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On a Semi-symmetric Metric Connection in Generalized Sasakian Space forms Satisfying Certain Curvature Conditions

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Abstract: In this paper,we investigate Ricci pseudo-symmetric and Ricci generalized pseudo-symmetric semi-symmetric metric connection in generalized sasakian space forms satisfying the curvature condition $\tilde{S}.\tilde{R} = 0$.

Keywords: Generalized Sasakian Space, Curvature Conditions, Semi-symmetric Metric Connection.

1. Introduction

A $(2n+1)$ dimensional Riemannian manifold of (M^n, g) is said to be an almost contact metric manifold if their exist on M^n a $(1,1)$ type tensor field ϕ, a vector field ξ and a $1-form\ \eta$ such that

$$\phi^2 X = X - \eta(X)\xi,\ \phi\xi = 0,\ \eta(\phi X) = 0. \tag{1}$$

for any vector field X , Y on M. An almost contact metric structure of M is a contact metric manifold if

$$d\eta(X,Y) = g(X,\phi Y),\ g(\xi,X) = \eta(X),\ g(\phi(X),\phi(Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2}$$

A normal contact metric manifold is called a Sasakian manifold. It is well know that an almost contact metric manifold is Sasakian manifold if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{3}$$

for any X , Y. We define endomorphisms $\tilde{R}(X,Y)$ and $X \wedge_A Y$ by

$$\tilde{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{XY} Z \tag{4}$$

and

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y \tag{5}$$

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respectively, where $X, Y, Z\epsilon\ \chi\ M,\ \chi\ M$ is the set of all differential vector field on M, A is the symmetric $(0,2)$-tensor, $\tilde{R}$ is the Riemannian curvature tensor of type (1,3) and ∇ is the Levi-Civita connection. For a $(0,k)$ - tensor field $T,\ k \geq 1$, on (M^n, g) we define the tensor $\tilde{R}.T$ and $Q(g,T)$ by

$$(\tilde{R}(X,Y).T)(X_1, X_2...X_k) = -T(\tilde{R}(X,Y)(X_1, X_2...X_k) - T(X_1, \tilde{R}(X,Y)X_2...X_k) - \cdots - T(X_1, X_2...\tilde{R}(X,Y)X_k) \quad (6)$$

and

$$Q(g,T)(X_1, X_2...X_k; X, Y) = -T((X\wedge Y)X_1, X_2...X_k) - T(X_1(X\wedge Y)X_2...X_k) - \cdots - T(X_1, X_2, ...(X\wedge Y)X_k) \quad (7)$$

respectively [12]. If the tensors $\tilde{R}.\tilde{S}$ and $Q(g,\tilde{S})$ are linearly dependent then M^n is called Ricci pseudo-symmetric [12]. This is equivalent to

$$\tilde{R}.\tilde{S} = fQ(g,\tilde{S}), \quad (8)$$

holding on the set $U_{\tilde{S}} = [x\epsilon(M) : \tilde{S} \neq (0)\ at\ x]$, where f is some function on $U_{\tilde{S}}$. Analogously, if the tensors $\tilde{R}.\tilde{R}$ and $Q(\tilde{S},\tilde{R})$ are linearly dependent then M^n is called Ricci generalized pseudo-symmetric [12]. This is equivalent to

$$\tilde{R}.\tilde{R} = fQ(\tilde{S},\tilde{R}), \quad (9)$$

holding on the set $U_{\tilde{R}} = [x\epsilon(M) : R \neq (0)\ at\ x]$,where f is some function on $U_{\tilde{R}}$. A very important subclass of this class of manifolds realizing the condition is

$$\tilde{R}.\tilde{R} = Q(\tilde{S},\tilde{R}). \quad (10)$$

Every three dimensional manifold satisfies the above equation identically. Other example are the semi-Riemannian manifolds (M,g) admitting a non-zero 1-form ω such that the equality $\omega(X)\tilde{R}(Y,Z) + \omega(Y)\tilde{R}(Z,X) + \omega(Z)\tilde{R}(X,Y) = 0$, holds on M. The condition $\tilde{R}.\tilde{R} = Q(\tilde{S},\tilde{R})$ also appears in the theory of plane gravitational waves. Furthermore we define the tensors $\tilde{R}.\tilde{R}$ and $\tilde{R}.\tilde{S}$ on (M^n, g) by

