International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(1)**(2023), 15–20
Available Online: http://ijmaa.in

On a Conjecture of Graph Parameters Ramsey Theory

Yuan Si[1,*]

[1]*Center for Combinatorics, Nankai University, Tianjin, China.*

Abstract

For graph parameters $f_1, f_2, \ldots, f_k$ and positive integers $n_1, n_2, \ldots, n_k$, the graph parameters Ramsey number $(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k)$ is the minimum positive integer n such that for any factorization of complete graph $K_n = \bigcup\limits_{i=1}^{k} G_i$, K_n contains at least one subgraph G_i satisfying $f_i(G_i) \geq n_i$, $1 \leq i \leq k$. In this paper, we focus on a conjecture of graph parameters Ramsey number $(a_1, \chi_1)(m, n)$, where $a_1(G)$ is edge arboricity of graph G and $\chi_1(G)$ is edge chromatic number of graph G. We prove that this conjecture is true in some special cases and discuss a possible way to solve this conjecture.

Keywords: Ramsey theory; Graph parameter; Edge arboricity; Edge chromatic number.
2020 Mathematics Subject Classification: 05C55, 05C70.

1. Introduction

Let G be a finite, simple and undirected graph, $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ be the vertex set, edge set, minimum degree, maximum degree of G, respectively. For $v \in V(G)$, let $d_G(v)$ be the degree of v in G. Let $A \subseteq V(G)$. Denote $E(A)$ be an edge subset of $E(G)$ such that endpoints of each edge in $E(A)$ are in A. For $v \in V(G)$, we use $G \setminus v$ to denote the subgraph of G obtained by removing the vertex v and the edges incident with v. Edge arboricity $a_1(G)$ is the minimum number of edge set partition of $E(G)$ such that each edge subset induces an acyclic graph. Edge chromatic number $\chi_1(G)$ is the minimum number of colors such that each adjacent edge of $E(G)$ does not have the same color. For the terminology and notations not defined in this paper, please refer to [1].

For k graph parameters $f_1, f_2, \ldots, f_k$ and positive integers $n_1, n_2, \ldots, n_k$, the graph parameters Ramsey number $(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k)$ is the minimum positive integer n such that for any factorization of complete graph $K_n = \bigcup\limits_{i=1}^{k} G_i$, K_n contains at least one subgraph G_i satisfying $f_i(G_i) \geq n_i$, $1 \leq i \leq k$. If $f_1 = f_2 = \ldots = f_k = f$, then we write $(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k)$ as $f(n_1, n_2, \ldots, n_k)$ briefly.

In 1977, Lesniak-Foster and Roberts studied Ramsey theory on vertex partition parameters and edge partition parameters with co-hereditary property (that is, if H is a subgraph of G, then $f(H) \leq f(G)$)

*Corresponding author (yuan_si@aliyun.com)

and $\lim_{n\to\infty} f(K_n) = \infty$. They proposed a conjecture of $(a_1, \chi_1)(m, n)$ and proved that the upper bound is true for all integers $m \geq 2$ and $n \geq 2$, and the lower bound is true for all integer $m \geq 2$ and odd integer $n \geq 3$. For more details, please refer to [3].

Conjecture 1.1. [3] For integers $m \geq 2$ and $n \geq 2$,

$$(a_1, \chi_1)(m, n) = 2m + n - 2.$$

In this paper, we focus on the case of integer $m \geq 2$ and even integer $n \geq 2$ of $(a_1, \chi_1)(m, n)$, and we prove that the conjecture is true when integer $m = 2$ or $n = 2$, and it is also true when integer $m = 3$ and even integer $n \geq 2$.

2. Preliminary

Our proof will use the following results.

Theorem 2.1. [4] A graph G has k edge disjoint forests decomposition if and only if for any $A \subseteq V(G)$,

$$|E(A)| \leq k(|A| - 1).$$

Theorem 2.2. [2] Let G be an even order regular graph and degree $d(G)$ equal to $|V(G)| - 3$, $|V(G)| - 4$ or $|V(G)| - 5$. If $d(G) \geq \frac{1}{2}|V(G)|$, then $\chi_1(G) = \Delta(G)$. In particular, if G is an even order regular graph with $|V(G)| < 10$ and $d(G) = |V(G)| - 5$, then $\chi_1(G) = \Delta(G)$.

