

A Mathematical Model for Unemployment Control-An Analysis with and without Delay

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Abstract: The paper presented a mathematical model to control unemployment using dynamic system. In this process we consider the situation where job search is open for native unemployed and new migrant workers enter into the territory as unemployed. So, government of that territory feels more burden of unemployment and make efforts for creating new vacancies with the help of private sector. Also, we assume that native unemployed and new migrant workers both make efforts for self-employment. We studied the local stability of the non-negative equilibrium point with and without delay. Numerical simulation has been carried out to illustrate the analytical result.

Keywords: Employed persons, unemployed persons, self-employment, migration, delay, dynamic variables.

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1. Introduction

Nowadays, world is facing so many burning problems like poverty, climate change, pollution, inequality, corruption etc. Unemployment is one of the serious problem among them. Unemployment is an undemonstrative issue for whole world because it effects directly or indirectly to other problems like poverty, inequality etc. Many people have to leave their own territory and go to the other territory just to find the job. So, the government of that other territory has to faces more burden of unemployment. In [6] Nikolopoulos and Tzanetis described a model for a housing allocation of homeless families due to natural disaster. Based on some concept of this paper, in [1, 2] Misra and Singh developed a nonlinear mathematical model for unemployment. In [2] the model considered three dynamic variables number of unemployed persons, employed persons and newly created vacancies by government intervention. Using concept of this paper G.N.Pathan and P.H.Bhathawala [7] developed a mathematical model for unemployment with effect of self-employment. M. Neamtu [4] presented a model for unemployment using some concept of [2] with adding two new variables number of present jobs in the market and number of immigrants. In [11, 12] G.N.Pathan and P.H.Bhathawala developed a mathematical model by adding a variable in [7] says present jobs in the market [11] and number of migrant workers [12] and analyzed the result with and without time delay. Using concept of above models we developed a new model to control unemployment with five dynamic variables: (i) Number of unemployed persons $U(t)$, (ii) Number of new migrant workers become part of labour workforce at territory denoted by $M(t)$, (iii) Number of Employed persons $E(t)$, (iv) Number of present jobs in the market $P(t)$ and (v) Number of newly created vacancies by government and private sector $V(t)$ and analyzed the result with delay and without delay. We assumed

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that available vacancies are equally open for native unemployed and new migrant workers. So, new migrant attracts to the territory and government of that territory feels more burden of native unemployed workers and migrant workers. Therefore, government of that territory tried to create new vacancies with the help of private sector. We assume the situation that at some level native unemployed and migrant workers both try to become self-employed to survive.

The paper is organized as follows: Model for unemployment describes in Section 2.1, Section 2.2 describes an equilibrium analysis, Section 2.3 describes the stability of equilibrium point, Numerical simulation describes in Section 2.4 and Conclusion is given in Section 2.5.

2. Main Result

2.1. Mathematical Model

To develop a model we consider that all entrants of the category unemployment are fully qualified to do any job at any time t . Number of unemployed persons $U(t)$, increases with constant rate a_1 . Present jobs in the market provided by government and private sector is denoted by $P(t)$ which is proportional to $U(t)$ with rate c_1 and depreciation rate in present jobs is c_2 . $V(t)$ is the new vacancies created by the efforts of government and private sector. Number of employed persons is denoted by $E(t)$. So, total available vacancies in the market are $(P(t) + V(t) - E(t))$. The rate of movement from unemployed class to employed class is jointly proportional to $U(t)$ and number of available vacancies in the market $(P(t) + V(t) - E(t))$. We assumed that job search is open for native unemployed as well as new migrant. So, new migrant attracts to the territory and become part of the labor workforce of the territory denoted by $M(t)$. Number of migrant increases with constant rate m_1 . The rate of movement of migrant workers in employment class is jointly proportional to $M(t)$ and $(P(t) + V(t) - E(t))$. Native unemployed and migrant both make efforts for self-employment to survive which is proportional to number of unemployed and migrant with the rate a_5 and a_7 respectively. Employed persons fired from their job or leave the job and joint unemployed class with rate a_4 . The death and retirement rate of employed person is a_8 . The death rate of unemployed and migrant are a_3 and m_3 respectively. The rate of migrant who registered as unemployed or fired from their job is a_6 .

$$\frac{dU}{dt} = a_1 - a_2U(P + V - E) - a_3U + a_4E - a_5U + a_6M \quad (1)$$

$$\frac{dM}{dt} = m_1 - m_2M(P + V - E) - a_6M - m_3M - a_7M \quad (2)$$

$$\frac{dE}{dt} = a_2U(P + V - E) + m_2M(P + V - E) - a_4E + a_5U + a_7M - a_8E \quad (3)$$

$$\frac{dP}{dt} = c_1U - c_2P \quad (4)$$

$$\frac{dV}{dt} = \alpha U(t - \tau) + \beta M(t - \tau) - \delta V \quad (5)$$

Here, α and β are the rate of newly created vacancies by government and Private sector proportional to $U(t)$ and $M(t)$ respectively. δ is the diminution rate of newly created vacancies. τ describes the delay in creating new vacancies.

