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Some Fixed Point Theorems for a Pair of Mappings Under the Rational Inequality in Complex Valued b-Metric Space

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Abstract: In this problem we improve a common fixed point theorem for a pair of mappings under the rational inequality and obtain sufficient conditions for existence and uniqueness in complex valued b-metric spaces.

Keywords: Common fixed point, Complex valued b-metric space, Cauchy sequence, Coincidence point, Weakly compatible mappings.

1. Introduction

Fixed point theory is one of the fundamental theories in nonlinear analysis, which has various applications in different branches of mathematics. The famous Banach contraction principle states that if (X, d) is a complete metric space and $A : X \to X$ is contraction mapping (i.e. $d(Ax, Ay) \leq kd(x, y)$ for all $x, y \in X$, where k is a non-negative real number such that $k < 1$), then A has a unique fixed point. Many researchers have carried out fixed point results for b-metric spaces. The concept of b-metric space as a generalization of metric spaces, this has been introduced by Bakhtin [2] in 1989. The most utilization of metric spaces in the nature enlargement of functional analysis is enormously. Several researchers discussed various generalization of this notion like rectangular metric spaces, semi-metric spaces, quasi-metric spaces, semi-quasi metric spaces, pseudo metric spaces, probabilistic metric spaces, 2-metric spaces, D-metric spaces, G-metric spaces, K-metric spaces, cone metric spaces etc. In 1998, Czerwik [5] introduced the concept of b-metric space. On the other hand Azam et al. [1] proved and established the idea of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed point of a pair of mappings satisfying contractive type conditions. Rao et al. [8] and Mukheimer [6] discussed the notion of complex valued b-metric spaces, which were more generalized then the theorem proved by Azam et al. [1], Rouzkard & Imdad [11], Nashine et al. [7], and Singh et al. [12].

In this paper, we improved some common fixed point theorems involving two pairs of weakly compatible mappings satisfying certain rational expression with contractive type of condition in complex valued b-metric space.

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2. Preliminaries

Let C be the set of complex numbers and let $z_1, z_2 \in C$. Define a partial order $\preceq$ on C as follows:

$z_1 \preceq z_2$ iff $Re(z_1) \preceq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. If follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

(1). $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2). $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3). $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(4). $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$,

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Definition 2.1 ([1]). *Let X be a non-empty set. Suppose that the mapping $d : X \times X \to C$ is called a complex valued metric on X if the following conditions are holds:*

(a). $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(b). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c). $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$;

Then d is called complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 2.2 ([8]). *Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to C$ satisfies the following conditions:*

(a). $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$;

(b). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c). $d(x, y) \preceq s\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called complex valued b-metric on X and (X, d) is called a complex valued b-metric space.

Example 2.3. *Let $X = [0, 1]$. Define a mapping $d : X \times X \to C$ such that*

$$d(x, y) = |x - y|^2 \sqrt{2} e^{i\frac{\pi}{4}}, \ \forall\ x, y \in X.$$

Then (X, d) is a complex valued b-metric space with $s = 2$.

Lemma 2.4 ([8]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.5 ([8]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$ where $m \in N$.*

Definition 2.6. *Let u and v be two self-maps defined on a set X. If $w = ux = vx$ for some x in X, then x is called a coincidence point of u and v, and w is called a point of coincidence of u and v.*

Definition 2.7. *Let u and v be two self-maps defined on a set X. Then u and v are said to be weakly compatible if they commute at coincidence points.*

3. Main Results

In this section, now we have proved common fixed point theorem for contraction conditions described by rational expressions. Which are generalization of Azam et al. [1], Bhatt et al. [4], and Singh et al. [12].

Theorem 3.1. *Let P, Q, R and S be four self-mappings of a complete complex valued b-metric spaces (X, d) satisfying*

(1). $P(X) \subseteq S(X)$ *and* $Q(X) \subseteq R(X)$.

(2). $d(Px, Qy) \preceq \alpha d(Rx, Sy) + \beta \frac{d(Rx,Px)d(Qy,Sy)}{1+d(Rx,Sy)} + \gamma \frac{d(Rx,Qy)d(Px,Sy)}{1+d(Rx,Sy)} + \eta \frac{[d(Px,Rx)d(Px,Sy)+d(Qy,Sy)d(Qy,Rx)]}{d(Px,Sy)+d(Qy,Rx)}$, $x, y \in X$, *where* $\alpha, \beta, \gamma, \eta$ *are non-negative real numbers such that* $s(\alpha + \gamma + \eta) + \beta < 1$.