$$(\tilde{R}(X,Y).\tilde{R})(U,V)W = \tilde{R}(X,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(X,Y)U,V)W - \tilde{R}(U,\tilde{R}(X,Y)V)W - \tilde{R}(U,V)\tilde{R}(X,Y)W, \quad (11)$$

and

$$(\tilde{R}(X,Y).\tilde{S})(U,V) = -\tilde{S}(\tilde{R}(X,Y)U,V) - \tilde{S}(U,\tilde{R}(X,Y)V). \quad (12)$$

respectively. Recently, Kowalczyk [13] studied semi-riemannian manifolds satisfying $Q(\tilde{S},\tilde{R}) = 0$ and $Q(\tilde{S},g) = 0$, where $\tilde{S},\tilde{R}$ are the Ricci tensor respectively. An almost paracontact Riemannian manifold M is said to be η - Einstein manifold if the Ricci tensor $\tilde{S}$ satisfies the condition

$$\tilde{S}(X,Y) = ag(X,Y) + b\eta(y)\eta(X). \quad (13)$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then manifold is Einstein Manifold.

2. Preliminaries

In a Semi-Symmetric Metric Connection in Generalized Sasakian space forms the following relations holds [1].

$$\tilde{S}(Y,\xi) = 2n(f_1 - f_3)\eta(Y), \tag{14}$$

$$\tilde{R}(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \eta(Y)\phi(X) - \eta(X)\phi(Y), \tag{15}$$

$$\tilde{R}(\xi,Y)Z = (f_1 - f_3)[g(Y,Z)\xi - \eta(Z)Y] + g(\phi(Y),Z)\xi - \eta(Z)\phi(Y), \tag{16}$$

$$\tilde{R}(X,\xi)Z = R(X,\xi)Z + \eta(Z)\phi(X) - g(\phi(X),Z)\xi, \tag{17}$$

$$\tilde{R}(\xi,Y)\xi = \tilde{R}(\xi,Y)\xi - \phi(Y), \tag{18}$$

$$\eta(\tilde{R}(X,Y)\xi = 0, \tag{19}$$

$$\tilde{S}(Y,Z) = S(Y,Z) + (2n-1)[g(\phi Y,Z) + \eta(Y)\eta(Z) - g(Y,Z)], \tag{20}$$

$$S(Y,Z) = (2nf_1 + 3f_2 - f_3)g(Y,Z) - (3f_2 + (2n-1)f_3)\eta(Y)\eta(Z), \tag{21}$$

$$\tilde{S}(Y,Z) = [2n(f_1 - 1) + 3f_2 - f_3 + 1]g(Y,Z) - [3f_2 + (2n-1)(f_3 - 1)]\eta(Y)\eta(Z) + (2n-1)g(\phi Y,Z). \tag{22}$$

for any vector fields $X, Y, Z \in \chi(M)$.

3. Ricci Pseudo-symmetric Semi-symmetric Connection in Generalised Sasakian Space Forms

In this section we study Ricci pseudo-symmetric manifold, that is the manifold satisfying the condition $\tilde{R}.\tilde{S} = fQ(g,\tilde{S})$. Assume that M is Ricci Pseudo-symmetric in Semi-symmetric metric connection in generalized sasakian space forms and $X, Y, U, V \in \chi(M)$. We have from (8),

$$(\tilde{R}(X,Y).\tilde{S})(U,V) = fQ(g,\tilde{S})(X,Y;U,V) \tag{23}$$

It is equivalent to

$$(\tilde{R}(X,Y).\tilde{S})(U,V) = f((X \wedge_g Y).\tilde{S})(U,V). \tag{24}$$

from (7) and (12), we obtain

$$-\tilde{S}(\tilde{R}(X,Y)U,V) - \tilde{S}(U,\tilde{R}(X,Y)V) = f[-\tilde{S}((X \wedge_g Y)U,V) - \tilde{S}(U,(X \wedge_g Y)V), \tag{25}$$

Using (5) equation (25) reduces to

$$-\tilde{S}(\tilde{R}(X,Y)U,V) - \tilde{S}(U,\tilde{R}(X,Y)V) = f[-g(Y,U)\tilde{S}(X,V) + g(X,U)\tilde{S}(Y,V) - g(Y,V)\tilde{S}(U,X) + g(X,V)\tilde{S}(U,Y)]. \tag{26}$$