Lemma 2.1. *The set $\Omega = \left\{ (U, M, E, P, V) : 0 \leq U + M + E \leq \frac{a_1+m_1}{\gamma}, 0 \leq P \leq \frac{c_1(a_1+m_1)}{c_2\gamma}, 0 \leq V \leq \frac{(\alpha+\beta)(a_1+m_1)}{\gamma\delta} \right\}$, where $\gamma = \min(a_3, m_3, a_8)$ is a region of attraction for the system (1)-(5) and it attracts all solutions initiating in the interior of the positive octant.*

Proof. From equation (1)-(3) we get,

$$\frac{d}{dt}(U(t) + M(t) + E(t)) = a_1 + m_1 - a_3U(t) - m_3M(t) - a_8E(t)$$

Which gives

$$\frac{d}{dt}(U(t) + M(t) + E(t)) \leq a_1 + m_1 - \gamma(U(t) + M(t) + E(t))$$

Where $\gamma = \min(a_3, m_3, a_8)$. By taking limit supremum

$$\limsup_{t \rightarrow \infty} (U(t) + M(t) + E(t)) \leq \frac{a_1 + m_1}{\gamma}$$

From equation (4) we get

$$\frac{dP}{dt} = c_1 U(t) - c_2 P(t)$$

Which gives

$$\frac{dP}{dt} \leq \frac{c_1(a_1 + m_1)}{\gamma} - c_2 P(t)$$

By taking limit supremum

$$\limsup_{t \rightarrow \infty} P(t) \leq \frac{c_1(a_1 + m_1)}{c_2 \gamma}$$

from (5) we have

$$\begin{aligned} \frac{dV}{dt} &= \alpha U(t) + \beta M(t) - \delta V(t) \\ \frac{dV}{dt} &\leq (\alpha + \beta)U(t) - \delta V(t) \end{aligned}$$

By taking limit supremum which leads to,

$$\limsup_{t \rightarrow \infty} V(t) \leq \frac{(\alpha + \beta)(a_1 + m_1)}{\delta \gamma}$$

This proves the lemma. □

2.2. Equilibrium Analysis

The model system (1)-(5) has only one non negative equilibrium point $E_0(U^*, M^*, E^*, P^*, V^*)$ which obtained by solving the following set of algebraic equations.

$$a_1 - a_2 U(P + V - E) - a_3 U + a_4 E - a_5 U + a_6 M = 0 \quad (6)$$

$$m_1 - m_2 M(P + V - E) - a_6 M - m_3 M - a_7 M = 0 \quad (7)$$

$$a_2 U(P + V - E) + m_2 M(P + V - E) - a_4 E + a_5 U + a_7 M - a_8 E = 0 \quad (8)$$

$$c_1 U - c_2 P = 0 \quad (9)$$

$$\alpha U + \beta M - \delta V = 0 \quad (10)$$

Taking addition of equation (6), (7) and (8)

$$a_1 + m_1 - a_3 U - m_3 M - a_8 E = 0$$

Therefore

$$E = \frac{a_1 + m_1 - a_3 U - m_3 M}{a_8} \quad (11)$$

From equation (9) and (10) we have

$$P = \frac{c_1 U}{c_2} \quad (12)$$

$$V = \frac{\alpha U + \beta M}{\delta} \quad (13)$$

Therefore

$$P + V - E = \frac{a_8 a U + b a_8 M - (a_1 + m_1)}{a_8} \quad (14)$$

Where $a = \frac{c_1}{c_2} + \frac{\alpha}{\delta} + \frac{a_3}{a_8}$, $b = \frac{\beta}{\delta} + \frac{m_3}{a_8}$. Put values of equations (11) and (14) in (6) and (7) we get,

$$A_0 U^2 + A_1 U M - A_2 U - A_3 M - A_4 = 0 \quad (15)$$

$$B_0 M^2 + B_1 U M - B_2 M - B_3 = 0 \quad (16)$$

Where,

$$\begin{aligned} A_0 &= a a_2 a_8, & B_0 &= b a_8 m_2, \\ A_1 &= a_2 a_8 b, & B_1 &= a a_8 m_2, \\ A_2 &= [a_2(a_1 + m_1) - a_3(a_4 + a_8) - a_5 a_8], & B_2 &= [m_2(a_1 + m_1) - a_8(a_6 + m_3 + a_7)], \\ A_3 &= [a_6 a_8 - m_3 a_4], & B_3 &= m_1 a_8. \\ A_4 &= [a_1 a_8 + a_4(a_1 + m_1)], \end{aligned}$$

Equation (15) and (16) represent the equation of hyperbolas. From equation (15)

$$M = \frac{A_4 + A_2 U - A_0 U^2}{A_1 U - A_3} \quad (17)$$

put value of equation (17) in (16) we get,

$$H_0 U^3 + H_1 U^2 - H_2 U - H_3 = 0 \quad (18)$$

Where

$$\begin{aligned} H_0 &= A_2 A_0 B_0 - A_0(A_3 B_1 + A_1 B_2), \\ H_1 &= A_0 B_0 A_4 - B_0 A_2^2 + A_3(A_2 B_1 + A_0 B_2) + A_1 A_2 B_2 + B_3 A_1^2, \\ H_2 &= 2 A_2 A_4 B_0 - A_4(A_3 B_1 + A_1 B_2) + A_2 A_3 B_2 + 2 A_1 A_3 B_3, \\ H_3 &= B_0 A_4^2 + A_3 A_4 B_2 - A_3^2 B_3. \end{aligned}$$

Since all H_i , $i = 0, 1, 2, 3$ are positive and number of changes in signs of equation (18) is only one. By Descart's rule equation (18) has only one positive solution say U^* . So, we get the non-negative equilibrium point of model with coordinates:

$$M^* = \frac{A_4 + A_2 U^* - A_0 (U^*)^2}{A_1 U^* - A_3}, \quad E^* = \frac{a_1 + m_1 - a_3 U^* - m_3 M^*}{a_8}, \quad P^* = \frac{c_1 U^*}{c_2}, \quad V^* = \frac{\alpha U^* + \beta M^*}{\delta}$$

So, $E_0(U^*, M^*, E^*, P^*, V^*)$ is required non negative solution of the Model.