(3). Pairs (P, R) *and* (Q, S) *are weakly compatible and* $Q(x)$ *is a closed subspace of* X.

Then P, Q, R *and* S *have a unique common fixed point.*

Proof. Consider a sequence $\{y_n\}$ in X such that

$$y_{2n} = Px_{2n} = Sx_{2n+1}$$

$$y_{2n+1} = Qx_{2n+1} = Rx_{2n+2}$$

Where $\{x_n\}$ is another sequence in X. First we have to show that $\{y_n\}$ is a Cauchy sequence in X, now consider

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Px_{2n}, Qx_{2n+1}) \\ &\preceq \alpha d(Rx_{2n}, Sx_{2n+1}) + \beta d\frac{(Rx_{2n}, Px_{2n})d(Qx_{2n+1}, Sx_{2n+1})}{1 + d(Rx_{2n}, Sx_{2n+1})} + \gamma \frac{d(Rx_{2n}, Qx_{2n+1})d(Px_{2n}, Sx_{2n+1})}{1 + d(Rx_{2n}, Sx_{2n+1})} \\ &+ \eta \frac{[d(Px_{2n}, Rx_{2n})d(Px_{2n}, Sx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n+1})d(Qx_{2n+1}, Rx_{2n})]}{d(Px_{2n}, Sx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n})} \\ &= \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} + \gamma \frac{d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \\ &+ \eta \frac{[d(y_{2n}, y_{2n-1})d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n-1})]}{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})} \\ &= \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} + \eta d(y_{2n+1}, y_{2n}) \\ |d(y_{2n+1}, y_{2n})| &\leq \alpha |d(y_{2n-1}, y_{2n})| + \beta |d(y_{2n+1}, y_{2n})| \left| \frac{d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \right| + \eta |d(y_{2n+1}, y_{2n})| \end{aligned}$$

since $|d(y_{2n-1}, y_{2n})| < |1 + d(y_{2n-1}, y_{2n})|$, we have

$$|d(y_{2n+1}, y_{2n})| \leq \alpha |d(y_{2n-1}, y_{2n})| + \beta |d(y_{2n+1}, y_{2n})| + \eta |d(y_{2n+1}, y_{2n})|$$

or $|d(y_{2n}, y_{2n+1})| \leq \frac{\alpha}{1-(\beta+\eta)} |d(y_{2n-1}, y_{2n})|$ we obtain

$$|d(y_{2n+1}, y_{2n+2})| \leq \delta |d(y_{2n}, y_{2n+1})|, \tag{1}$$

where $\delta = \frac{\alpha}{1-(\beta+\eta)}$. Now we have

$$|d(y_{2n}, y_{2n+1})| \leq \delta |d(y_{2n-1}, y_{2n})| \leq \delta^2 |d(y_{2n-2}, y_{2n-1})| \leq \ldots \leq \delta^{2n} |d(y_0, y_1)| \tag{2}$$

Finally, we can conclude that $|d(y_n, y_{n+1})| \preceq \delta^n |d(y_0, y_1)|$. For all $m > n$, $m, n \in N$ and since $s\delta = \frac{s\alpha}{1-(\beta+\eta)} < 1$, we get

$$|d(y_n, y_m)| \leq s|d(y_n, y_{n+1})| + s^2|d(y_{n+1}, y_{n+2})| + ... + s^{m-n}|d(y_{m-1}, y_m)|$$

By using equation (2), we get

$$|d(y_n, y_m)| \leq s\delta^n |d(y_0, y_1)| + s^2\delta^{n+1} |d(y_0, y_1)| + ... + s^{m-n}\delta^{m-1} |d(y_0, y_1)|$$

This implies that

$$|d(y_n, y_m)| \leq (s\delta^n + s^2\delta^{n+1} + ... + s^{m-n}\delta^{m-1})\, |d(y_0, y_1)|$$

This implies that

$$\begin{aligned}
|d(y_n, y_m)| &\leq (s^n\delta^n + s^{n+1}\delta^{n+1} + ... + s^{m-1}\delta^{m-1})\, |d(y_0, y_1)| \\
&= \sum_{u=n}^{m-1} s^u\delta^u\, |d(y_0, y_1)| \\
&\leq \sum_{u=n}^{\infty} (s\delta)^u\, |d(y_0, y_1)| \\
&= \frac{(s\delta)^n}{(1-s\delta)}\, |d(y_0, y_1)|
\end{aligned}$$