Substituting $X = U = \xi$ in (26) and using (14) and (15), we obtain

$$((f_1 - f_3 - f)^2 + 1)[\tilde{S}(Y,V) - 2n(f_1 - f_3)g(Y,V)] = 0 \tag{27}$$

Then either $(f_1 - f_3 - f)^2 = -1$ or the manifold is an Einstein Manifold of the from

$$\tilde{S}(Y,V) = 2n(f_1 - f_3)g(Y,V) \tag{28}$$

By the above discussions we have the following theorem,

Theorem 3.1. *An $(2n+1)$ - dimensional Ricci pseudo - symmetric in Semi - symmetric metric connection in generalized sasakian space forms satisfying $\tilde{R}.\tilde{S} = Q(g, \tilde{S})$ is an Einstein Manifold if $(f_1 - f_3 - f)^2 \neq -1$.*

From the above discussion we state the following theorem,

Theorem 3.2. *An $(2n+1)$ - dimensional Ricci pseudo - symmetric in Semi - symmetric metric connection in generalized sasakian space forms satisfying $\tilde{R}.\tilde{S} = Q(g, \tilde{S})$ is Not Einstein Manifold if $(f_1 - f_3 - f)^2 = 1$.*

If $Q(g, \tilde{S}) = 0$ then we can state the following corollaries,

Corollary 3.3. *An $(2n+1)$ - dimensional Ricci pseudo - symmetric in Semi - symmetric metric connection in generalized sasakian space forms satisfying $Q(g, \tilde{S}) = 0$ is an Einstein Manifold if $(f_1 - f_3) - f \neq 0)$.*

Corollary 3.4. *An $(2n+1)$ - dimensional Ricci pseudo - symmetric in Semi - symmetric metric connection in generalized sasakian space forms satisfying $Q(g, \tilde{S} = 0)$ is Not Einstein Manifold if $(f_1 - f_3) - f = 0)$.*

4. Ricci Generalized Pseudo-symmetric Semi-symmetric Connection in Generalised Sasakian Space Forms

In this section we study Ricci Generalized pseudo-symmetric in Semi - symmetric metric connection in generalized space forms. Let us assume that M be an $(2n+1)$ - dimensional Ricci Generalized Pseudo-symmetric in Semi - symmetric connection in generalised sasakian space forms manifold then from (9), we have

$$\tilde{R}.\tilde{R} = fQ(\tilde{S}.\tilde{R}) \tag{29}$$

that is

$$(\tilde{R}(X,Y)).\tilde{R})(U,V)W = f((X \wedge_{\tilde{S}} Y).\tilde{R})(U,V)W \tag{30}$$

using (7) and (11) in (30), we obtain

$$\begin{aligned}\tilde{R}(X,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(X,Y)U,V)W - \tilde{R}(U,\tilde{R}(X,Y)V)W - \tilde{R}(U,V)\tilde{R}(X,Y)W = f[(X \wedge_{\tilde{S}} Y)\tilde{R}(U,V)W \\ - \tilde{R}((X \wedge_{\tilde{S}} Y)U,V)W - \tilde{R}(U,(X \wedge_{\tilde{S}} Y)V)W - \tilde{R}(U,V)(X \wedge_{\tilde{S}} Y)W],\end{aligned} \tag{31}$$

In view of (5) and (31), we obtain

$$\begin{aligned}&\tilde{R}(X,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(X,Y)U,V)W - \tilde{R}(U,\tilde{R}(X,Y)V)W - \tilde{R}(U,V)\tilde{R}(X,Y)W \\ &= f[\tilde{S}(Y,\tilde{R}(U,V)W)X - \tilde{S}(X,\tilde{R}(U,V)W)Y - \tilde{S}(Y,U)\tilde{R}(X,V)W + \tilde{S}(X,U)\tilde{R}(Y,V)W - \\ &\tilde{S}(Y,V)\tilde{R}(U,X)W + \tilde{S}(X,V)\tilde{R}(U,Y)W - \tilde{S}(Y,W)\tilde{R}(U,V)X + \tilde{S}(X,W)\tilde{R}(U,V)Y),\end{aligned} \tag{32}$$