2.3. Stability Analysis

Stability of equilibrium point without any delay: To check the local stability for $\tau = 0$ i.e. in absence of delay at equilibrium point $E_0(U^*, M^*, E^*, P^*, V^*)$ we calculate the variational matrix T of the model system (1)-(5) corresponding to $E_0(U^*, M^*, E^*, P^*, V^*)$.

$$T = \begin{bmatrix} r_{11} & a_6 & r_{13} & -p_3 & -p_3 \\ 0 & r_{22} & p_4 & -p_4 & -p_4 \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \\ c_1 & 0 & 0 & -c_2 & 0 \\ \alpha & \beta & 0 & 0 & -\delta \end{bmatrix}$$

Where

$$\begin{aligned} p_1 &= a_2(P + V - E), & p_2 &= m_2(P + V - E), & p_3 &= a_2U, & p_4 &= m_2M, \\ r_{11} &= -p_1 - a_3 - a_5, & r_{31} &= p_1 + a_5, & r_{22} &= -p_2 - a_6 - m_3 - a_7, & r_{32} &= p_2 + a_7, \\ r_{13} &= p_3 + a_4, & r_{33} &= -p_3 - p_4 - a_4 - a_8, & r_{34} &= p_3 + p_4, & r_{35} &= p_3 + p_4 \end{aligned}$$

The characteristic equation of above matrix is

$$\lambda^5 + d_1\lambda^4 + d_2\lambda^3 + d_3\lambda^2 + d_4\lambda + d_5 = 0 \quad (19)$$

Where

$$\begin{aligned} d_1 &= \delta + c_2 - r_{11} - r_{22} - r_{33}, \\ d_2 &= p_4(\beta + a_6 + m_3 + \delta) + p_3\alpha + a_8\delta + r_{13}(a_3 + \delta) - r_{11}(a_8 + p_4 + \delta) - r_{22}(r_{13} - r_{11} + a_8 + \delta) \\ &\quad + c_2(\delta - r_{11} - r_{22} - r_{33}) + c_1p_3, \\ d_3 &= (\alpha + c_1)(p_4(a_6 - a_4) + p_3a_8) - r_{22}\{p_3(\alpha + c_1) + a_3r_{13} - a_8r_{11} + c_2\delta + (r_{13} - r_{11} + a_8)(\delta + c_2)\} \\ &\quad - r_{11}\{p_4(\beta + m_3) + (a_8 + p_4)(\delta + c_2) + c_2\delta\} + p_4\{\delta(a_6 + m_3) + \beta(a_4 + a_8) + a_6m_3 \\ &\quad + c_2(\beta + a_6 + m_3 + \delta)\} + \delta\{a_3r_{13} + c_2a_8 + c_1p_3 + c_2r_{13}\} + c_2(\alpha p_3 + r_{13}a_3) \\ d_4 &= p_4(a_6a_8 - m_3a_4)(\alpha + c_1) - r_{22}\{\alpha p_3(c_2 + a_8) + (c_2 + \delta)(a_3r_{13} - a_8r_{11}) + c_2\delta(r_{13} - r_{11} + a_8) \\ &\quad + c_1p_3(a_8 + \delta)\} - r_{11}\{p_4m_3(\delta + c_2) + c_2(\beta p_4 + \delta(a_8 + p_4))\} + \{p_4(a_6 - a_4) + p_3a_8\}(\alpha c_2 + c_1\delta) \\ &\quad + \beta p_4(a_4 + a_8)(p_3 + c_2) + p_4m_3a_6(\delta + c_2) + c_2\delta\{p_4(a_6 + m_3) + a_3r_{13}\} + \beta p_4a_8r_{31}, \\ d_5 &= p_4(a_6a_8 - m_3a_4)(\alpha c_2 + c_1\delta) - r_{22}\{p_3a_8(\alpha c_2 + c_1) + c_2\delta(a_3r_{13} - a_8r_{11})\} \\ &\quad + c_2p_4\{\delta m_3(a_6 - r_{11}) + \beta(a_8r_{31} + p_3(a_4 + a_8))\} \end{aligned}$$

Since, d_1, d_2, d_3, d_4, d_5 are positive then all coefficients of equation (19) are positive and some algebraic manipulation convey that $d_1d_2 > d_3$ and $d_1d_2d_3 > d_3^2d_4$ and $(d_1d_2 - d_3)(d_3d_4 - d_2d_5) > (d_1d_4 - d_5)^2$. So, by Routh Hurwitz criteria all roots of equation (19) are negative or having a negative real part. Therefore equilibrium point $E_0 = (U^*, M^*, E^*, P^*, V^*)$ is locally asymptotically stable.

