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Ricci Solitons on (ϵ)-Para Sasakian Manifolds

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Abstract: The object of the present paper is to study some semi-symmetry conditions on (ϵ)-para Sasakian manifolds admitting Ricci solitons.

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1. Introduction

In the differential geometry, the Ricci flow is an intrinsic geometric flow, which was introduced by R. Hamilton ([10, 11]). The Ricci flow is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing our irregularities in the metric. The Ricci flow equation is the evolution equation

$$\frac{d}{dt}g_{ij}(t) = -2R_{ij}$$

for a Riemannian metric g_{ij}, where R_{ij} is the Ricci curvature tensor. Hamilton showed that there is a unique solution to this equation for an arbitrary smooth metric g_{ij}. He also showed that Ricci flow preserves positivity of Ricci curvature tensor in three dimensions and the curvature operator in all dimensions. Ricci solitons are Ricci flows that may change their size but not their shape up to diffeomorphisms. A significant 2-dimensional example of Ricci soliton is the cigar solution which is given by the metric $(dx^2 + dy^2)/(e^{4t} + x^2 + y^2)$ on the Euclidean plane. Although this metric shrinks under the Ricci flow, its geometry remains the same. Such solutions are called steady Ricci solitons. A Ricci soliton is a triple (g, v, λ) with g a Riemannian metric, V a vector field andλ a real scalar such that

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1}$$

where S is a Ricci tensor of M^n and L_V denote the Lie-derivative along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as real scalar λ is negative, zero and positive respectively. Ricci solitons were studied by several authors in contact and Lorentzian manifold such as Sharma [16], He and Zhu [12], Nagaraja and Premalatha [13], Bagewadi et all [1], and others.

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On the other hand, the study of manifolds with indefinite metrics is of interest from the stand point of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1993, Bejancu and Duggal [3] introduced the concept of (ϵ)-Sasakian manifolds and Xufeng and Xiaoli [19] established that these manifolds are real hyper-surfaces of indefinite Kahlerian manifolds. De and Sarkar [5] introduced (ϵ)-Kenmotsu manifolds and studied some curvature conditions on it. Singh, Pandey, Pandey and Tiwari [18], established the relation between semi-symmetric metric connection and Riemannian connection on (ϵ)-Kenmotsu manifolds and have studied several curvature conditions. Singh, Pandey and Tiwari [17] studied quarter symmetric metric connection in an indefinite Sasakian manifold. The relation between the semi-symmetric non-metric connection and Levi-Civita connection in an indefinite para Sasakin manifold have been established by Pandey, Pandey, Tiwari and Singh [14]. Motivated by these studies, we study Ricci solitons in (ϵ)-para Sasakian manifolds. In this paper, we have studied Ricci solitons in (ϵ)-para sasakian manifolds satisfying

$$R(\xi, X).R = 0, \tag{2}$$

$$R(\xi, X).S = 0, \tag{3}$$

$$R(\xi, X).P = 0, \tag{4}$$

$$P(\xi, X).R = 0, \tag{5}$$

$$S(\xi, X).P = 0, \tag{6}$$

where P is projective curvature tensor of the manifold.

2. (ϵ)-Para Sasakian Manifolds

Let M^n be an almost paracontact manifold equipped with an almost paracontact structure (φ, ξ, η) consisting of a tensor field φ of type (1,1), a vector field ξ and a one form η satisfying

$$\varphi^2 X = X - \eta(X)\xi, \tag{7}$$

$$\eta(\xi) = 1, \tag{8}$$

$$\varphi(\xi) = 0, \tag{9}$$

$$\eta ø \varphi = 0. \tag{10}$$

Let M^n be an n-dimensional almost paracontact manifold and g be semi-Riemannian metric with index (g) = ν such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{11}$$

where $\epsilon = \pm 1$. In this case, M^n is called an (ϵ)-almost paracontact metric manifold equipped with an (ϵ)-almost paracontact structure (φ, ξ, η, g) [19]. In particular, if $ndex(g) = 1$, then an (ϵ)-almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. If in case, the metric is positive definite, then an (ϵ)-almost paracontact metric manifold is the almost paracontact metric manifold. In view of equations (8), (9) and (11), we have