substituting $X = U = \xi$ in (32) and taking inner product with Z and using (14),(15) and (17), we obtain

$$\begin{aligned}&-(f_1 - f_3)^2 g(V,W)g(Y,Z) - (f_1 - f_3)g(\phi V,W)g(Y,Z) - (f_1 - f_3)g(\phi Y,Z)g(V,W) \\ &(f_1 - f_3)g(\phi Y,Z)\eta(V)\eta(W) - g(\phi V,W)g(\phi(Y),Z) + (f_1 - f_3)^2 \\ &\eta(W)\eta(Y)g(V,Z) + (f_1 - f_3)R(Y,V,W,Z) + R(\phi(Y),V,W,Z) + (f_1 - f_3)^2 \\ &\eta(V)\eta(Z)(f_1 - f_3)g(\phi Y,W)\eta(V)\eta(Z) - g(V,W)\eta(V)\eta(Z) + g(Y,Z)\end{aligned}$$

$$\eta(V)\eta(W) - (f_1 - f_3)^2\eta(V)\eta(Z)g(Y,W) + (f_1 - f_3)^(2)g(Y,W)g(V,Z)$$
$$- (f_1 - f_3)^2\eta(W)\eta(Y)g(V,Z) + (f_1 - f_3)g(\phi(Y),W)g(V,Z) + (f_1 - f_3)$$
$$g(\phi(V),Z)g(Y,W) + g(\phi(Y),W)g(\phi(V),Z) = f[-(f_1 - f_3)\tilde{S}(Y,V)\eta(W)\eta(Z)$$
$$- \tilde{S}(Y,\phi(V))\eta(W)\eta Z - 2n(f_1 - f_3)^2 g(V,W)g(Y,Z) - 2n(f_1 - f_3)$$
$$g(\phi(V),W)g(Y,Z) + 2n(f_1 - f_3)R(Y,V,W,Z) + 2n((f_1 - f_3)^2)g(Y,W)\eta(V)\eta(Z)$$
$$- 2n(f_1 - f_(3))^2 g(\phi(Y),Z)\eta(V)\eta(W) - (f_1 - f_3)\tilde{S}(Y,W)\eta(V)\eta(Z)$$
$$+ (f_1 - f_3)\tilde{S}(Y,W)g(V,Z) + \tilde{S}g(\phi(V),Z) + 2n(f_1 - f_3)^2 g(V,Y)\eta(W)\eta(Z)$$
$$+ 2n(f_1 - f_3)g(\phi(V),Y)\eta(W)\eta(Z)] \tag{33}$$

Let $e_i \quad (1 \leq i \leq (2n+1))$ be an orthonormal basis of the tangent space at any point. Now taking summation over $i = 1,2,3,4...(2n+1)$ of the relation (33) for $\; V = W = e_i$ gives

$$\begin{aligned}
&- (2n+1)(f_1 - f_3)^2 g(Y,Z) + (f_1 - f_3)\tilde{S}(Y,Z) - \eta(Y)\eta(Z) + g(Y,Z) + (f_1 - f_3)^2 g(Y,Z) - \\
&(2n+1)(f_1 - f_3)g(\phi Y,Z) + \tilde{S}(\phi Y,Z) + (f_1 - f_3)g(\phi e_i,Z)g(Y,e_i) + g(\phi Y,e_i)g(\phi e_i,Z) = \\
&f[-(f_1 - f_3)\tilde{S}(Y,e_i)\eta(e_i)\eta(Z) - \tilde{S}(Y,\phi e_i)\eta(e_i)\eta(Z) - 2n(f_1 - f_3)^2 g(e_i,e_i) \\
&g(Y,Z) + 2n(f_1 - f_3)\tilde{R}(Y,e_i,e_i,Z) + 2n(f_1 - f_3)^2 g(Y,e_i)\eta(e_i)\eta(Z) - 2n(f_1 - f_3)g(\phi Y,Z) - \\
&(f_1 - f_3)\tilde{S}(Y,e_i)\eta(e_i)\eta(Z) + (f_1 - f_3)\tilde{S}(Y,e_i)g(e_i,Z) + \tilde{S}(Y,e_i)g(\phi e_i,Z) \\
&2n(f_1 - f_3)^2 g(e_i,Y)\eta(e_i)\eta(Z) + 2n(f_1 - f_3)g(\phi e_i,Y)\eta(e_i)\eta(Z)],
\end{aligned} \tag{34}$$