Stability of equilibrium point with delay: To check the local stability for $\tau \neq 0$ (in presence of delay) at equilibrium point $E_0 = (U^*, M^*, E^*, P^*, V^*)$ we calculate the variational matrix T_1 and T_2 of the model system (1)-(5) corresponding to $E_0 = (U^*, M^*, E^*, P^*, V^*)$.

$$\frac{dx}{dt} = T_1x(t) + T_2x(t - \tau) \quad (20)$$

Where $x(t) = [u(t) \ m(t) \ e(t) \ p(t) \ v(t)]^T$, $u(t)$, $m(t)$, $e(t)$, $p(t)$ and $v(t)$ are small perturbations around the equilibrium point E_0 .

$$T_1 = \begin{bmatrix} r_{11} & a_6 & r_{13} & -p_3 & -p_3 \\ 0 & r_{22} & p_4 & -p_4 & -p_4 \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \\ c_1 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 & -\delta \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 \end{bmatrix}$$

The characteristic equation of system (20) is

$$\psi^5 + j_1\psi^4 + j_2\psi^3 + j_3\psi^2 + j_4\psi + j_5 + (k_1\psi^3 + k_2\psi^2 + k_3\psi + k_4)e^{-\psi\tau} = 0 \quad (21)$$

Where

$$\begin{aligned} j_1 &= \delta + c_2 - r_{11} - r_{22} - r_{33}, \\ j_2 &= p_4(a_6 + m_3 + \delta) + a_8\delta + r_{13}(a_3 + \delta) - r_{11}(a_8 + p_4 + \delta) - r_{22}(r_{13} - r_{11} + a_8 + \delta) \\ &\quad + c_2(\delta - r_{11} - r_{22} - r_{33}) + c_1p_3, \\ j_3 &= c_1\{p_4(a_6 - a_4) + p_3a_8\} - r_{22}\{a_3r_{13} - a_8r_{11} + (r_{13} - r_{11} + a_8)(\delta + c_2) + c_2\delta + c_1p_3\} \\ &\quad - r_{11}\{p_4m_3 + (a_8 + p_4)(c_2 + \delta) + c_2\delta\} + (c_2 + \delta)\{p_4(a_6 + m_3) + a_3r_{13}\} + p_4a_6m_3 \\ &\quad + c_1p_3\delta + c_2\delta\{p_4 + a_8 + r_{13}\}, \\ j_4 &= c_1p_4(a_6a_8 - m_3a_4) - r_{22}\{(c_2 + \delta)(a_3r_{13} - a_8r_{11}) + c_2\delta(r_{13} - r_{11} + a_8) + c_1p_3(a_8 + \delta)\} \\ &\quad - r_{11}\{p_4m_3(\delta + c_2) + c_2\delta(a_8 + p_4)\} + c_1\delta\{p_4(a_6 - a_4) + p_3a_8\} + p_4m_3a_6(\delta + c_2) \\ &\quad + c_2\delta\{p_4(a_6 + m_3) + a_3r_{13}\}, \\ j_5 &= c_1\delta p_4(a_6a_8 - m_3a_4) - r_{22}\{c_1p_3a_8 + c_2\delta(a_3r_{13} - a_8r_{11})\} + c_2p_4\delta m_3(a_6 - r_{11}) \\ k_1 &= p_4\beta + p_3\alpha, k_2 = \alpha\{p_4(a_6 - a_4) + p_3(a_8 - r_{22} + c_2)\} + p_4\beta\{a_4 + a_8 + c_2 - r_{11}\}, \\ k_3 &= \alpha p_4(a_6a_8 - m_3a_4) - r_{22}\alpha p_3(c_2 + a_8) - r_{11}c_2\beta p_4 + \alpha c_2\{p_4(a_6 - a_4) + p_3a_8\} \\ &\quad + \beta p_4(a_4 + a_8)(p_3 + c_2) + \beta p_4a_8r_{31}, \\ k_4 &= \alpha c_2p_4(a_6a_8 - m_3a_4) - r_{22}\alpha c_2p_3a_8 + \beta c_2p_4\{a_8r_{31} + p_3(a_4 + a_8)\}. \end{aligned}$$

Now to check the stability of Equation (21) we should not directly use Routh-Hurwitz criterion. We should check that Hopf-bifurcation occurs and for that we have to show that Equation (21) has a pair of purely imaginary roots. For this we substitute $\psi = i\omega$ in Equation (21) and we get

$$i\omega^5 + j_1\omega^4 - i j_2\omega^3 - j_3\omega^2 + i j_4\omega + j_5 + (-ik_1\omega^3 - k_2\omega^2 + ik_3\omega + k_4)e^{-i\omega\tau} = 0 \quad (22)$$

$$\begin{aligned}
i\omega^5 + j_1\omega^4 - ij_2\omega^3 - j_3\omega^2 + ij_4\omega + j_5 + (-ik_1\omega^3 - k_2\omega^2 + ik_3\omega + k_4)(\cos\omega\tau - i\sin\omega\tau) = 0 \\
j_1\omega^4 - j_3\omega^2 + j_5 - (k_2\omega^2 - k_4)\cos\omega\tau - (k_1\omega^3 - k_3\omega)\sin\omega\tau + i(\omega^5 - j_2\omega^3 \\
+ j_4\omega - (k_1\omega^3 - k_3\omega)\cos\omega\tau + (k_2\omega^2 - k_4)\sin\omega\tau) = 0
\end{aligned} \tag{23}$$