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{12}$$

and

$$g(X, \xi) = \epsilon\eta(X) \tag{13}$$

for every $X, Y \in TM^n$. From equation (13), it follows that

$$\epsilon = g(\xi, \xi), \tag{14}$$

i.e. the structure of vector field ξ is never light-like. An (ϵ)-almost paracontact metric manifold (respectively a Lorentzian almost paracontact manifold $(M^n, \varphi, \xi, g, \epsilon)$) is said to space-like (ϵ)-almost paracontact metric manifold (respectverly a space-like Lorentzian almost paracontact manifold), if $\epsilon = 1$ and M^n is said to be a time-like (ϵ)-almost paracontact metric manifold (respectively, a Lorentzian almost paracontact manifold) if $\epsilon = -1$. An (ϵ)-almost paracontact metric structure is called an (ϵ)-Para Sasakian structure if

$$(\nabla_X \varphi)(Y) = -g(X, \varphi Y)\xi - \epsilon\eta(Y)\varphi^2 X, \ \ X, Y \in TM^n, \tag{15}$$

where ∇ is the Levi-Civita connection. A manifold M^n endowed with an (ϵ)-para Sasakian structure is called an (ϵ)-para Sasakian manifold. For $\epsilon = 1$ and g Riemannian, M^n is the usual para Sasakian manifold [15]. For $\epsilon = -1$, g Lorentzian and ξ replaced by $-\xi$, M^n becomes a Lorentzian para Sasakian manifold [9]. In an (ϵ)-para Sasakian manifold, we have

$$\nabla_X \xi = \epsilon\varphi X, \tag{16}$$

$$\Omega(X, Y) = \epsilon g(\varphi X, Y) = (\nabla_X \eta)(Y), \tag{17}$$

for every $X, Y \in TM^n$, where Ω is the fundamental 2-form. In an (ϵ)-almost para Sasakian manifold M^n, the following relations holds.

$$R(\xi, X)Y = -\epsilon g(X, Y)\xi + \epsilon\eta(Y)X, \tag{18}$$

$$R(X, Y)\xi = -\epsilon\eta(Y)X + \epsilon\eta(X)Y, \tag{19}$$

$$\eta(R(X, Y)Z) = \epsilon[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \tag{20}$$

In an n-dimensional (ϵ)-para Sasakian manifold M^n, the Ricci tensor S satisfies

$$S(\varphi Y, \varphi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z), \tag{21}$$

for all $X, Y \in TM^n$. By the virtue of above equation, we have

$$S(\varphi Y, Z) = S(Y, \varphi Z) \tag{22}$$

and

$$S(Y, \xi) = -(n-1)\eta(Y) \tag{23}$$

Let (g, V, λ) be a Ricci soliton in an indefinite Sasakian manifold. Then from equation (15), we have

$$(L_\xi g)(X, Y) = -2[\epsilon g(X, Y) - \eta(X)\eta(Y)]. \tag{24}$$

In view of equations (1) and (24), we have

$$S(X, Y) = [\epsilon g(\varphi X, Y) - \lambda g(X, Y)], \tag{25}$$

which gives

$$S(X,\xi) = -\epsilon\lambda\eta(X), \tag{26}$$

$$QX = \epsilon\varphi X - \lambda X, \tag{27}$$

$$r = n[\varphi X - \lambda X]. \tag{28}$$

The projective curvature tensor P is defined as [6].

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{29}$$

By virtue of equations (20), (26) and (28) the projective curvature tensor P on (ϵ)-Para Sasakian manifold takes the form

$$P(\xi,Y)Z = [-\epsilon g(Y,Z)\xi + \epsilon\eta(Z)Y] + \frac{\lambda}{n-1}[\epsilon\eta(Z)\xi - \epsilon g(Y,Z)\xi] - \frac{\epsilon}{n-1}g(\varphi Y,Z)\xi, \tag{30}$$

Putting $Y = \xi$ in equation (29) and using equation (8), (20), (26) and (27), we get

$$P(Y,\xi)Z = \left[1 + \frac{\lambda}{n-1}\right][\epsilon\eta(Z)X - \epsilon g(X,Z)\xi] + \frac{\epsilon}{n-1}g(\varphi X,Z)\xi. \tag{31}$$