Lemma 2.1. [2] Let G be an even order regular graph and G is not a complete graph. For $w \in V(G)$, $\chi_1(G) = \Delta(G)$ if and only if $\chi_1(G \setminus w) = \Delta(G \setminus w)$.

3. Main Results

For integer $m \geq 2$ and even integer $n \geq 2$, based on the work of Lesniak-Foster and Roberts [3], we only need to prove that the lower bound of the conjecture holds.

Theorem 3.1. *For even integer $n \geq 2$,*

$$(a_1, \chi_1)(2, n) = n + 2.$$

Proof. Since $K_{n+1} = K_{1,n} \cup K_n$ and n is even, it follows that $a_1(K_{1,n}) = 1$ and $\chi_1(K_n) = n - 1$. □

Theorem 3.2. *For even integer $n \geq 4$,*

$$(a_1, \chi_1)(3, n) = n + 4.$$

Proof. Let $V(K_{n+3}) = \{v_1, v_2, \ldots, v_{n+3}\}$ and consider the factorization $K_{n+3} = G_1 \cup G_2$ with $G_1 = P_1 \cup P_2$ where $P_1 = v_1 v_2 \ldots v_{n+3}$ and $P_2 = v_{n/2+1} v_{n/2-1} \ldots v_1 v_{n+2} \ldots v_{n/2+4} v_{n/2+2} \ldots v_{n+3} v_{n+1}$

$\ldots v_{n/2+5}v_{n/2+3}$, as shown in Figure 1. Since P_1 and P_2 are spanning paths of K_{n+3}, it follows that $a_1(G_1) \leq 2$.

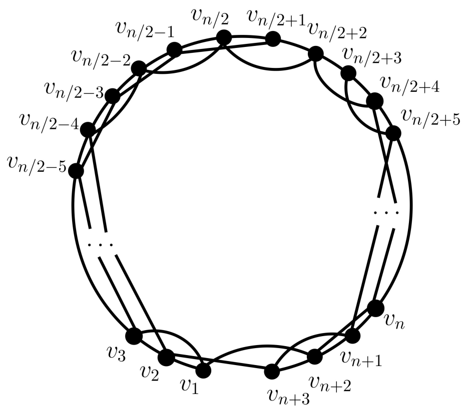

Figure 1: $G_1 = P_1 \cup P_2$

Obviously, only $d_{G_2}(v_{n/2+1}) = d_{G_2}(v_{n/2+3}) = d_{G_2}(v_1) = d_{G_2}(v_{n+3}) = n-1$ and the other vertices in $V(G_2)$ have degree $n-2$. Denote $V' = \{v' \in V(G_2) | d_{G_2}(v') = n-2\}$. We add a vertex w to G_2 to construct $n-1$ regular graph G_2', that is $w \notin V(G_2)$, $E(G_2') = E(G_2) \cup \{wv' | v' \in V'\}$ and $V(G_2') = V(G_2) \cup \{w\}$. Since $d(G') = n-1 = |V(G')| - 5$ and $|V(G')|$ is even, it follows from Theorem 2.2 and Lemma 2.1 that $\chi_1(G_2') = \Delta(G_2') = n-1$ and $\chi_1(G_2) = \Delta(G_2) = n-1$. □

Theorem 3.3. *For integer $m \geq 2$,*

$$(a_1, \chi_1)(m, 2) = 2m.$$

Proof. Let $K_{2m-1} = G_1 \cup G_2$, where $V(G_1) = V(G_2) = V(K_{2m-1})$ and $E(G_2)$ contains $m-1$ matching edges. Since G_2 has no adjacent edge, it follows that $\chi_1(G_2) = 1$. Therefore, we only need to prove that $a_1(G_1) \leq m-1$, that is, G_2 has $m-1$ edge disjoint forests decomposition. According to Theorem 2.1 of Nash-Williams, we know that the necessary and sufficient condition is for all $V \subseteq V(G_1)$, $|E(A)| \leq (m-1)(|A|-1)$.