Separating real and imaginary part of Equation (23) we get

$$j_1\omega^4 - j_3\omega^2 + j_5 = (k_2\omega^2 - k_4)\cos\omega\tau + (k_1\omega^3 - k_3\omega)\sin\omega\tau \tag{24}$$

$$\omega^5 - j_2\omega^3 + j_4\omega = (k_1\omega^3 - k_3\omega)\cos\omega\tau - (k_2\omega^2 - k_4)\sin\omega\tau \tag{25}$$

By squaring and adding Equation (24) and Equation (25)

$$(j_1\omega^4 - j_3\omega^2 + j_5)^2 + (\omega^5 - j_2\omega^3 + j_4\omega)^2 = (k_2\omega^2 - k_4)^2 + (k_1\omega^3 - k_3\omega)^2 \tag{26}$$

By taking expansion of this

$$\omega^{10} + e_1\omega^8 + e_2\omega^6 + e_3\omega^4 + e_4\omega^2 + e_5 = 0, \tag{27}$$

where

$$\begin{aligned}
e_1 &= (j_1^2 - 2j_2), & e_2 &= (j_2^2 + 2j_4 - 2j_1j_3 - k_1^2), \\
e_3 &= (j_3^2 + 2j_1j_5 - 2j_2j_4 + 2k_1k_3 - k_2^2), & e_4 &= (j_4^2 - k_3^2 - 2j_3j_5 + 2k_2k_4), \\
e_5 &= (j_5^2 - k_4^2).
\end{aligned}$$

Substituting $\omega^2 = \sigma$ in Equation (27) then we have

$$f(\sigma) = \sigma^5 + e_1\sigma^4 + e_2\sigma^3 + e_3\sigma^2 + e_4\sigma + e_5 = 0 \tag{28}$$

If all $e_i > 0$ ($i = 1, 2, 3, 4, 5$) and satisfies the Routh-Hurwitz criterion then there is no positive root of Equation (28) i.e. Equation (21) has not pair of purely imaginary root. So, by Routh-Hurwitz criterion equilibrium E_0 is asymptotically stable for all delay $\tau > 0$. Contrary assume that e_i in Equation (28) does not satisfy the Routh-Hurwitz criterion then there is at least one positive root ω_0 exists of Equation (28) for $e_5 < 0$. From this we get that $(j_5^2 - k_4^2) < 0$. Since $j_5 + k_4 > 0$ so, $j_5 - k_4 < 0$. Which gives the condition for the existence of pair of purely imaginary roots $(\pm i\omega_0)$ of Equation (21) is $j_5 - k_4 < 0$ i.e.

$$\begin{aligned}
p_4(a_6a_8 - m_3a_4)(c_1\delta - \alpha c_2) - r_{22}\{p_3a_8(c_1 - \alpha c_2) + c_2\delta(a_3r_{13} - a_8r_{11})\} \\
+ c_2p_4\{\delta m_3(a_6 - r_{11}) - \beta(a_8r_{31} + p_3(a_4 + a_8))\} < 0
\end{aligned} \tag{29}$$

From Equation (24) and Equation (25) we get

$$\tan\omega\tau = \frac{(k_1\omega^3 - k_3\omega)(j_1\omega^4 - j_3\omega^2 + j_5) - (k_2\omega^2 - k_4)(\omega^5 - j_2\omega^3 + j_4\omega)}{(k_1\omega^3 - k_3\omega)(\omega^5 - j_2\omega^3 + j_4\omega) + (k_2\omega^2 - k_4)(j_1\omega^4 - j_3\omega^2 + j_5)}$$

For positive ω_0 we have corresponding τ_0 is given by

$$\tau_n = \frac{n\pi}{\omega_0} + \frac{1}{\omega_0} \tan^{-1} \left(\frac{(k_1\omega^3 - k_3\omega)(j_1\omega^4 - j_3\omega^2 + j_5) - (k_2\omega^2 - k_4)(\omega^5 - j_2\omega^3 + j_4\omega)}{(k_1\omega^3 - k_3\omega)(\omega^5 - j_2\omega^3 + j_4\omega) + (k_2\omega^2 - k_4)(j_1\omega^4 - j_3\omega^2 + j_5)} \right); \quad n = 0, 1, 2, 3, \dots \tag{30}$$

By Butler's lemma we can say that equilibrium E_0 remains stable for $\tau < \tau_0$. Now to check that Hopf- bifurcation occurs as τ increases through τ_0 . For this we have to check that τ_0 satisfies the transversality condition.