Again, on putting $Z = \xi$ in equation (29) and by the use of equation (8), (9), (24) and (27), we obtain

$$P(X,Y)\xi = \left[1 - \frac{\lambda}{(n-1)}\right][\epsilon\eta(X)Y - \epsilon\eta(Y)X] \tag{32}$$

and

$$\eta(P(X,Y)Z) = \left[\frac{\lambda}{(n-1)} - \epsilon\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - \frac{\epsilon}{n-1}[g(\varphi X,Z)\eta(X) - g(\varphi X,Z)\eta(Y)]. \tag{33}$$

Example 2.1. *Let R^3 be the 3-dimensional real number space with a co-ordinate system (x,y,z). We define*

$$\eta = dz - ydx,\ \xi = \frac{\partial}{\partial Z},\ \varphi(\frac{d}{dx}) = -\frac{d}{dx} - y\frac{d}{dz},\ \varphi(\frac{d}{dy}) = -\frac{d}{dy},\ \varphi(\frac{d}{dz}) = 0,$$

$$g_1 = (dx)^2 + dy^2 - \eta\otimes\eta,$$

$$g_2 = (dx)^2 + (dy)^2 + (dz)^2 - y(dx\otimes dz + dz\otimes dx),$$

$$g_3 = -(dx)^2 + (dy)^2 + (dz)^2 - y(dx\otimes dz + dz\otimes dx).$$

Then the (φ,ξ,η) is an almost paracontact structure in R^3. The set the (φ,ξ,η,g_1) is a time -like Lorentzian paracontact structure. Moreover, trajectories of the time-like structure vector ξ are geodesics. The set (φ,ξ,η,g_2) is space-like Lorentzian almost paracontact structure. The set (φ,ξ,η,g_3) is a space-like (ϵ)-almost paracontact metric structure $(\varphi,\xi,\eta,g_3,\epsilon)$ with index $(g_3) = 2$.

3. Ricci Solitons on (ϵ)-para Sasakian Manifolds Satisfying (2)

Let us suppose $R(\xi,X).R = 0$, that is $(R(\xi,X).R)(Y,Z)W = 0$, which gives

$$R(\xi,X)R(Y,Z)U - R(R(\xi,X)Y,Z)U - R(Y,R(\xi,X)Z)U - R(Y,Z)R(\xi,X)U = 0. \tag{34}$$

In view of equation (20), above equation reduces to

$$\epsilon\eta(R(Y,Z)U)X - \epsilon g(X,R(Y,Z)U)\xi + \epsilon g(X,Y)R(\xi,Z)U - \epsilon\eta(Y)R(X,Z)U \\ + \epsilon g(X,Z)R(Y,\xi)U - \epsilon\eta(Z)R(Y,X)U + \epsilon g(X,U)R(Y,Z)\xi - \epsilon\eta(U)R(Y,Z)X = 0. \tag{35}$$

Now, taking the inner product of above equation with ξ,and using equations (8) and (9), we get

$$\eta(X)\eta(R(Y,Z)U)X - \epsilon g(X,R(Y,Z)U) + g(X,Y)\eta(R(\xi,Z)U) - \eta(Y)\eta(R(X,Z)U) \\ + g(X,Z)\eta(R(Y,\xi)U) - \eta(Z)\eta(R(Y,X)U) + g(X,U)\eta(R(Y,Z)\xi) - \eta(U)\eta(R(Y,Z)X) = 0. \tag{36}$$

By virtue of equation (25), above equation takes the form

$$\epsilon g(X,R(Y,Z)W) = [(\epsilon-1)\{g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}\eta(U) \\ + \epsilon\{-2g(Z,U)\eta(X)\eta(Y) + g(X,Z)g(Y,U)g(X,Y)g(Z,U)\}. \tag{37}$$

Putting $X = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$S(Y,Z) = -2n\epsilon\eta(Y)\eta(Z) = 0, \tag{38}$$

which on putting equation Y $= \xi$, gives

$$S(\xi,Z) = -2n\epsilon\eta(Z) = 0, \tag{39}$$

Using equation (25) in above equation, we get $\lambda > 0$, which shows that λ is expanding. Thus, we can state as follows.

