de Vries Equation By Using New Analytical Approach

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Abstract

This paper is concerned with the approximate analytical solution of non-linear time-fractional generalized Hirota-Satsuma Coupled Korteweg-de Vries equation (GHS-cKdV) using an efficient analytical approach, namely the Sumudu transform iterative approach. The proposed approach is an elegant amalgam of the Sumudu transform method and the Iterative method. The time-fractional derivative are described in Caputo sense. The results obtained are graphically shown and demonstrate that the approach is simple to apply and highly efficient to analyze the behavior of non-linear coupled fractional differential equations.

Keywords: Generalized Hirota-Satsuma coupled KdV equation; Caputo fractional derivative; Sumudu transform; Sumudu transform iterative method; Fractional differential equations.
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1. Introduction

Fractional calculus is a branch of mathematical study concerning derivatives and integrals of arbitrary orders. In recent years, it has been found very helpful in several fields such as rheology, viscoelasticity, biomedicine, diffusion, statistics, engineering and other fields of science ([2, 11, 21]). It is always challenging to find the exact and approximate solutions of the fractional differential equations(FDEs). Therefore it is needed to find some efficient and effective methods to solve FDEs. As a result, many different methods and approaches have been adopted so far. The Korteweg-de Vries (KdV) equations are the important equations which plays great role in different fields of science and engineering. It occures in the study of nonlinear dispersive waves. This equation was introduced by Kortweg and de Vries for modelling of shallow water waves in canal in 1895. Firstly, Gardner *et al.* derived an analytical approach to solve this equation, which describes interactions of two long waves with

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different dispersion relations. Later, Hirota and Satsuma proposed a coupled Korteweg-de Vries (cKdV) equation

$$u_t - a(u_{\xi\xi\xi} + 6uu_\xi) = 2b\phi\phi_\xi, \tag{1}$$

$$\phi_t + \phi_{\xi\xi\xi} + 3u\phi_\xi = 0, \tag{2}$$

where a and b are arbitrary constants ([15, 16, 24]). In equation (1) $2b\phi\phi_\xi$ acts as a force term on the KdV wave system with the linear dispersion relation $\omega = ak^3$. In absence of the effect of ϕ, equation (1) reduced to the ordinary KdV equation

$$u_t - 6uu_\xi + u_{\xi\xi} = 0. \tag{3}$$

with soliton solution

$$u = 2P^2 sech^2 P(x + 4aP^2 t). \tag{4}$$

Consider the Generalized Hirota-Satsuma Coupled Korteweg-de Vries (GHS-cKdV) equations

$$\begin{aligned} u_t &= \frac{1}{2}u_{\xi\xi\xi} - 3uu_\xi + 3(vw)_\xi, \\ v_t &= -v_{\xi\xi\xi} + 3uv_\xi, \\ w_t &= -w_{\xi\xi\xi} + 3uw_\xi. \end{aligned}$$

which was introduced by Wu *et al.* [22]. Recently, using an extended tanh-function method, Fan [6] constructed different solition solutions for GHS-cKdV. In this study , we solve the GHS-cKdV of time-fractional order

$$\left.\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{2}\frac{\partial^3 u}{\partial \xi^3} - 3u\frac{\partial u}{\partial \xi} + 3v\frac{\partial w}{\partial \xi} + 3w\frac{\partial v}{\partial \xi}, \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= -\frac{\partial^3 v}{\partial \xi^3} + 3u\frac{\partial v}{\partial \xi}, \\ \frac{\partial^\alpha w}{\partial t^\alpha} &= -\frac{\partial^3 w}{\partial \xi^3} + 3u\frac{\partial w}{\partial \xi}, \\ & 0 < \alpha < 1, \end{aligned}\right\} \tag{5}$$

subject to initial conditions

$$\left.\begin{aligned} u(\xi, 0) &= \frac{\beta - 2k^2}{3} + 2k^2 tanh^2(k\xi), \\ v(\xi, 0) &= \frac{-4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2) tanh(k\xi)}{3c_1}, \\ w(\xi, 0) &= c_0 + c_1 tanh(k\xi), \end{aligned}\right\} \tag{6}$$

where $k, c_0, c_1 \neq 0$ and β are the arbitrary constants. A lot of analytical or numerical methods have been developed to solve generalised cKdV equations such as Adomian Decomposition Method (ADM)

[4], Variational Iteration Method (VIM) [10], Reduced Differential Transform Method (RDTM) [14], Homotopy Analysis Method (HAM) [17], Homotopy Pertubation Method (HPM) [3] and many others ([5, 9, 23]). Recently, Wang and Liu used Sumudu transform (ST) in conjunction with the iterative method and became a well-known approach known as the Sumudu transform iterative method (STIM) to find approximate analytical solutions for time-fractional Cauchy reaction-diffusion equations. More recent, STIM approach was successfully implemented by Kumar and Daftardar- Gejji [13] to obtain solutions for various time and space fractional partial differential equations as well as their systems. The main objective of this study is to extand the application of Sumudu transform iterative method (STIM) to derive the approximate analytical solution of GHS-cKdV.

2. Preliminaries and Notations

In this portion, some fundamental definitions and properties related to fractional calculus and sumudu transform are given which are used further.

Definition 2.1. *A fractional integral of Riemann-Liouville's order $\alpha > 0$ of the real value function $w(\xi, t)$ is defined as [8]*

$$I_t^{\alpha} w(\xi,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(\xi,s) ds, \tag{7}$$

where $\Gamma(.)$ is known as the Gamma function.