Lemma 2.2. *Transversality condition is*

$$\text{sgn} \left[\frac{d(\text{Re}(\psi))}{d\tau} \right]_{\tau=\tau_0} > 0$$

Proof. By differentiating Equation (21) with respect to τ , we have

$$\left(\frac{d\psi}{d\tau} \right)^{-1} = \frac{5\psi^4 + 4j_1\psi^3 + 3j_2\psi^2 + 2j_3\psi + j_4 + (3k_1\psi^2 + 2k_2\psi + k_3)e^{-\psi\tau}}{\psi(k_1\psi^3 + k_2\psi^2 + k_3\psi + k_4)e^{-\psi\tau}} - \frac{\tau}{\psi}$$

Now,

$$\begin{aligned} \text{sgn} \left[\frac{d(\text{Re}(\psi))}{d\tau} \right]_{\tau=\tau_0} &= \text{sgn} \left[\frac{d(\text{Re}(\psi))}{d\tau} \right]_{\tau=\tau_0}^{-1} \\ &= \text{sgn} \left[\text{Re} \left(\frac{d\psi}{d\tau} \right)^{-1} \right]_{\psi=i\omega_0} \\ &= \text{sgn} \left[\frac{5\omega_0^8 + 4e_1\omega_0^6 + 3e_2\omega_0^4 + 2e_3\omega_0^2 + e_4}{(k_1\omega_0^3 - k_3\omega_0)^2 + (k_2\omega_0^2 - k_4)^2} \right] \\ &= \text{sgn} \left[\frac{h'(\omega_0^2)}{(k_1\omega_0^3 - k_3\omega_0)^2 + (k_2\omega_0^2 - k_4)^2} \right] \end{aligned}$$

Here, $h'(\omega_0^2) > 0$ if the condition (29) is satisfied. This shows that if condition (29) satisfies then equilibrium E_0 of the model system (1)-(5) is asymptotically stable for $\tau < \tau_0$ (i.e. $\tau \in [0, \tau_0)$) and unstable for $\tau > \tau_0$. The condition of Hopf-bifurcation is satisfied so, periodic solution occurs when τ passes the τ_0 for equilibrium point E_0 . \square

2.4. Numerical Simulation

For the Numerical simulation using MATLAB 7.6.0 we consider the following data, $a_1 = 5000$, $a_2 = 0.00004$, $a_3 = 0.04$, $a_4 = 0.004$, $a_5 = 0.03$, $a_6 = 0.1$, $a_7 = 0.01$, $a_8 = 0.07$, $m_1 = 3000$, $m_2 = 0.00002$, $m_3 = 0.05$, $\alpha = 0.2$, $\beta = 0.001$, $\delta = 0.08$, $c_1 = 0.08$, $c_2 = 0.3$. The equilibrium values of the model are: $U^* = 32387$, $M^* = 13164$, $E^* = 86376$, $P^* = 8637$, $V^* = 81132$. The eigenvalues of the variational matrix corresponding to the equilibrium point $E_0 = (U^*, M^*, E^*, P^*, V^*)$ of model system (1)-(5) are -1.5228 , $-0.3923 + 0.088i$, $-0.3923 - 0.088i$, $-0.0694 + 0.0965i$ and $-0.0694 - 0.0965i$. All eigenvalues are negative or having negative real part. So, equilibrium $E_0 = (U^*, M^*, E^*, P^*, V^*)$ is locally asymptotically stable. Using above parameter in equation (27) and (30) we get $\omega_0 = 0.16$ and $\tau_0 = 11$.

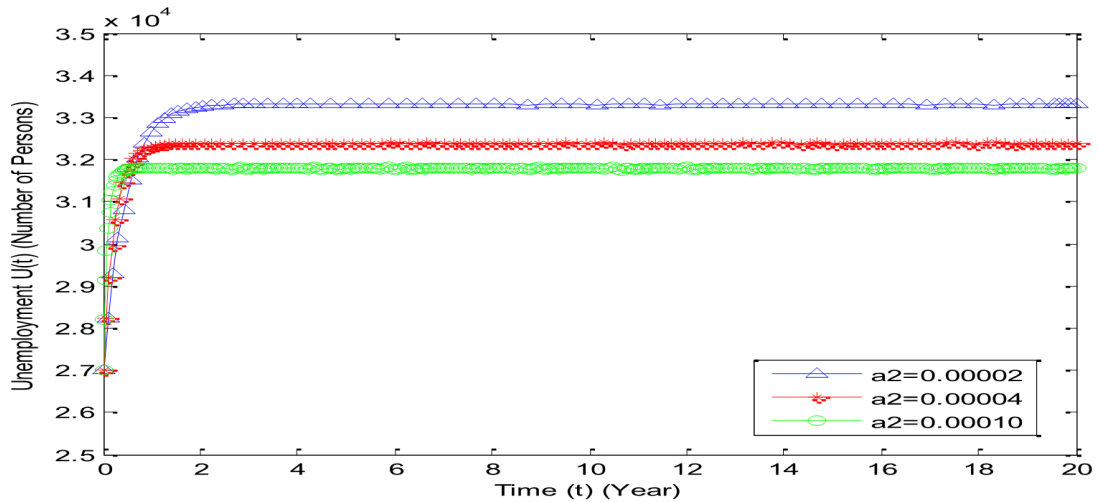


Figure 1.