Since $E(G_1) \cap E(G_2) = \emptyset$, only one vertex $v \in V(G_1)$ has degree $2m-2$ and the other vertices in G_1 have degree $2m-3$. Suppose that $d_{G_1}(v) = 2m-2$ and $v \notin A \subseteq V(G_1)$, then $|E(A)| \leq \frac{|A|(|A|-1)}{2} \leq (m-1)(|A|-1)$. Suppose that $d_{G_1}(v) = 2m-2$ and $v \in A \subseteq V(G_1)$. Denote $A = A' \cup \{v\} \subseteq V(G_1)$ where $v \notin A'$, then we only need to prove that $|E(A')| \leq (m-2)|A'|$. Let c be the number of edges of $E(G_2)$ which contained in the induced subgraph of A'. Thus this problem is equivalent to proving that $\frac{|A'|(|A'|-1)}{2} - c \leq (m-2)|A'|$ for all $A' \subseteq V(G_1)$. Note that $|A'| \geq 2c \geq 0$, then we have $0 \leq \frac{2c}{|A'|} \leq 1$. If $|A'| = 0$, then $0 = |E(A')| = (m-2)|A'| = 0$ and if $|A'| = 1$, then $0 = |E(A')| \leq (m-2)|A'| = m-2$. Let function $g(|A'|) = \frac{1}{2}(|A'| - \frac{2c}{|A'|} - 1)$. Since $g(|A'|)$ strictly monotonically increases in interval $2 \leq |A'| \leq 2m-2$, it follows that we only consider the case $|A'| = 2m-2$. If $|A'| = 2m-2$, then $c = m-1$ and $\frac{|A'|(|A'|-1)}{2} - c = |E(A')| = (m-2)|A'| = 2(m-1)(m-2)$, the proof is done. □

We use the same method to generalize the special case of Conjecture 1.1.

Theorem 3.4. *Let integers $n_i \geq 2$ for all $1 \leq i \leq t$ and odd integers $n_i \geq 3$ for all $t+1 \leq i \leq k$, where $1 \leq t < k$. If $f_1 = f_2 = \ldots = f_t = a_1$ and $f_{t+1} = f_{t+2} = \ldots = f_k = \chi_1$, then*

$$(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k) = 2\sum_{i=1}^{t} n_i + \sum_{i=t+1}^{k} n_i - k - t + 1.$$

Proof. Let $n = 2\sum_{i=1}^{t} n_i + \sum_{i=t+1}^{k} n_i - k - t$. If $(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k) \leq n+1$ does not hold, then there exists a factorization $K_{n+1} = \bigcup_{i=1}^{k} G_i$ such that $a_1(G_i) \leq n_i - 1$ for all $1 \leq i \leq t$ and $\chi_1(G_i) \leq n_i - 1$ for all $t+1 \leq i \leq k$. This implies that $\bigcup_{i=1}^{t} G_i$ has at most $n\sum_{i=1}^{t}(n_i - 1)$ edges and $\bigcup_{i=t+1}^{k} G_i$ has at most $\frac{n+1}{2}\sum_{i=t+1}^{k}(n_i - 1)$ edges. Note that

$$\begin{aligned}
|E(K_{n+1})| &= |E(\bigcup_{i=1}^{k} G_i)| \\
&\leq n\sum_{i=1}^{t}(n_i - 1) + \frac{n+1}{2}\sum_{i=t+1}^{k}(n_i - 1) \\
&= \frac{1}{2}\Big(n^2 + \sum_{i=t+1}^{k} n_i - k + t\Big) \\
&= \frac{1}{2}\Big(n^2 + n + 2\big(t - \sum_{i=1}^{t} n_i\big)\Big) \\
&< \frac{1}{2}(n^2 + n) = |E(K_{n+1})|,
\end{aligned}$$