Definition 2.2. *The Caputo fractional derivative of function $w(\xi, t)$ of order α is defined as ([12, 18])*

$$\begin{aligned} \frac{\partial^{\alpha}}{\partial t^{\alpha}} w(\xi,t) &= I_t^{m-\alpha}\Big[\frac{\partial^m}{\partial t^m} w(\xi,t)\Big] \\ &= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \int_0^t \dfrac{w^m(\xi,\tau)}{(\xi-\tau)^{\alpha-m+1}} d\rho, m-1 < \alpha \leq m, m \in \mathbb{N}, \\ \frac{\partial^m}{\partial t^m} w(\xi,t) \qquad\qquad\qquad , \alpha = m \end{cases} \end{aligned} \tag{8}$$

Definition 2.3. *The Sumudu transform over the set of functions $A = \{f(t) | \exists M, \rho_1 > 0, \rho_2 > 0$ such that $|f(t)| < Me^{|t|/\rho_j}$ if $t \in (-1)^j \times [0,\infty), j = 1,2\}$ is defined as [7]*

$$S[f(t)] = F(\omega) = \int_0^{\infty} e^{-t} f(\omega t) dt \ , \ \omega \in (-\rho_1, \rho_2). \tag{9}$$

One of the basic property of Sumudu transform is given by

$$S\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big] = \omega^{\alpha} \ , \ \alpha > -1. \tag{10}$$

The inverse Sumudu transform of ω^{α} is defined as

$$S^{-1}[\omega^{\alpha}] = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \ , \ \alpha > -1. \tag{11}$$

Definition 2.4. *The Sumudu transform of Caputo time fractional derivative of $w(\xi,t)$ of order $\alpha > 0$ is defined as [20]*

$$S[\frac{\partial^{\alpha} w(\xi,t)}{\partial t^{\alpha}}] = \omega^{-\alpha} S[w(\xi,t)] - \sum_{k=0}^{m-1} \Big[\omega^{-\alpha+k} \frac{\partial^{k} w(\xi,0)}{\partial t^{k}}\Big], m-1 < \alpha \leq m \ , \ m \in \mathbb{N}. \tag{12}$$

3. Basic Concept of STIM

In this section, the basic idea of Sumudu transform iterative method (STIM) to solve system of time and space fractional PDEs has been given. Consider the following system of general time and space fractional differential equations

$$\begin{gathered} \frac{\partial^{\gamma_i} v_i(\xi,t)}{\partial t^{\gamma_i}} = \zeta_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big), m_i - 1 < \gamma_i \leq m_i, \\ i = 1,2,...,q, n-1 < \beta \leq n, \ \ m_i, l, n, q \in \mathbb{N}, \end{gathered} \tag{13}$$

with initial conditions

$$\frac{\partial^{j} v_i(\xi,0)}{\partial t^{j}} = g_{ij}(\xi), \quad j = 0,1,2,...,m_i - 1, \tag{14}$$

where $\bar{v} = (v_1, v_2, ..., v_q)$ and $\zeta_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big)$ is a linear/ nonlinear operator. Taking the Sumudu transform on both sides of equation (13) and using equation (14), we have

$$S[v_i(\xi,t)] = \sum_{j=0}^{m_i-1} [w^{j} g_{ij}(\xi)] + w^{\gamma_i} S\Big[\zeta_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big)\Big]. \tag{15}$$

On taking inverse Sumudu transform of equation (15) ,we get following system of equations

$$v_i(\xi,t) = S^{-1}\Big(\sum_{j=0}^{m_i-1} [w^{j} g_{ij}(\xi)]\Big) + S^{-1}\Big[w^{\gamma_i} S\Big(\zeta_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big)\Big)\Big], \ \ i = 1,2,...,q. \tag{16}$$

Equation (16) can be written in following form

$$v_i(\xi,t) = f_i(\xi,t) + M_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big), \tag{17}$$

where

$$\begin{gathered} f_i(\xi,t) = S^{-1}\Big(\sum_{j=0}^{m_i-1} [w^{j} g_{ij}(\xi)]\Big), \\ M_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big) = S^{-1}\Big[w^{\gamma_i} S\Big(\zeta_i\Big(\xi, \bar{v}, \frac{\partial^{\beta}\bar{v}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}}{\partial \xi^{l\beta}}\Big)\Big)\Big]. \end{gathered} \tag{18}$$

Here f_i is known function and M_i is a linear/nonlinear operator. Equation (17) can be solved by the DJM decomposition technique introduced by Daftardar-Gejji and Jafari [19], which represents the solution as an infinite series

$$v_i = \sum_{j=0}^{\infty} v_i^{(j)}, \quad 1 \le i \le q, \tag{19}$$

where the terms $v_i^{(j)}$ are calculated recursively. Here, following abbreviations are used

$$\begin{aligned}
\bar{v}^{(j)} &= (v_1^{(j)}, v_2^{(j)}, ..., v_q^{(j)}),\\
\sum_{j=0}^{r} \bar{v}^{(j)} &= \left(\sum_{j=0}^{r} v_1^{(j)}, \sum_{j=0}^{r} v_2^{(j)}, ..., \sum_{j=0}^{r} v_q^{(j)} \right), \quad r \in \mathbb{N} \cup \{\infty\},\\
\frac{\partial^{k\beta}\left(\sum\limits_{j=0}^{r} \bar{v}^{(j)}\right)}{\partial \xi^{k\beta}} &= \left(\frac{\partial^{k\beta}\left(\sum\limits_{j=0}^{r} v_1{}^{(j)}\right)}{\partial \xi^{k\beta}}, \frac{\partial^{k\beta}\left(\sum\limits_{j=0}^{r} v_2{}^{(j)}\right)}{\partial \xi^{k\beta}}, ..., \frac{\partial^{k\beta}\left(\sum\limits_{j=0}^{r} v_q{}^{(j)}\right)}{\partial \xi^{k\beta}} \right), \; k \in \mathbb{N}.
\end{aligned}$$

The operator M_i can be decomposed as

$$\begin{aligned}
M_i\left(\xi, \sum_{j=0}^{\infty} \bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum\limits_{j=0}^{\infty} \bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\left(\sum\limits_{j=0}^{\infty} \bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right) &= M_i\left(\xi, \bar{v}^{(0)}, \frac{\partial^{\beta}\bar{v}^{(0)}}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\bar{v}^{(0)})}{\partial \xi^{l\beta}}\right)\\
&+ \sum_{p=1}^{\infty}\left(M_i\left(\xi, \sum_{j=0}^{p} \bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum\limits_{j=0}^{p} \bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\left(\sum\limits_{j=0}^{p} \bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right)\\
&- \sum_{p=1}^{\infty}\left(M_i\left(\xi, \sum_{j=0}^{p-1} \bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum\limits_{j=0}^{p-1} \bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\left(\sum\limits_{j=0}^{p-1} \bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right).
\end{aligned} \tag{20}$$