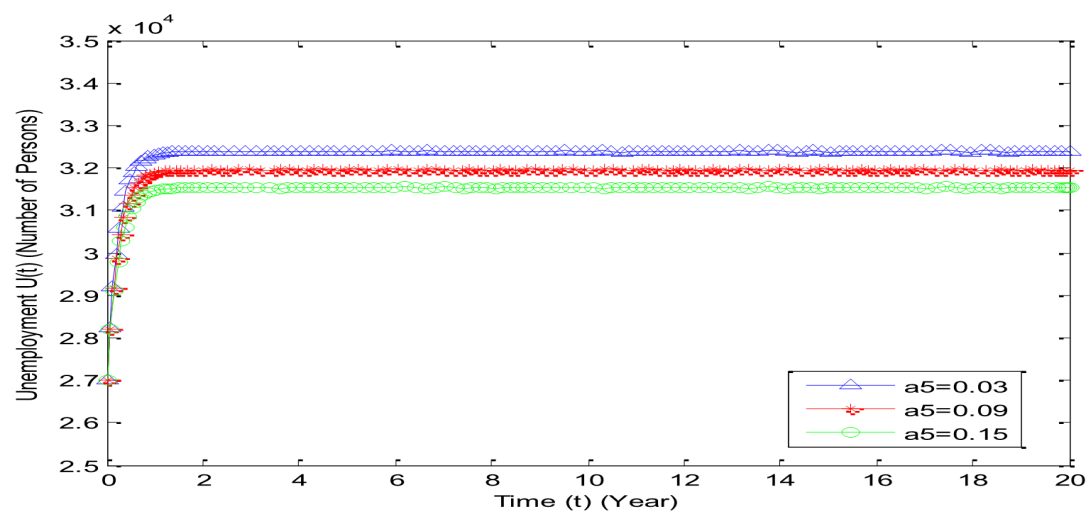


Figure 2.

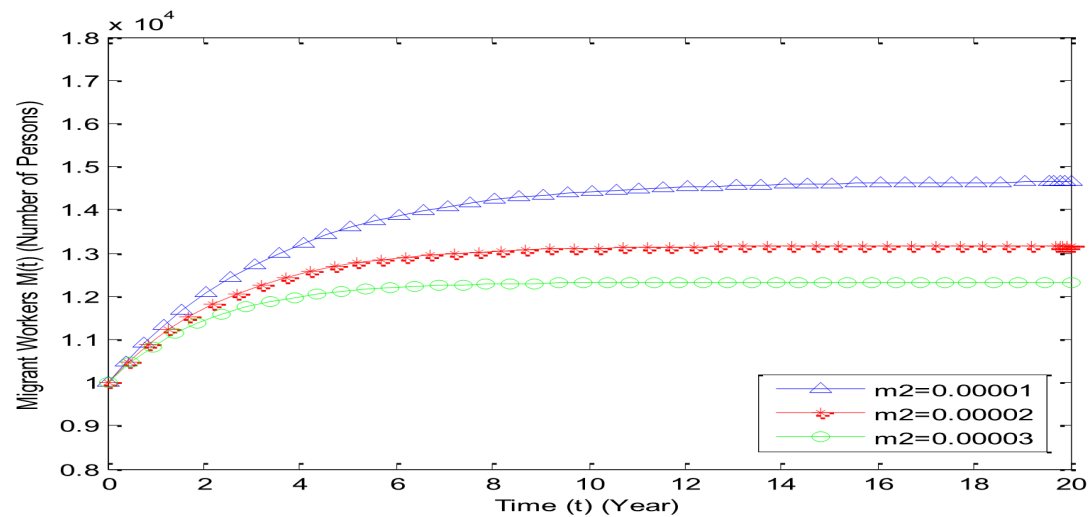


Figure 3.

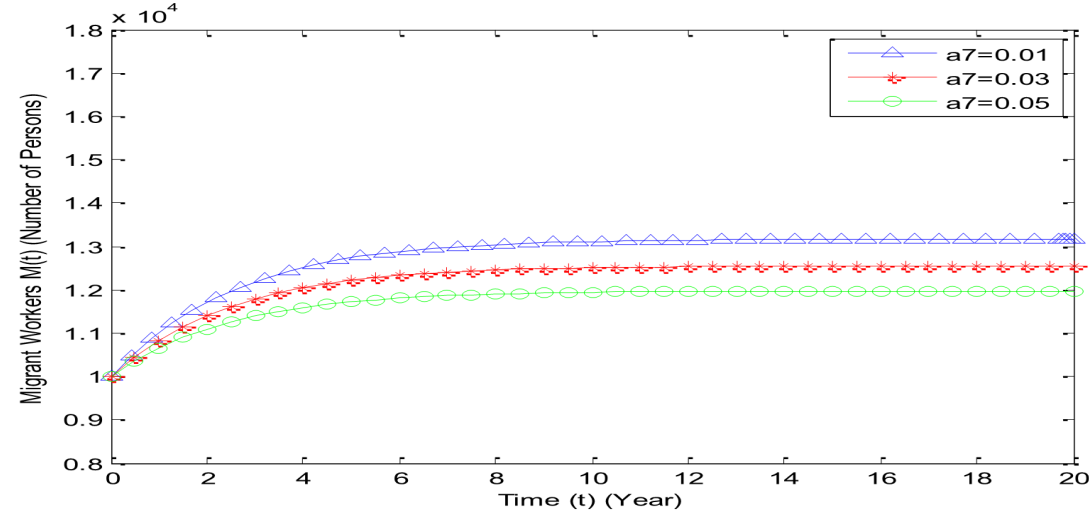


Figure 4.