Theorem 3.1. *Ricci Solitons on (ϵ)-para Sasakian manifolds with ξ as space-like vector field satisfying $R(\xi,X).R = 0$, is expanding.*

4. Ricci Solitons on (ϵ)-para Sasakian Manifolds Satisfying (3)

Let us suppose that $R(\xi,X).S = 0$. Then we have

$$S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z) = 0. \tag{40}$$

By virtue of equation (20) above equation takes the form

$$\epsilon\eta(Y)S(X,Z) - \epsilon g(X,Y)S(Z,\xi) + \epsilon\eta(Z)S(Y,X) - \epsilon g(X,Z)S(Y,\xi) = 0. \tag{41}$$

In view of equations (26) and (27), above equation takes the form

$$(\lambda - \lambda\epsilon)\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\} + \{g(\varphi X,Z)\eta(Y) + g(\varphi Y,X)\eta(Z)\} = 0. \tag{42}$$

Now, putting $X = Y = \xi$, in above equation and by use of equations (8), (9) and (10), we obtain

$$\lambda = \frac{1}{\epsilon}, \tag{43}$$

Now, suppose ξ is space-like vector field in (ϵ)-para Sasakian manifolds, then from equation (43), we obtain $\lambda > 0$, which shows that λ is expanding. Thus, we can state as follows.

Theorem 4.1. *Ricci Solitons on (ϵ)-para Sasakian manifolds with ξas space-like vector field satisfying $R(\xi, X).S = 0$, is expanding. Again, if we assume vector field ξ as time-like vector field in (ϵ)-para Sasakian manifolds then, in view of equation (43), we obtain $\lambda < 0$, which shows that λ is shrinking.*

Thus, we can state as follows.

Theorem 4.2. *Ricci Solitons on (ϵ)-para Sasakian manifolds admitting ξ as time like vector field satisfying $R(\xi, X).S = 0$, is shrinking.*

5. Ricci Solitons on (ϵ)-para Sasakian Manifolds Satisfying (4)

Let $R(\xi, X).P = 0$, then we have

$$R(\xi, X)P(Y, Z)U - P(R(\xi, X)Y, Z)U - P(Y, R(\xi, X)Z)U - P(Y, Z)R(\xi, X)U = 0. \tag{44}$$

By virtue of equation (20) above equation reduces to

$$\begin{aligned}&[\eta(P(Y, Z)U)X - \epsilon g(X, P(Y, Z)U)\xi + \epsilon g(X, Y)P(\xi, Z)U - \epsilon\eta(Y)P(X, Z)U\\&+ \epsilon g(X, Z)P(Y, \xi)U) - \epsilon\eta(Z)P(Y, X)U + \epsilon g(X, U)P(Y, Z)\xi - \epsilon\eta(U)P(Y, Z)X] = 0.\end{aligned} \tag{45}$$

Taking the inner product of above equation with ξ and using equation (8) and (9), we get

$$\begin{aligned}&[\epsilon\eta(X)\eta(P(Y, Z)U) - \epsilon g(X, P(Y, Z)U) + g(X, Y)\eta(P(\xi, Z)U) - \eta(Y)\eta(P(X, Z)U)\\&+ g(X, Z)\eta(P(Y, \xi)U) - \eta(Z)\eta(P(Y, X)U) + g(X, U)\eta(P(Y, Z)\xi) - \eta(U)\eta(P(Y, Z)X)] = 0.\end{aligned} \tag{46}$$

Using equation (33) in above equation, we obtain

$$\begin{aligned}\epsilon g(X, P(Y, Z)U) = &\left[\frac{\lambda}{n-1} - \epsilon\right][(1-\epsilon)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y)\eta(U)\} + \{g(X, U)\eta(Y) - g(Y, U)\eta(X)\}\eta(Z)\\&+ \{g(X, Y)g(Z, W) - g(Y, U)g(X, Z)\} - \frac{\epsilon}{n-1}[\epsilon\{g(X, U)\eta(Y)\eta(\varphi Z) - g(X, U)\eta(\varphi Y)\eta(Z)\}\\&+ \{g(X, Y)g(\varphi Z, U) - g(X, Z)g(\varphi Y, U)\} + \{g(X, \varphi Y)\eta(Z) - g(\varphi Z, X)\}\eta(U)].\end{aligned} \tag{47}$$