Therefore

$$\begin{aligned}
S^{-1}\left[w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{\infty} \bar{v}^{(j)}, \frac{\partial^{\beta}(\sum\limits_{j=0}^{\infty} \bar{v}^{(j)})}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}(\sum\limits_{j=0}^{\infty} \bar{v}^{(j)})}{\partial \xi^{l\beta}}\right)\right)\right] &= S^{-1}\left[w^{\gamma_i} S\left(\zeta_i\left(\xi, \bar{v}^{(0)}, \frac{\partial^{\beta}\bar{v}^{(0)}}{\partial \xi^{\beta}}, ...,\right.\right.\right.\\
\left.\left.\left.\frac{\partial^{l\beta}\bar{v}^{(0)}}{\partial \xi^{l\beta}}\right)\right)\right] + \sum_{p=1}^{\infty} S^{-1}\left[w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{p} \bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum\limits_{j=0}^{p} \bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\left(\sum\limits_{j=0}^{p} \bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right)\right]&\\
- \sum_{p=1}^{\infty} S^{-1}\left[w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{p-1} \bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum\limits_{j=0}^{p-1} \bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta}\left(\sum\limits_{j=0}^{p-1} \bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right)\right]&.
\end{aligned} \tag{21}$$

Using equations (19) and (21) in equation (17), we get

$$
\begin{aligned}
\sum_{j=0}^{\infty} v_i^{(j)} &= S^{-1}\left(\sum_{j=0}^{m_i-1}\left[w^j g_{ij}(\xi)\right]\right) + S^{-1}w^{\gamma_i} S\left(\zeta_i\left(\xi, \bar{v}^{(0)}, \frac{\partial^{\beta}\bar{v}^{(0)}}{\partial \xi^{\beta}}, \ldots, \frac{\partial^{l\beta}\bar{v}^{(0)}}{\partial \xi^{l\beta}}\right)\right) \\
&+ \sum_{p=1}^{\infty}\left(S^{-1}w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{p}\bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum_{j=0}^{p}\bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, \ldots, \frac{\partial^{l\beta}\left(\sum_{j=0}^{p}\bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right)\right. \\
&\left. - S^{-1}w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{p-1}\bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum_{j=0}^{p-1}\bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, \ldots, \frac{\partial^{l\beta}\left(\sum_{j=0}^{p-1}\bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right)\right).
\end{aligned} \tag{22}
$$

The recurrence relation are defined as follows

$$
\left.
\begin{aligned}
v_i^{(0)} &= S^{-1}\left(\sum_{j=0}^{m_i-1}\left[w^j g_{ij}(\xi)\right]\right), \\
v_i^{(1)} &= S^{-1}w^{\gamma_i} S\left(\zeta_i\left(\xi, \bar{v}^{(0)}, \frac{\partial^{\beta}\bar{v}^{(0)}}{\partial \xi^{\beta}}, \ldots, \frac{\partial^{l\beta}\bar{v}^{(0)}}{\partial \xi^{l\beta}}\right)\right), \\
v_i^{(m+1)} &= S^{-1}w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{m}\bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum_{j=0}^{m}\bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, \ldots, \frac{\partial^{l\beta}\left(\sum_{j=0}^{m}\bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right) \\
&- S^{-1}w^{\gamma_i} S\left(\zeta_i\left(\xi, \sum_{j=0}^{m-1}\bar{v}^{(j)}, \frac{\partial^{\beta}\left(\sum_{j=0}^{m-1}\bar{v}^{(j)}\right)}{\partial \xi^{\beta}}, \ldots, \frac{\partial^{l\beta}\left(\sum_{j=0}^{m-1}\bar{v}^{(j)}\right)}{\partial \xi^{l\beta}}\right)\right), m \geq 1.
\end{aligned}
\right\} \tag{23}
$$

Therefore, the m-term approximate analytical solution of equations (13) and (14) is given by

$$
v_i \approx v_i^{(0)} + v_i^{(1)} + \ldots + v_i^{(m-1)} \quad or \quad v_i \approx v_{i0} + v_{i1} + \ldots + v_{i(m-1)}.
$$

4. The STIM Solution of the GHS-cKdV

Considering system (5), we use the application of Sumudu transform iterative method (STIM) for solving this system with initial condition (6). Now, taking the sumudu transform on the both sides of the equation (5), we obtain

$$
\left.
\begin{aligned}
S\left[\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right] &= S\left[\frac{1}{2}\frac{\partial^3 u}{\partial \xi^3} - 3u\frac{\partial u}{\partial \xi} + 3v\frac{\partial w}{\partial \xi} + 3w\frac{\partial v}{\partial \xi}\right], \\
S\left[\frac{\partial^{\alpha} v}{\partial t^{\alpha}}\right] &= S\left[-\frac{\partial^3 v}{\partial \xi^3} + 3u\frac{\partial v}{\partial \xi}\right], \\
S\left[\frac{\partial^{\alpha} w}{\partial t^{\alpha}}\right] &= S\left[-\frac{\partial^3 w}{\partial \xi^3} + 3u\frac{\partial w}{\partial \xi}\right],
\end{aligned}
\right\} \tag{24}
$$

Using formula (12), we get

$$\left.\begin{aligned} p^{-\alpha}S[u(\xi,t)]-p^{-\alpha}u(\xi,0)&=S\left[\frac{1}{2}\frac{\partial^3 u}{\partial\xi^3}-3u\frac{\partial u}{\partial\xi}+3v\frac{\partial w}{\partial\xi}+3w\frac{\partial v}{\partial\xi}\right],\\ p^{-\alpha}S[v(\xi,t)]-p^{-\alpha}v(\xi,0)&=S\left[-\frac{\partial^3 v}{\partial\xi^3}+3u\frac{\partial v}{\partial\xi}\right],\\ p^{-\alpha}S[w(\xi,t)]-p^{-\alpha}w(\xi,0)&=S\left[-\frac{\partial^3 w}{\partial\xi^3}+3u\frac{\partial w}{\partial\xi}\right],\end{aligned}\right\}\tag{25}$$

so the equations become

$$\left.\begin{aligned} S[u(\xi,t)]&=u(\xi,0)+p^{\alpha}S\left[\frac{1}{2}\frac{\partial^3 u}{\partial\xi^3}-3u\frac{\partial u}{\partial\xi}+3v\frac{\partial w}{\partial\xi}+3w\frac{\partial v}{\partial\xi}\right],\\ S[v(\xi,t)]&=v(\xi,0)+p^{\alpha}S\left[-\frac{\partial^3 v}{\partial\xi^3}+3u\frac{\partial v}{\partial\xi}\right],\\ S[w(\xi,t)]&=w(\xi,0)+p^{\alpha}S\left[-\frac{\partial^3 w}{\partial\xi^3}+3u\frac{\partial w}{\partial\xi}\right].\end{aligned}\right\}\tag{26}$$