Hence $|d(y_n, y_m)| = \frac{(s\delta)^n}{(1-s\delta)} |d(y_0, y_1)| \to 0$ as $m, n \to \infty$, since $s\delta < 1$. Thus, $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete, therefore exists point $z \in X$. Such that $Px_{2n} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Rx_{2n+2} = z$. Because $Q(X)$ is closed sub space of X and so $z \in Q(X)$. So $Q(X) \subseteq R(X)$, then their exist a point $u \in X$, such that $z = Ru$. Now we have to show that $Pu = Ru = z$ by condition (2) of Theorem 3.1, we have $d(Pu, z) \leq s[d(Pu, Qx_{2n+1}) + d(Qx_{2n+1}, z)]$. This implies that

$$\begin{aligned}
\frac{1}{s}d(Pu, z) \leq \alpha d(Ru, Sx_{2n+1}) + \beta\frac{d(Ru, Pu)d(Qx_{2n+1}, Sx_{2n+1})}{1 + d(Ru, Sx_{2n+1})} + \gamma\frac{d(Ru, Qx_{2n+1})d(Pu, Sx_{2n+1})}{1 + d(Ru, Sx_{2n+1})} \\
+ \eta[\frac{d(Pu, Ru)d(Pu, Sx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n+1})d(Qx_{2n+1}Ru)}{d(Pu, Sx_{2n+1}) + d(Qx_{2n+1}, Ru)}] + d(Qx_{_{2n+1}}, z)
\end{aligned}$$

Letting $n \to \infty$, we have

$$\frac{1}{s}d(Pu, z) \leq \alpha(0) + \beta(0) + \gamma(0) + \eta\, d(Pu, z)$$
$$\frac{1}{s}d(Pu, z) - \eta d(Pu, z) = 0 \quad \text{or} \quad \Rightarrow |d(Pu, z)| = 0 \quad \text{or} \quad Pu = z.$$

Thus $Pu = Ru = z$. u is a coincidence point of (P, R). Since $P(X) \subseteq S(X)$ and now $z \in P(X)$, then their exist a point $v \in X$ such that $z = Sv$. At this time we have to show that $Qv = z$. By condition (2) of the Theorem 3.1, and by $Pu = Ru = Sv = z$, we have $d(Pu, Qv) = d(z, Qv) \preceq s[d(z, Px_{2n}) + d(Px_{2n}, Qv)]$, we have

$$\begin{aligned}
\frac{1}{s}d(z, Qv) \preceq d(z, Px_{2n}) + \alpha d(Rx_{2n}, Sv) + \beta\frac{d(Rx_{2n}, Px_{2n})d(Qv, Sv)}{1 + d(Rx_{2n}, Sv)} + \gamma\frac{d(Rx_{2n}, Qv)d(Px_{2n}, Sv)}{1 + d(Rx_{2n}, Sv)} \\
+ \eta\frac{[d(Px_{2n}, Rx_{2n})d(Px_{2n}, Sv) + d(Qv, Sv)d(Qv, Rx_{2n})}{d(Px_{2n}, Sv) + d(Qv, Rx_{2n})} \\
\frac{1}{s}d(z, Qv) \preceq d(z, y_{2n}) + \alpha d(y_{2n-1}, Sv) + \beta\frac{d(y_{2n-1}, y_{2n})d(Qv, Sv)}{1 + d(y_{2n-1}, Sv)} + \gamma\frac{d(y_{2n-1}, Qv)d(y_{2n}, Sv)}{1 + d(y_{2n-1}, Sv)} \\
+ \eta\frac{[d(y_{2n}, y_{2n-1})d(y_{2n}, Sv) + d(Qv, Sv)d(Qv, y_{2n-1})}{d(y_{2n}, Sv) + d(Qv, y_{2n-1})}
\end{aligned}$$

Taking $n \to \infty$, we have $|d(z, Qv)| = 0 \Rightarrow Qv = z$. Hence $Qv = Sv = z \Rightarrow v$ is a coincidence point of (Q, S). Now we have $Pu = Ru = Sv = Qv = z$. Because P and R are weakly compatible mapping then $PRu = RPu = Pz = Rz$. Now we have to show that z is a fixed point of P, i.e. $Pz = z$. If $Pz \neq z$ then the condition (2) of Theorem 3.1.