which is a contradiction. Therefore, $(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k) \leq 2\sum_{i=1}^{t} n_i + \sum_{i=t+1}^{k} n_i - k - t + 1$. Next, we consider the lower bound. Since n_i is odd for every $t+1 \leq i \leq k$, $k-t$ and $k+t$ have the same parity, it follows that $\sum_{i=t+1}^{k} n_i - k - t$ is even, thus n is even. Therefore, there exists a factorization $K_n = \bigcup_{i=1}^{\frac{n}{2}} P_i$ where P_i is a spanning path (see [1] p. 342). For $1 \leq i \leq t$, let G_i be the union of $n_i - 1$ edge disjoint spanning paths of K_n. For $t+1 \leq i \leq k$, let G_i be the union of $\frac{1}{2}(n_i - 1)$ edge disjoint spanning paths of K_n, that is

$$G_1 = \bigcup_{i=1}^{n_1-1} P_i, \quad G_2 = \bigcup_{i=n_1}^{n_1+n_2-2} P_i, \ldots, \quad G_t = \bigcup_{i=\sum_{j=1}^{t-1}(n_j-1)+1}^{\sum_{j=1}^{t}(n_j-1)} P_i \quad \text{and}$$

$$G_{t+1} = \bigcup_{i=\sum_{j=1}^{t}(n_j-1)+1}^{\sum_{j=1}^{t}(n_j-1)+\frac{1}{2}(n_{t+1}-1)} P_i, \quad G_{t+2} = \bigcup_{i=\sum_{j=1}^{t}(n_j-1)+\frac{1}{2}(n_{t+1}-1)+1}^{\sum_{j=1}^{t}(n_j-1)+\frac{1}{2}(n_{t+1}+n_{t+2}-2)} P_i, \ldots, \quad G_k = \bigcup_{i=\sum_{j=1}^{t}(n_j-1)+\frac{1}{2}\sum_{j=t+1}^{k-1}(n_j-1)+1}^{\sum_{j=1}^{t}(n_j-1)+\frac{1}{2}\sum_{j=t+1}^{k}(n_j-1)} P_i.$$

We can see that $a_1(G_i) \leq n_i - 1$ for all $1 \leq i \leq t$ and $\chi_1(G_i) \leq n_i - 1$ for all $t+1 \leq i \leq k$. □

Similarly, the generalized form of Conjecture 1.1 is given below.

Conjecture 3.1. *Let integers $n_i \geq 2$ for all $1 \leq i \leq k$ and integer $1 \leq t < k$. If $f_1 = f_2 = \ldots = f_t = a_1$ and $f_{t+1} = f_{t+2} = \ldots = f_k = \chi_1$, then*

$$(f_1, f_2, \ldots, f_k)(n_1, n_2, \ldots, n_k) = 2\sum_{i=1}^{t} n_i + \sum_{i=t+1}^{k} n_i - k - t + 1.$$

4. Further Discussion

For integer $m \geq 3$ and even integer $n \geq 4$, we consider constructing a factorization $K_{2m+n-3} = G_1 \cup G_2$ to satisfy $a_1(G_1) \leq m-1$ and $\chi_1(G_2) \leq n-1$. One idea is to make $G_1 = \bigcup_{i=1}^{m-1} P_i$, and for every $i \neq j$, P_i and P_j are edge disjoint spanning paths of K_{2m+n-3}, which can ensure that $a_1(G_1) \leq m-1$. So if we can prove $\chi_1(G_2) \leq n-1$, then Conjecture 1.1 is true. We should pay attention to the fact that the choice of $m-1$ edge disjoint spanning paths of K_{2m+n-3} is not arbitrary, the following example will illustrate this fact.