Taking the inverse sumudu transform of above equations, we obtain

$$\left.\begin{aligned} u(\xi,t)&=S^{-1}[u(\xi,0)]+S^{-1}\left[p^{\alpha}S\left(\frac{1}{2}\frac{\partial^3 u}{\partial\xi^3}-3u\frac{\partial u}{\partial\xi}+3v\frac{\partial w}{\partial\xi}+3w\frac{\partial v}{\partial\xi}\right)\right],\\ v(\xi,t)&=S^{-1}[v(\xi,0)]+S^{-1}\left[p^{\alpha}S\left(-\frac{\partial^3 v}{\partial\xi^3}+3u\frac{\partial v}{\partial\xi}\right)\right],\\ w(\xi,t)&=S^{-1}[w(\xi,0)]+S^{-1}\left[p^{\alpha}S\left(-\frac{\partial^3 w}{\partial\xi^3}+3u\frac{\partial w}{\partial\xi}\right)\right].\end{aligned}\right\}\tag{27}$$

In the view of recurrence relations

$$u_0=S^{-1}[u(\xi,0)]=\frac{\beta-2k^2}{3}+2k^2tanh^2(k\xi),\tag{28}$$

$$v_0=S^{-1}[v(\xi,0)]=-\frac{4k^2c_0(\beta+k^2)}{3c_1^2}+\frac{4k^2(\beta+k^2)tanh(k\xi)}{3c_1},\tag{29}$$

$$w_0=S^{-1}[w(\xi,0)]=c_0+c_1tanh(k\xi),\tag{30}$$

Therefore we determine the other components of the STIM solutions as follows:

First order solutions

$$\begin{aligned} u_1&=S^{-1}p^{\alpha}S\left(\frac{1}{2}\frac{\partial^3 u_0}{\partial\xi^3}-3u_0\frac{\partial u_0}{\partial\xi}+3v_0\frac{\partial w_0}{\partial\xi}+3w_0\frac{\partial v_0}{\partial\xi}\right)\\ &=S^{-1}p^{\alpha}S\Big(-16k^5tanh(k\xi)sech^4(k\xi)+8k^5tanh^3(k\xi)sech^2(k\xi)\\ &-4(\beta-2k^2)k^3tanh(k\xi)sech^2(k\xi)-24k^5tanh^3(k\xi)sech^2(k\xi)\\ &-4k^3\frac{c_0}{c_1}(\beta+k^2)sech^2(k\xi)+4k^3(\beta+k^2)tanh(k\xi)sech^2(k\xi)\\ &+(c_0+c_1tanh(k\xi))(\frac{4k^3(\beta+k^2)}{c_1}sech^2(k\xi))\Big)\end{aligned}$$

$$= 4\beta k^3 tanh(k\xi) sech^2(k\xi) \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{31}$$

$$v_1 = S^{-1} p^\alpha S\left(-\frac{\partial^3 v_0}{\partial \xi^3} + 3u_0 \frac{\partial v_0}{\partial \xi}\right),$$

$$= S^{-1} p^\alpha S\left(8k^5 \frac{(\beta+k^2)}{3c_1} sech^2(k\xi) + 4k^3(\beta - 2k^2)\frac{(\beta+k^2)}{3c_1} sech^2(k\xi)\right)$$

$$= 4\beta k^3 \frac{(\beta+k^2)}{3c_1} sech^2(k\xi) \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{32}$$

$$w_1 = S^{-1} p^\alpha S\left(-\frac{\partial^3 w_0}{\partial \xi^3} + 3u_0 \frac{\partial w_0}{\partial \xi}\right)$$

$$= S^{-1} p^\alpha S\left(-4c_1 k^3 sech^2(k\xi) tanh^2(k\xi) + 2c_1 k^3 sech^4(k\xi) + c_1 k(\beta - 2k^2) sech^2(k\xi)\right.$$

$$\left. + 6k^3 c_1 sech^2(k\xi) tanh^2(k\xi)\right)$$

$$= c_1 \beta k sech^2(k\xi) \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{33}$$

Second order solutions

$$u_2 = S^{-1} p^\alpha S\left(\frac{1}{2}\frac{\partial^3(u_0+u_1)}{\partial \xi^3} - 3(u_0+u_1)\frac{\partial(u_0+u_1)}{\partial \xi} + 3(v_0+v_1)\frac{\partial(w_0+w_1)}{\partial \xi}\right.$$

$$\left. + 3(w_0+w_1)\frac{\partial(v_0+v_1)}{\partial \xi}\right) - S^{-1} p^\alpha S\left(\frac{1}{2}\frac{\partial^3 u_0}{\partial \xi^3} - 3u_0\frac{\partial u_0}{\partial \xi} + 3v_0\frac{\partial w_0}{\partial \xi} + 3w_0\frac{\partial v_0}{\partial \xi}\right)$$

$$= S^{-1} p^\alpha S\left(4k^3(\beta+k^2) sech^2(k\xi) tanh(k\xi) + 4\beta^2 k^4 sech^4(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)} - 8\beta^2 k^4 sech^2(k\xi)\right.$$

$$tanh^2(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)} - 16k^5 sech^2(k\xi) tanh(k\xi) - 16\beta^3 k^5 sech^4(k\xi) tanh(k\xi)\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2}$$