In view of equation (30), above equation reduces to

$$\begin{aligned}\epsilon g(X, R(Y, Z)U) - \frac{\epsilon}{n-1}[S(Z, U)g(X, Y) - S(Y, U)g(X, Z)] = &\left[\frac{\lambda}{n-1} - \epsilon\right][(1-\epsilon)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y)\eta(U)\\&+ \{g(X, U)\eta(Y) - g(Y, U)\eta(X)\}\eta(Z) + \{g(X, Y)g(Z, W) - g(Y, U)g(X, Z)\}\\&- \frac{\epsilon}{n-1}[\epsilon\{g(X, U)\eta(Y)\eta(\varphi Z) - g(X, U)\eta(\varphi Y)\eta(Z)\}\\&+ \{g(X, Y)g(\varphi Z, U) - g(X, Z)g(\varphi Y, U)\} + \{g(X, \varphi Y)\eta(Z) - g(\varphi Z, X)\}\eta(U)].\end{aligned} \tag{48}$$

Now, putting $X = U = \xi$, in above equation and by use of equations (8), (9) and (10), we obtain

$$\epsilon S(Y, Z) = 4n\left(\frac{\lambda}{n-1} - \epsilon\right)(1-\epsilon)\eta(Y)\eta(Z) \tag{49}$$

In view of equation (26) above equation takes the form

$$g(\varphi Y, Z) - \lambda\epsilon g(Y, Z) = 4n\left[\frac{\lambda}{n-1} - \epsilon\right](1-\epsilon)\eta(Y)\eta(Z). \tag{50}$$

Now, putting $Y = Z = \xi$ in above equation and by use of equations (8), (9) and (10), we obtain

$$\lambda = \frac{(4n - 4n\epsilon)(n-1)}{(5n - 4n\epsilon - 1)} \tag{51}$$

Now, suppose ξ is space-like vector field in (ϵ)-para Sasakian manifolds, then from equation (51), we obtain $\lambda > 0$, which shows that λ is expanding. Thus, we can state as follows.

Theorem 5.1. *Ricci Solitons on (ϵ)-para Sasakian manifolds with ξ as space-like vector field satisfying $R(\xi, X).P = 0$, is expanding. Again, if we assume vector field ξ as time-like vector field in (ϵ)-para Sasakian manifolds then, in view of equation (51), we obtain $\lambda < 0$, which shows that λ is shrinking.*

Thus, we can state as follows.

Theorem 5.2. *Ricci Solitons on (ϵ)-para Sasakian manifolds admitting ξ as time-like vector field satisfying $R(\xi, X).P = 0$, is shrinking.*

6. Ricci Solitons on (ϵ)-para Sasakian Manifolds Satisfying (5)

Let $P(\xi, X).R = 0$, then we have

$$P(\xi, X)R(Y, Z)W - R(P(\xi, X)Y, Z)W - R(Y, P(\xi, X)Z)W - P(Y, Z)R(\xi, X)W = 0. \tag{52}$$

By virtue of equation (31) above equation reduces to

$$\begin{aligned}
&[\eta(R(Y, Z)U)X - \epsilon g(X, R(Y, Z)U)\xi + \epsilon g(X, Y)R(\xi, Z)U - \epsilon\eta(Y)R(X, Z)U \\
&\quad + \epsilon g(X, Z)R(Y, \xi)U) - \epsilon\eta(Z)R(Y, X)U] + \frac{\lambda}{n-1}[\epsilon\eta(R(Y, Z)U)\xi \\
&\quad - g(X, R(Y, Z)U)\xi - \epsilon\eta(Y)R(\xi, Z)U + g(X, Y)R(\xi, Z)U \\
&\quad + \epsilon g(X, Z)R(Y, \xi)U - \epsilon\eta(Z)R(Y, X)U + \epsilon g(X, Y)R(Y, Z)\xi \\
&\quad - \epsilon\eta(U)R(Y, Z)\xi] - \frac{\epsilon}{n-1}[g(\varphi X, R(Y, Z)U)\xi - g(\varphi X, Y)R(\xi, Z)U \\
&\quad - g(\varphi X, Z)R(Y, \xi)U - g(\varphi X, U)R(Y, Z)\xi] = 0.
\end{aligned} \tag{53}$$