Further on simplifying, we obtain

$$\begin{aligned}
&[1 + (f_1 - f_3)^2 - (2n+1)(f_1 - f_3)^2]g(Y,Z) - [(2n+1)(f_1 - f_3) - (f_1 - f_3)]g(\phi Y,Z) + \\
&(f_1 - f_3)\tilde{S}(Y,Z) - \eta(Y)\eta(Z) + \tilde{S}(\phi Y,Z) - g(\phi Y,\phi Z) = -2n(2n+1)f(f_1 \\
&- f_3)^2 g(Y,Z) + 2nf(f_1 - f_3)\tilde{S}(Y,Z) - 2nf(f_1 - f_3)g(\phi Y,Z)f(f_1 - f_3)\tilde{S}(Y,Z) \\
&- f\tilde{S}(Y,\phi Z),
\end{aligned} \tag{35}$$

Using (22) in (35), we obtain

$$\begin{aligned}
&[1 + (f_1 - f_3)^2 - (2n+1)(f_1 - f_3)^2 + 2n(2n+1)f(f_1 - f_3)^2]g(Y,Z) + [(f_1 - f_3) \\
&(1 - 2nf - f)]\tilde{S}(Y,Z) - \eta(Y)\eta(Z) = -g(\phi(Y),Z)[2nf(f_1 - f_3) - 2n(f_1 - f_3)] - \tilde{S} \\
&(\phi Y,Z) + g(\phi Y,\phi Z),
\end{aligned} \tag{36}$$

Further on simplifying, we obtain

$$\begin{aligned}
&[1 + (f_1 - f_3)^2 - (2n+1)(f_1 - f_3)^2 + 2n(2n+1)f(f_1 - f_3)^2]g(Y,Z) + [(f_1 - f_3)(1 - 2nf - f)] \\
&\tilde{S}(Y,Z) - \eta(Y)\eta(Z) = [2n(f_1 - f_3)(1 - f)]g(\phi Y,Z) - [2n(f_1 - 1) + 3f_2 - f_3 + 1] \\
&g(\phi Y,Z) - (2n-1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z) + g(Y,Z) - \eta(Y)\eta(Z) - 2(2n+1)f(f_1 - f_3) \\
&- (f_1 - f_3) + 2nff_1 + 3ff_2 - ff_2 - f].
\end{aligned} \tag{37}$$

From (37), we obtain

$$Ag(Y,Z) + B\tilde{S}(Y,Z) + C\eta(Y)\eta(Z) = Dg(\phi Y,Z), \tag{38}$$

where

$$\begin{aligned} A &= [(f_1 - f_3)^2 - (2n+1)(f_1 - f_3)^2 + 2n(2n+1)f(f_1 - f_3)^2 - (2n-1)], \\ B &= (1 - 2nf - f)(f_1 - f_3), \\ C &= (2n-1), \\ D &= [2n(f_1 - f_3)(1 - f) - (2n(f_1 - 1) + 3f_2 - f_3 + 1)] \end{aligned} \tag{39}$$

Further, we obtain

$$g(\phi Y, Z) = \frac{Ag(Y,Z) + B\tilde{S}(Y,Z) + C\eta(Y)\eta(Z)}{D}, \tag{40}$$

Using (22) in (40), we obtain

$$g(\phi Y, Z) = \frac{g(Y,Z)[A + (2n(f_1 - 1) + 3f_2 - f_3 + 1)] + \eta(Y)\eta(Z)[C - 3f_2 + (2n-1)(f_3 - 1)]}{D - (2n-1)}, \tag{41}$$

Using (41) in (38), we obtain

$$\tilde{S}(Y,Z) = Pg(Y,Z) + Q\eta(Y)\eta(Z). \tag{42}$$

where

$$\begin{aligned} P &= \frac{D[2n(f_1 - 1) + 3f_2 - f_3 + 1] - A(2n-1)}{B(D - (2n-1))} \\ Q &= \frac{2C - 3f_2 + (2n-1)(f_3 - 1) - CD - 2nC}{B(D - (2n-1))} \end{aligned} \tag{43}$$