$$- 16\beta^2 k^7 sech^4(k\xi) tanh(k\xi)\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2} + 16\beta k^6 sech^4(k\xi) tanh^2(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$- 16\beta k^6 sech^6(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)} + 16\beta k^6 sech^4(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)} + 32\beta k^6 sech^2(k\xi) tanh^4(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$- 32\beta k^6 sech^2(k\xi) tanh^2(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)} + 96\beta^2 k^7 sech^4(k\xi) tanh^3(k\xi)\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2}$$

$$\left. - 48\beta^2 k^7 sech^6(k\xi) tanh(k\xi)\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2}\right) - 4\beta k^3 tanh(k\xi) sech^2(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$= 4k^4\beta^2 sech^2(k\xi)\left(sech^2(k\xi) - 2tanh^2(k\xi)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$- 16\beta^3 k^5 sech^4(k\xi) tanh(k\xi)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}$$

$$+ 16\beta^2 k^7 sech^4(k\xi) tanh(k\xi)\left(5tanh^2(k\xi) - 4sech^2(k\xi)\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} \tag{34}$$

$$v_2 = S^{-1}\left[p^\alpha S\left(-\frac{\partial^3(v_0+v_1)}{\partial \xi^3} + 3(u_0+u_1)\frac{\partial(v_0+v_1)}{\partial \xi}\right)\right] - S^{-1}\left[p^\alpha S\left(-\frac{\partial^3 v_0}{\partial \xi^3} + 3u_0\frac{\partial v_0}{\partial \xi}\right)\right]$$

$$= S^{-1} p^\alpha S\left(\frac{4k^2(\beta+k^2)}{3c_1}[2k^3 sech^4(k\xi) - 2k^3 sech^2(k\xi) tanh^2(k\xi) + 8\beta k^4 sech^2(k\xi) tanh(k\xi)\right.$$

$$(tanh^2(k\xi) - 2sech^2(k\xi))\frac{t^\alpha}{\Gamma(\alpha+1)} + (\beta - 2k^2) k sech^2(k\xi) - 2\beta k^2(\beta - 2k^2) sech^2(k\xi) tanh(k\xi)$$

$$\frac{t^\alpha}{\Gamma(\alpha+1)} + 6k^3 sech^2(k\xi) tanh^2(k\xi) - 12\beta k^4 sech^2(k\xi) tanh^3(k\xi)\frac{t^\alpha}{\Gamma(\alpha+1)} + 12\beta k^4 sech^4(k\xi) tanh(k\xi)$$

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 24\beta^2 k^5 sech^4(k\xi) tanh^2(k\xi) \frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2}]\Big) - \frac{4\beta k^3(\beta+k^2)}{3c_1} sech^2(k\xi) \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$= -\frac{8k^4\beta^2(\beta+k^2)}{3c_1} tanh(k\xi) sech^2(k\xi) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$- \frac{32}{c_1}\beta^2 k^7(\beta+k^2) sech^4(k\xi) tanh^2(k\xi) \frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} \quad (35)$$

$$w_2 = S^{-1}\left[p^{\alpha} S\left(-\frac{\partial^3(w_0+w_1)}{\partial\xi^3} + 3(u_0+u_1)\frac{\partial(w_0+w_1)}{\partial\xi}\right)\right] - S^{-1}\left[p^{\alpha} S\left(-\frac{\partial^3 w_0}{\partial\xi^3} + 3u_0\frac{\partial w_0}{\partial\xi}\right)\right]$$
$$= S^{-1}\Big[p^{\alpha} S\Big(2c_1k^3 sech^4(k\xi) - 4c_1k^3 sech^2(k\xi) tanh^2(k\xi) - 8k^4 c_1\beta sech^2(k\xi) tanh(k\xi)$$
$$(2sech^2(k\xi) - tanh^2(k\xi))\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (\beta-2k^2)kc_1 sech^2(k\xi) - 2c_1\beta k^2(\beta-2k^2)$$
$$tanh(k\xi) sech^2(k\xi)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 6k^3c_1 sech^2(k\xi) tanh^2(k\xi) + 12k^4c_1\beta sech^2(k\xi) tanh(k\xi)$$
$$(sech^2(k\xi) - tanh^2(k\xi))\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 24c_1\beta^2k^5 sech^4(k\xi) tanh^2(k\xi)\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2}\Big)$$
$$- c_1\beta k sech^2(k\xi)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$= -2c_1\beta^2k^2 tanh(k\xi) sech^2(k\xi)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 24c_1\beta^2k^5 sech^4(k\xi) tanh^2(k\xi)\frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}. \quad (36)$$

Similarly more iterations can be obtained. The STIM yields the solution $u(\xi,t), v(\xi,t), w(\xi,t)$ as follows

$$u(\xi,t) = u_0 + u_1 + u_2 + ...$$
$$= \frac{\beta-2k^2}{3} + 2k^2 tanh^2(k\xi) + 4\beta k^3 tanh(k\xi) sech^2(k\xi)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$+ 4k^4\beta^2 sech^2(k\xi)\left(sech^2(k\xi) - 2tanh^2(k\xi)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$- 16\beta^3k^5 sech^4(k\xi) tanh(k\xi)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}$$
$$+ 16\beta^2k^7 sech^4(k\xi) tanh(k\xi)\left(5tanh^2(k\xi) - 4sech^2(k\xi)\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + ... \quad (37)$$

$$v(\xi,t) = v_0 + v_1 + v_2 + ...$$
$$= -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)tanh(k\xi)}{3c_1} + 4\beta k^3\frac{(\beta+k^2)}{3c_1} sech^2(k\xi)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$- \frac{8k^4\beta^2(\beta+k^2)}{3c_1} tanh(k\xi) sech^2(k\xi)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$- \frac{32}{c_1}\beta^2k^7(\beta+k^2) sech^4(k\xi) tanh^2(k\xi)\frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} + ... \quad (38)$$