Taking the inner product of above equation with ξ and using equation (8) and (9), we get

$$\begin{aligned}
&[\epsilon\eta(X)\eta(R(Y, Z)U) - \epsilon g(X, R(Y, Z)U) + g(X, Y)\eta(R(\xi, Z)U) - \eta(Y)\eta(R(X, Z)U) \\
&\quad + g(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, X)U)] + \frac{\lambda}{n-1}[\eta(R(Y, Z)U) \\
&\quad - g(X, R(Y, Z)U) - \eta(Y)\eta(R(\xi, Z)U) + g(X, Y)\eta(R(\xi, Z)U) \\
&\quad + g(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, X)U) + g(X, Y)\eta(R(Y, Z)\xi) \\
&\quad - \eta(U)\eta(R(Y, Z)\xi)] - \frac{\epsilon}{n-1}[g(\varphi X, R(Y, Z)U) - g(\varphi X, Y)\eta(R(\xi, Z)U)
\end{aligned}$$

$$- g(\varphi X, Z)\eta(R(Y,\xi)U) - g(\varphi X, U)\eta(R(Y,Z)\xi)] = 0. \tag{54}$$

Using equations (13) and (20) in above equation, we obtain

$$\left[\frac{\lambda}{n-1} - 1\right][g(Y,U) - \eta(U)\eta(Y)] = 0. \tag{55}$$

Since $g(Y,U) \neq \eta(U)\eta(Y)$ therefore we get $\lambda > 0$, which shows that λ is expanding. Thus, we can state as follows.

Theorem 6.1. *Ricci Solitons on (ϵ)-para Sasakian manifolds with ξas space-like vector field satisfying $P(\xi, X).R = 0$, is expanding.*

7. Ricci Solitons on (ϵ)-para Sasakian Manifolds Satisfying (6)

Define [1]

$$\begin{aligned}(S(X,\xi).P)(Y,Z)U &= ((X \wedge_S \xi).P)(Y,Z)U \\ &= (X \wedge_s \xi)P(Y,Z)U + P((X \wedge_s \xi)Y,Z)U + P(Y,(X \wedge_s \xi)Z)U + P(Y,Z)(X \wedge_s \xi)U,\end{aligned} \tag{56}$$

where the endomorphism $(X \wedge_S Y)$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{57}$$

Now, from equations (56) and (57), we have

$$\begin{aligned}(S(X,\xi).P)(Y,Z)U &= S(\xi, P(Y,Z)U)X - S(X,P(Y,Z)U)\xi + S(\xi,Y)P(X,Z)U) - S(X,Y)P(\xi,Z)U \\ &+ S(\xi,Z)P(Y,X)U - S(X,Z)P(Y,\xi)U + S(\xi,U)P(Y,Z)X - S(X,U)P(Y,Z)\xi\end{aligned} \tag{58}$$

Assuming $(S(X,\xi).P)(Y,Z)U = 0$, then above equation reduces to

$$\begin{aligned}&S(\xi, P(Y,Z)U)X - S(X,P(Y,Z)U)\xi + S(\xi,Y)P(X,Z)U - S(X,Y)P(\xi,Z)U \\ &\quad + S(\xi,Z)P(Y,X)U - S(X,Z)P(Y,\xi)U + S(\xi,U)P(Y,Z)X - S(X,U)P(Y,Z)\xi = 0.\end{aligned} \tag{59}$$

Taking the inner product of above equation with ξ and using equation (9), (13), we get

$$\begin{aligned}&\epsilon\eta(X)S(\xi,P(Y,Z)U) - S(X,P(Y,Z)U) + \epsilon S(\xi,Y)\eta(P(X,Z)U) - \epsilon S(X,Y)\eta(P(\xi,Z)U) \\ &\quad + \epsilon S(\xi,Z)\eta(P(Y,X)U) - \epsilon S(X,Z)\eta(P(Y,\xi)U) + \epsilon S(\xi,U)\eta(P(Y,Z)X) \\ &\quad - \epsilon S(X,U)\eta(P(Y,Z)\xi) = 0.\end{aligned} \tag{60}$$