Example 4.1. $(a_1, \chi_1)(4, 4) = 10.$ *Recall that we only need to prove the lower bound. Let $V(K_9) = \{v_1, v_2, \ldots, v_9\}$. Consider a factorization $K_9 = G_1 \cup G_2$ with $G_1 = P_1 \cup P_2 \cup P_3$ where $P_1 = v_1v_2v_3v_4v_5v_6v_7v_8v_9$, $P_2 = v_4v_2v_9v_7v_5v_3v_1v_8v_6$ and $P_3 = v_3v_7v_2v_6v_1v_5v_9v_4v_8$, as shown in Figure 2. One can easy to check that $a_1(G_1) \leq 3$ and $\chi_1(G_2) = \Delta(G_2) = 3$. But if we choose $G'_1 = P_1 \cup P'_2 \cup P'_3$ and consider a factorization $K_9 = G'_1 \cup G'_2$ where $P'_2 = v_1v_4v_7v_2v_5v_8v_3v_6v_9$ and $P'_3 = v_1v_3v_5v_7v_9v_2v_8v_4v_6$, as shown in Figure 3. One can easy to check that $a_1(G'_1) \leq 3$ and $\chi_1(G'_2) \geq \Delta(G'_2) = 5$.*

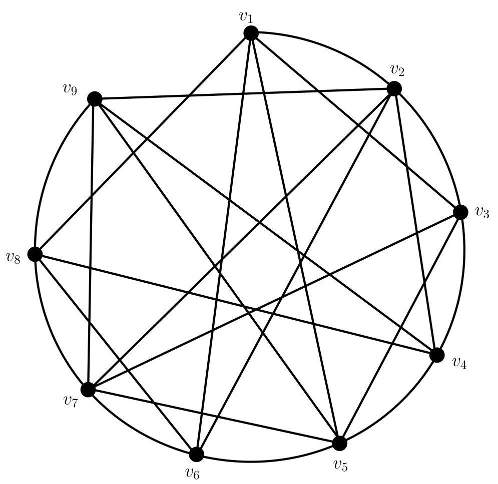

Figure 2: $G_1 = P_1 \cup P_2 \cup P_3$

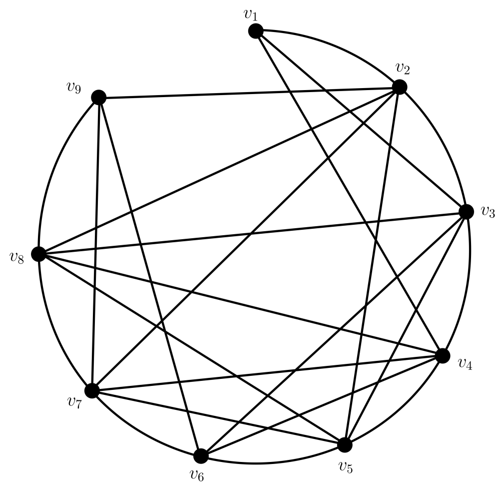

Figure 3: $G'_1 = P_1 \cup P'_2 \cup P'_3$

Based on the above discussion, we would better choose $m-1$ edge disjoint spanning paths of K_{2m+n-3} (see [1] p. 341) to make $\Delta(G_2) - \delta(G_2)$ minimum.

Problem 4.1. *For integer $m \geq 3$ and even integer $n \geq 4$, consider a factorization $K_{2m+n-3} = G_1 \cup G_2$ where G_1 is the union of $m-1$ edge disjoint spanning paths of K_{2m+n-3}.*

(1). How to choose $m-1$ *edge disjoint spanning paths to minimize* $\Delta(G_2)-\delta(G_2)$*?*

(2). If $\Delta(G_2)-\delta(G_2)$ *reaches the minimum value, whether* $\chi_1(G_2)\leq n-1$*?*

References

[1] G. Chartrand, L. Lesniak and P. Zhang, *Graphs* & *Digraphs*, CRC Press, (2016).

[2] A. G. Chetwynd and A. J. W. Hilton, *Regular graphs of high degree are 1-factorizable*, Proc. London Math. Soc., 50(3)(1985), 193-206.

[3] L. Lesniak-Foster and J. Roberts, *On Ramsey theory and graphical parameters*, Pacific J. Math., 68(1)(1977), 105-114.

[4] C. St. J. A. Nash-Williams, *Decompositions of finite graphs into forests*, J. London Math. Soc., 39(1964), 12.