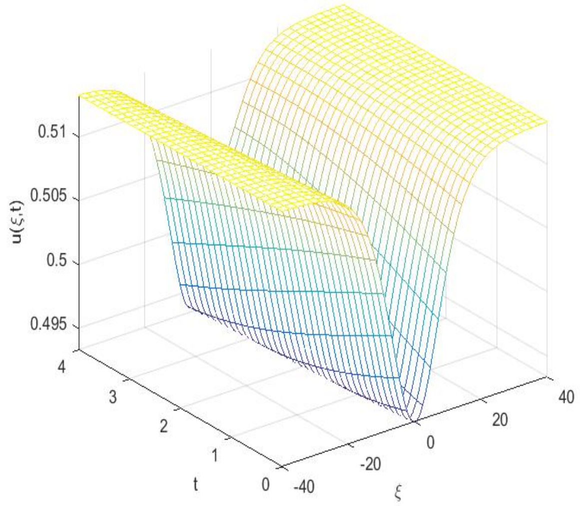
$$w(\xi,t) = w_0 + w_1 + w_2 + ...$$
$$= c_0 + c_1 tanh(k\xi) + c_1\beta k sech^2(k\xi)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2c_1\beta^2k^2 tanh(k\xi) sech^2(k\xi)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$- 24c_1\beta^2k^5 sech^4(k\xi) tanh^2(k\xi)\frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} + ... \quad (39)$$

Special case: for $\alpha = 1$

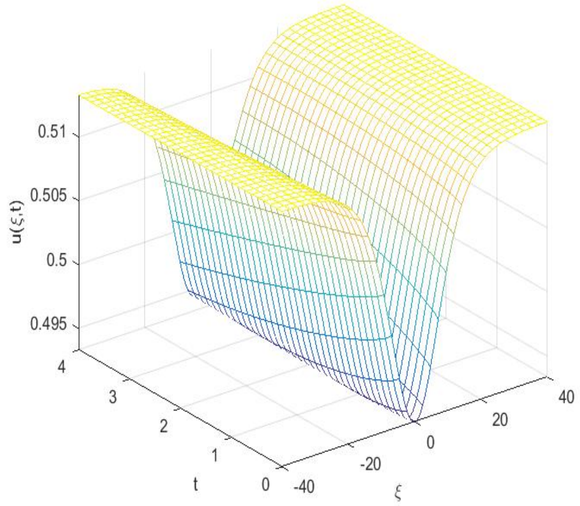
$$u = \frac{\beta - 2k^2}{3} + 2k^2 tanh^2(k\xi) + 4\beta k^3 tanh(k\xi) sech^2(k\xi) t + 2k^4\beta^2 sech^2(k\xi)\left(sech^2(k\xi) - 2tanh^2(k\xi)\right) t^2$$
$$- \frac{16}{3}\beta^3 k^5 sech^4(k\xi) tanh(k\xi) t^3 + \frac{16}{3}\beta^2 k^7 sech^4(k\xi) tanh(k\xi)\left(5tanh^2(k\xi) - 4sech^2(k\xi)\right) t^3 + ... \quad (40)$$

$$v = -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1} tanh(k\xi) + 4\beta k^3 \frac{(\beta + k^2)}{3c_1} sech^2(k\xi) t$$
$$- \frac{4k^4\beta^2(\beta + k^2)}{3c_1} tanh(k\xi) sech^2(k\xi) t^2 - \frac{32}{3c_1}\beta^2 k^7(\beta + k^2) sech^4(k\xi) tanh^2(k\xi) t^3 + ... \quad (41)$$

$$w = c_0 + c_1 tanh(k\xi) + c_1\beta k sech^2(k\xi) t - c_1\beta^2 k^2 tanh(k\xi) sech^2(k\xi) t^2$$
$$- 8c_1\beta^2 k^5 sech^4(k\xi) tanh^2(k\xi) t^3 + ... \quad (42)$$



(a)



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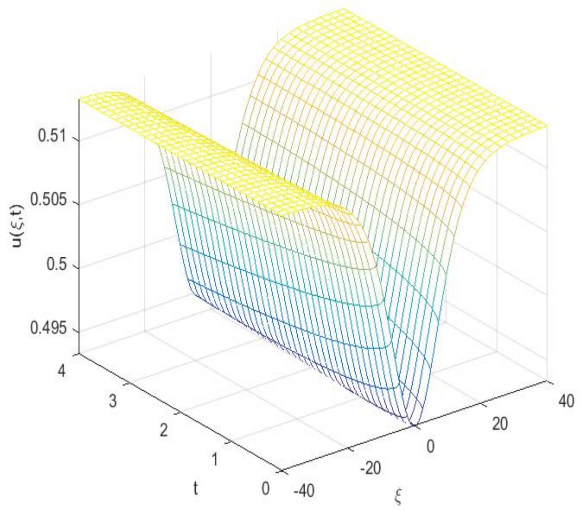
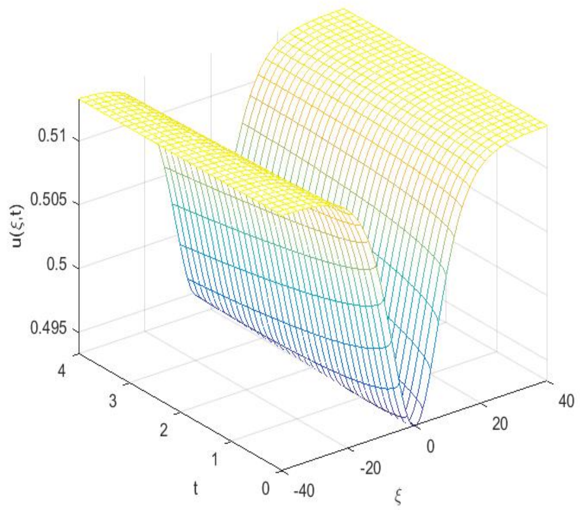
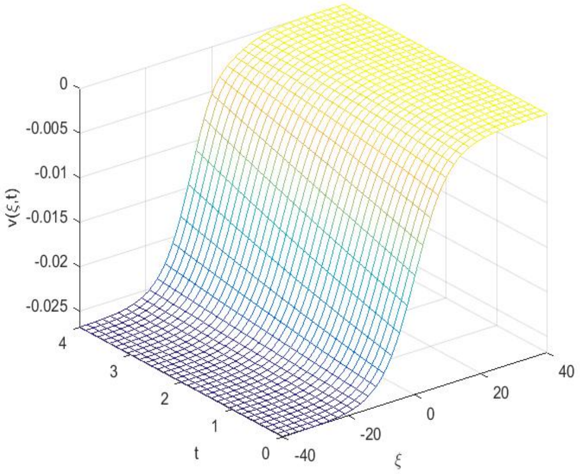