In virtue of above equations (32), (26) and (27), we get

$$\begin{aligned}S(\xi,P(Y,Z)U) = &\left[\lambda - \frac{\epsilon}{n-1}\right][-2\lambda\{g(Z,U)\eta(Y) - g(Y,U)\eta(Z)\} + \{S(\xi,Y)\eta(Z) - S(Z,\xi)\eta(Y)\}\eta(U) - \lambda g(Z,\xi)\eta(Y)\eta(U) \\ &+ \epsilon S(\xi,Z)g(Y,U) - S(\xi,Y)g(Z,U)] - \frac{\epsilon}{n-1}[-2g(\varphi Z,U)\eta(Y) + 2\lambda g(\varphi Y,W)\eta(Z) \\ &+ \epsilon S(\xi,Y)g(\varphi\xi,U)\eta(Z) - \lambda g(\varphi\xi,U)\eta(Y)\eta(Z) - \epsilon S(\xi,Z)g(\varphi\xi,U)\eta(Y) + \epsilon S(\xi,Z)g(\varphi Y,U)\end{aligned}$$

$$- \lambda g(\varphi Z, \xi)\eta(Y)\eta(U) + \lambda g(\varphi Y, \xi)\eta(Z)\eta(U) - \epsilon S(\xi, U)\eta(\varphi\xi)\eta(Y) + S(\xi, U)\eta(\varphi Y)\eta(Z)]. \tag{61}$$

Putting $Y = \xi$ in above equation and using equation (11), we get

$$\begin{aligned} -\epsilon\eta(P(\xi, Z)U) = \left[\lambda - \frac{\epsilon}{n-1}\right] &[-2\lambda\{g(Z, U) - \eta(U)\eta(Z)\} \\ &+ \lambda\epsilon\eta(Z) - \lambda\epsilon\eta(Z)\eta(U) + \lambda\epsilon\eta(Z)\eta(U) - \lambda\epsilon\eta(Z)\eta(U) + \lambda\epsilon\eta(Z)\eta(U)] \\ &- \frac{\epsilon}{n-1}[-2\lambda g(\varphi Z, U) + \lambda g(\varphi Z, U) - \lambda\eta(\varphi Z)\eta(U)] \end{aligned} \tag{62}$$

Using equation (31) in above equation, we get

$$\begin{aligned} \left[\lambda - \frac{\epsilon}{n-1}\right] &[-2\lambda g(Z, U) + 2\lambda\epsilon\eta(U)\eta(Z) + \lambda\epsilon\eta(Z) + \lambda\epsilon g(Z, U) - \lambda\eta(Z)\eta(U)] \\ &- \frac{\epsilon}{n-1}[-\lambda g(\varphi Z, U) + \lambda\epsilon g(\varphi Z, U) - \lambda\eta(\varphi Z)\eta(U) - \lambda\epsilon g(\varphi\xi, U)\eta(Z)] = 0. \end{aligned} \tag{63}$$

Putting $Z = U = \xi$ in above equation and using equations (8), (9) and (11), we get

$$\lambda\epsilon(\lambda - n\epsilon) = 0, \tag{64}$$

which gives

$$\text{either} \quad \lambda = 0 \quad \text{or} \quad \lambda = n\epsilon \tag{65}$$

Now, suppose ξ is space-like vector field in (ϵ)-para Sasakian manifolds, then from equation (65), we obtain $\lambda = 0$ or $\lambda > 0$, which shows that λ is either steady or expanding. Thus, we can state as follows.

Theorem 7.1. *Ricci Solitons on (ϵ)-para Sasakian manifolds with ξ as space -like vector field satisfying $S(\xi, X).P = 0$, is either steady or expanding. Again, if we assume vector field ξ as time-like vector field in (ϵ)-para Sasakian manifolds then, in view of equation (65), we obtain $\lambda = 0$ or $\lambda < 0$, which shows that λ is either steady or shrinking.*

Thus, we can state as follows

Theorem 7.2. *Ricci Solitons on (ϵ)-para Sasakian manifolds admitting ξas time like vector field satisfying $S(\xi, X).P = 0$, is either steady or shrinking.*

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