Theorem 4.1. *An $(2n+1)$ - dimensional Ricci pseudo - symmetric in Semi - symmetric metric connection in generalized sasakian space forms satisfying $\tilde{R}.\tilde{R} = f(Q(\tilde{S}, \tilde{R})$ is η Einstein Manifold.*

5. Semi-symmetric Connection in Generalized Sasakian Space form Manifold Satisfying the Curvature Condition S.R=0

In this section we consider a Semi-symmetric connection in generalized sasakian space form Manifold satisfying the curvature condition $\tilde{S}.\tilde{R} = 0$. Thus we have

$$(\tilde{S}(X,Y).\tilde{R})(U,V)W = 0, \tag{44}$$

which implies

$$(X \wedge_{\tilde{S}} Y)\tilde{R}(U,V)W + \tilde{R}((X \wedge_{\tilde{S}} Y)U,V)W + \tilde{R}(U,(X \wedge_{\tilde{S}} Y)V)W + \tilde{R}(U,V)(X \wedge_{\tilde{S}} Y)W = 0, \tag{45}$$

using (5) we have from (45)

$$\begin{aligned} &\tilde{S}(Y,\tilde{R}(U,V)W)X - \tilde{S}(X,\tilde{R}(U,V)W)Y + \tilde{S}(Y,U)\tilde{R}(X,V)W - \tilde{S}(X,U)\tilde{R}(Y,V)W \\ &\quad + \tilde{S}(Y,V)\tilde{R}(U,X)W - \tilde{S}(X,V)\tilde{R}(U,Y)W + \tilde{S}(Y,W)\tilde{R}(U,V)X - \tilde{S}(X,W)\tilde{R}(U,V)Y = 0. \end{aligned} \tag{46}$$

substituting $U = W = \xi$ in (46) and using (15) and (17)

$$\begin{aligned}
&(f_1 - f_3)\eta(V)\tilde{S}(Y,\xi)X - (f_1 - f_3)\tilde{S}(Y,V)X - \tilde{S}(Y,\phi(V))X - (f_1 - f_3)\eta(V)\tilde{S}(X,\xi)Y \\
&+ (f_1 - f_3)\tilde{S}(X,V)Y + \tilde{S}(X,\phi(V))Y + 2n(f_1 - f_3)^2\eta(Y)\eta(V)X + 2n(f_1 - f_3)\eta(Y)\eta(V)\phi(X) \\
&- 2n(f_1 - f_3)^2\eta(X)\eta(V)Y - 2n(f_1 - f_3)\eta(X)\eta(V)\eta(Y) + (f_1 - f_3)\eta(X)\tilde{S}(Y,V)\xi \\
&- (f_1 - f_3)\tilde{S}(Y,V)X - \tilde{S}(Y,V)\phi(X) - (f_1 - f_3)\eta(Y)\tilde{S}(X,V)\xi + (f_1 - f_3)\tilde{S}(X,V)Y \\
&+ \tilde{S}(X,V)\eta(Y) + 2n(f_1 - f_3)^2 g(V, X\eta(Y)\xi + 2n(f_1 - f_3)g(\phi(V),X)\eta(Y)\xi\xi - \\
&2n(f_1 - f_3)^2 g(V,Y)\eta(X)\xi - 2n(f_1 - f_3)g(\phi(V),Y)\eta(X)\xi = 0,
\end{aligned} \tag{47}$$

Taking inner product of (47) with ξ and then replacing $X = \xi$ and using (14), we obtain

$$[1 + \frac{1}{(f_1 - f_3)^2}][\tilde{S}(Y,V) - 4n(f_1 - f_3)\eta(V)\eta(Y) - 2n(f_1 - f_3)g(V,Y)] = 0, \tag{48}$$

Then either $\frac{1}{(f_1-f_3)^2} = -1$ or the manifold is an η - Einstein Manifold of the form

$$\tilde{S}(Y,V) = Gg(V,Y) + H\eta(V)\eta(Y). \tag{49}$$

where

$$G = -2n(f_1 - f_3), H = 4n(f_1 - f_3). \tag{50}$$

Theorem 5.1. *An $2n+1$-dimensional Semi-symmetric metric connection in Generalized sasakian space forms satisfying the curvature condition $\tilde{S}.\tilde{R} = 0$ then the manifold is an η-Einstein manifold.*

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