(c)

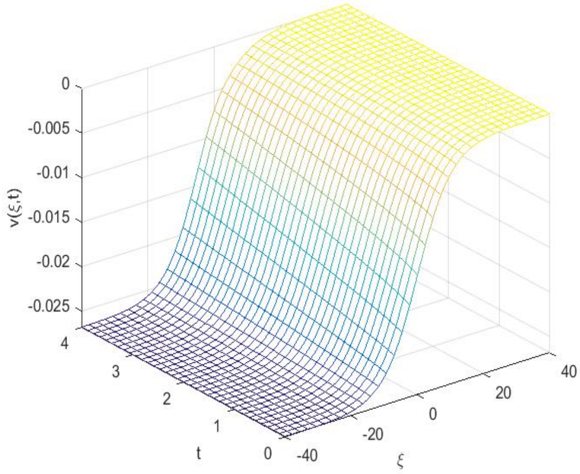


(d)

Figure 1: The surface shows the solution $u(\xi, t)$, when $\beta = 1.5$, $k = 0.1$, : (a) Approximate solution for $\alpha = 1$, (b) Approximate solution for $\alpha = 0.75$, (c) Approximate solution for $\alpha = 0.50$, (d) Approximate solution for $\alpha = 0.25$



(a)



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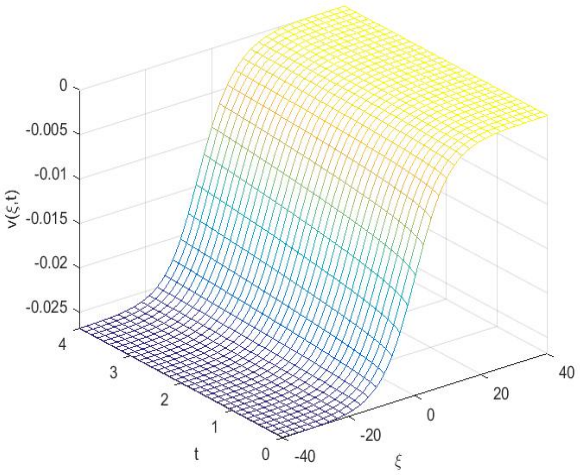
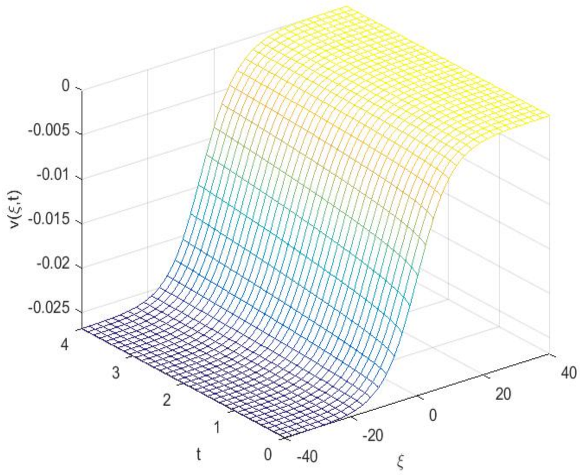
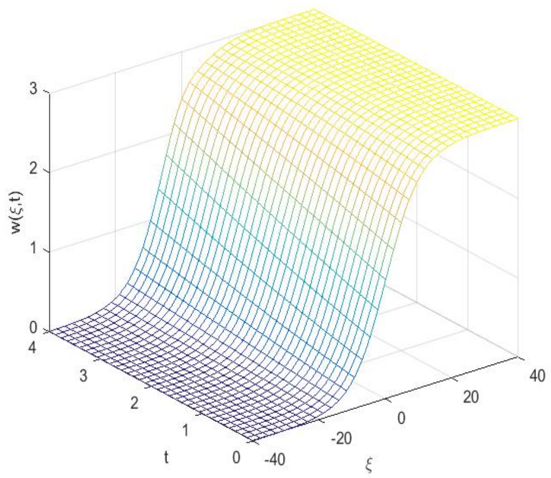


(c)

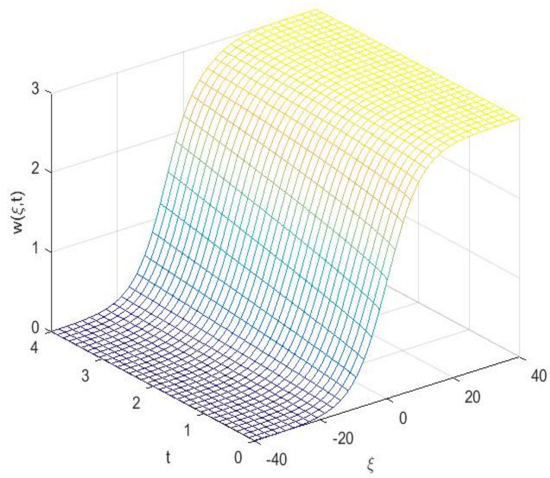


(d)

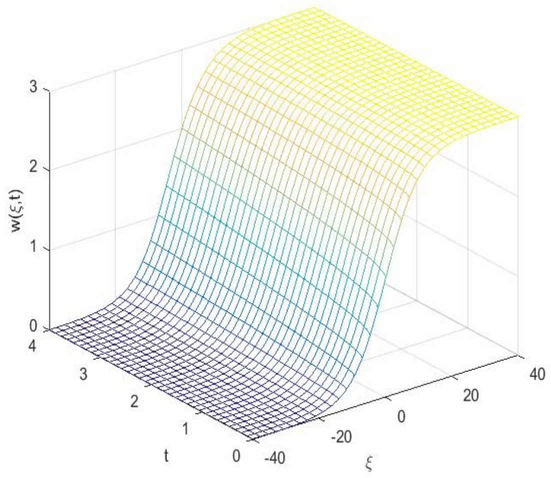
Figure 2: The surface shows the solution $v(\xi, t)$, when $\beta = 1.5$, $k = 0.1$, $c_0 = 1.5$,$c_1 = 1.5$: (a) Approximate solution for $\alpha = 1$, (b) Approximate solution for $\alpha = 0.75$, (c) Approximate solution for $\alpha = 0.50$, (d) Approximate solution for $\alpha = 0.25$