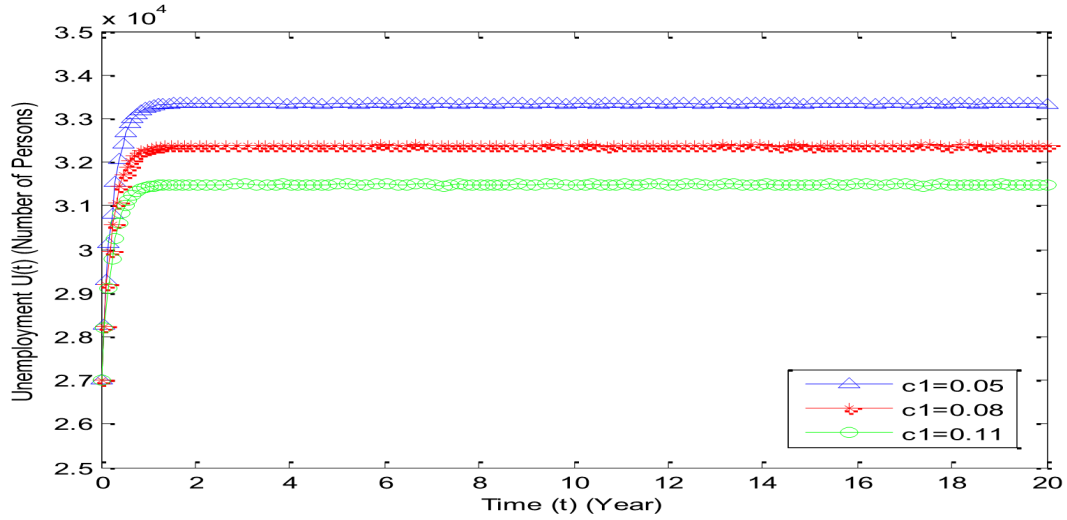


Figure 5.

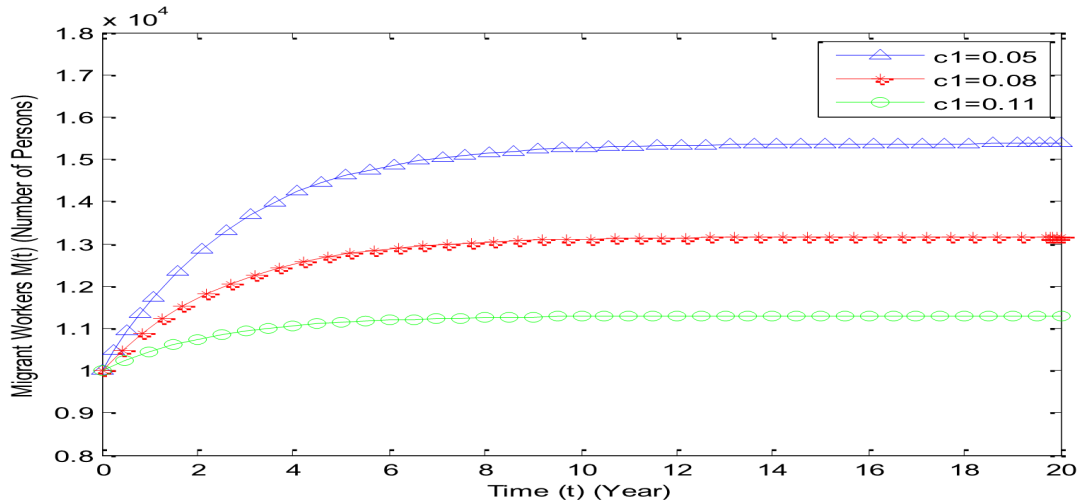


Figure 6.

3. Conclusion

In this paper, the dynamical system presented a nonlinear mathematical model to control unemployment with five variables. Theoretical calculation is compared with numerical simulation using MATLAB 7.6.0. Figure 1 indicates that if rate of a_2 goes higher then unemployment rate goes lower. That is if maximum unemployed persons joined employment class then unemployment rate goes lower. Similarly, Fig.3 shows that for higher value of m_2 the rate of unemployment of migrant worker become lower. From Figure 2 and Figure 4 we observe that for higher value of a_5 and a_7 the rate of unemployment for native unemployed as well as migrant workers become lower. Which is show that if rate of self-employment varies then rate of unemployment of native unemployed and migrant workers varies according to it. Figure 5 and Figure 6 indicates that variation of c_1 effects to rate of unemployment. If c_1 posses bigger value then rate of unemployment decreases. That is variation in present jobs inversely related to unemployment.

From the analysis of the model it can be seen that if maximum persons joined employment class then rate of unemployment goes lower. Also, it can be observe that variation in present jobs effects to the unemployment rate of native unemployed and also migrant workers. Therefore If number of present jobs is good then unemployment can control at some level. We can

see that attempt of government and private sector for creating new vacancies is very important. Because Equilibrium point is stable without any condition in absence of delay. But in presence of delay if the condition given by Equation (29) satisfies then equilibrium is stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. Where τ_0 is critical value which can be find by the Equation (30). From Figure 2 and Figure 3 it can be observe that Self-employment is very important tool to control unemployment. By making the efforts for self-employment native unemployed and migrant worker both can create chances for employment. By attempt of self-employment unemployment can control at some level.

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