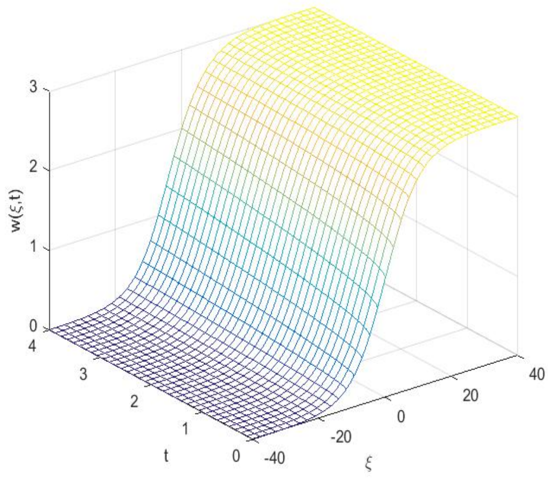
(a)



(b)



(c)



(d)

Figure 3: The surface shows the solution $w(\xi, t)$, when $\beta = 1.5$, $k = 0.1$, $c_0 = 1.5$,$c_1 = 1.5$: (a) Approximate solution for $\alpha = 1$, (b) Approximate solution for $\alpha = 0.75$, (c) Approximate solution for $\alpha = 0.50$, (d) Approximate solution for $\alpha = 0.25$

5. Conclusion

In this paper, the time-fractional generalised Hirota-Satsuma coupled Kortweg-de Vries (GHS-cKdV) equations were solved by new approach namely Sumudu Transform Iterative Method (STIM) within Caputo operator. By increasing the number of iteration , the accuracy of STIM can be further improved.

References

[1] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Korteweg-deVries equation and generalization. VI. Methods for exact solution*, Communications on Pure Applied Mathematics, 27(1)(1974), 97-133.

[2] D. Baleanu, K. Diethelm,E. Scalas and J. J. Trujillo, *Fractional Calculus, vol. 3 of Series on Complexity, Nonlinearity and Chaos*, World Scientific, Singapore, 3(2012), 428.

[3] D. D. Ganji and M. Rafei, *Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equations by homotopy perturbation method*, Physics Letters A, 356(2)(2006), 131-137.

[4] D. Kaya, *Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equations*, Applied Mathematics and Computation, 147(1)(2004), 69-78.

[5] E. C. Aslan, M. Inc, M. M. Al Qurashi and D. Baleanu, *On numerical solutions of time-fraction generalized Hirota Satsuma coupled KdV equation*, Journal of Nonlinear Sciences and Applications, 10(2017), 619-629.

[6] E. G. Fan, *Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation*, Physics Letters A, 282(1-2)(2001), 18-22.

[7] F. B. M. Belgacem and A. A. Karaballi, *Sumudu transform fundamental properties investigations and applications*, International Journal of Stochastic Analysis, 2006(2006), 1-23.

[8] I. Podlubny, *Fractional differential equations, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, Academic Press, San Diego, CA, (1999).

[9] J. C. Li and H. Li, *Approximate analytic solutions of time-fractional Hirota-Satsuma coupled KdV equation and coupled MKdV equation*, Abstract and Applied Analysis, (2013), 11 pages.

[10] J. H. He and X. H. Wu, *Construction of solitary solution and compacton–like solution by variational iteration method*, Chaos, Solitons and Fractals, 29(1)(2006), 108-113.

[11] J. Sabatier, O. P. Agrawal and J. A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer*, Springer, (2007).

[12] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley Sons, New York, NY, USA, (1993).

[13] M. Kuma and V. Daftardar-Gejji, *Exact solutions of fractional partial differential equations by Sumudu transform iterative method*, arXiv:1806.03057v1 [math.AP], (2018), 157-180.

[14] R. Abazari and M. Abazari, *Numerical simulation of generalized Hirota–Satsuma coupled KdV equation by RDTM and comparison with DTM*, Commun Nonlinear Sci Numer Simulat, 17(2012), 619-629.

[15] R. Hirota, *Exact solution of the Korteweg—de Vries equation for multiple collisions of solitons*, Physical Review Letters, 27(18)(1971), 1192-1194.

[16] R. Hirota and J. Satsuma, *Soliton solutions of a coupled Korteweg-de Vries equation*, Physics Letters A, 85(8-9)(1981), 407-408.

[17] S. Abbasbandy, *The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation*, Physics Letters A, 361(2007), 478-483.

[18] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives ,Theory and Applications*, Gordon and Breach, Yverdon, (1993).

[19] V. Daftardar-Gejji and H. Jafari, *An iterative method for solving nonlinear functional equations*, Journal of Mathematical Analysis and Applications, 316(2)(2006), 753-763.

[20] Y. Amer, A. Mahdy and E. Youssef, *Solving system of fractional nonlinear equations of Emden Fowler type by using sumudu transform method*, Global Journal of Pure and Applied Mathematics, 14(2018), 91-113.

[21] Y. Liu and B. Xin, *Numerical solutions of a fractional predatorprey system*, Advances in Difference Equations, 2011(2011), Article ID 190475, 11 Pages.

[22] Y. Wu, X. Geng, X. Hu and S. Zhu, *A generalized Hirota-Satsuma coupled Korteweg-de Vries equation and Miura transformations*, Physics Letters A, 255(4-6)(1999), 259-264.

[23] Z. Guo-Zhong, Y. Xi-Jun, X. Yun, Z. Jiang and W. Di, *Approximate analytic solutions for a generalized Hirota Satsuma coupled KdV equation and a coupled MKDV equation*, Chinese Physics B, 19(8)(2010), 1-9.

[24] Z. Z. Ganji, D. D. Ganji and Y. Rostamiyan, *Solitary wave solutions for a time-fraction generalized Hirota-Satsuma coupled KdV equation by an analytical technique*, Applied Mathematical Modelling, 33(7)(2009), 3107-3113.