$$\begin{aligned}
&d(Pz, z) \preceq s[d(Pz, Qx_{2n+1}) + d(Qx_{2n+1}, z)] \\
\Rightarrow \frac{1}{s}&d(Pz, z) \preceq \alpha d(Rz, Sx_{2n+1}) + \beta \frac{d(Rz, Pz)d(Qx_{2n+1}, Sx_{2n+1})}{1 + d(Rz, Sx_{2n+1})} + \gamma \frac{d(Rz, Qx_{2n+1})d(Pz, Sx_{2n+1})}{1 + d(Rz, Sx_{2n+1})} \\
&\quad + \eta \left[\frac{d(Pz, Rz)d(Pz, Sx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n+1})(Qx_{2n+1}, Rz)}{d(Pz, Sx_{2n+1}) + d(Qx_{2n+1}, Rz)} \right] + d(Qx_{2n+1}, z) \\
&\quad \preceq \alpha d(Rz, y_{2n}) + \beta \frac{d(Rz, Pz)d(y_{2n+1}, y_{2n})}{1 + d(Rz, y_{2n})} + \gamma \frac{d(Rz, y_{2n+1})d(Pz, y_{2n})}{1 + d(Rz, y_{2n})} \\
&\quad + \eta \left[\frac{d(Pz, Rz)d(Pz, y_{2n}) + d(y_{2n+1}, y_{2n})(y_{2n+1}, Rz)}{d(Pz, y_{2n}) + d(y_{2n+1}, Rz)} \right] + d(y_{2n+1}, z)
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
\frac{1}{s}d(Pz, z) &\preceq \alpha d(Pz, z) + \beta \frac{d(Rz, Pz)d(z, z)}{1 + d(Rz, z)} + \gamma \frac{d(Rz, z)d(Pz, z)}{1 + d(Rz, z)} \\
&\quad + \eta \left[\frac{d(Pz, Rz)d(Pz, z) + d(z, z)d(z, Rz)}{d(Pz, z) + d(z, Rz)} \right] + d(z, z) \\
&\quad \preceq \alpha d(Pz, z) + \gamma \frac{d(Rz, z)d(Pz, z)}{1 + d(Rz, z)} + \eta \left[\frac{d(Pz, z) + d(z, Rz)}{1 + d(Rz, z)} \right]
\end{aligned}$$

or

$$\frac{1}{s}|d(Pz, z)| \leq \alpha |d(Pz, z)| + \gamma \left| \frac{d(Rz, z)}{1 + d(Rz, z)} \right| |d(Pz, z)| + \eta \left| \frac{d(Pz, Rz)}{d(Pz, z) + d(z, Rz)} \right| |d(Pz, z)|$$

$\frac{1}{s}|d(Pz, z)| \leq (\alpha+\gamma+\eta)|d(Pz, z)| \Rightarrow |d(Pz, z)| \leq s(\alpha+\gamma+\eta)|d(Pz, z)|$ since $s(\alpha+\gamma+\eta)+\beta < 1 \Rightarrow |d(Pz, z)| = 0 \Rightarrow Pz = z$. Therefore $Pz = Rz = z$. Since Q and S are weakly compatible mapping then, $QSv = SQv = Qv = Sv = z$, we have $Qz = Sz$.

$$\begin{aligned}
d(Qz, z) &\preceq s[d(Qz, Px_{2n}) + d(Px_{2n}, z)] \\
\frac{1}{s}d(Qz, z) &\preceq d(Px_{2n}, Qz) + d(Px_{2n}, z) \\
&\preceq \alpha d(Rx_{2n}, Sz) + \beta \frac{d(Rx_{2n}, Px_{2n})d(Qz, Sz)}{1 + d(Rx_{2n}, Sz)} + \gamma \frac{d(Rx_{2n}, Qz)d(Px_{2n}, Sz)}{1 + d(Rx_{2n}, Sz)} \\
&\quad + \eta \left[\frac{d(Px_{2n}, Rx_{2n})d(Px_{2n}, Sz) + d(Qz, Sz)d(Qz, Rx_{2n})}{d(Px_{2n}, Sz) + d(Qz, Rx_{2n})} \right] + d(x_{2n}, z) \\
&\preceq \alpha d(y_{2n-1}, Sz) + \beta \frac{d(y_{2n-1}, y_{2n})d(Qz, Sz)}{1 + d(y_{2n-1}, Sz)} + \gamma \frac{d(y_{2n-1}, Qz)d(y_{2n}, Sz)}{1 + d(y_{2n-1}, Sz)} \\
&\quad + \eta \left[\frac{d(y_{2n}, y_{2n-1})d(y_{2n}, Sz) + d(Qz, Sz)d(Qz, y_{2n-1})}{d(y_{2n}, Sz) + d(Qz, y_{2n-1})} \right] + d(y_{2n}, z)
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
\frac{1}{s}|d(Qz, z)| &\leq \alpha |d(z, Sz)| + \beta \left| \frac{d(z, z)d(Qz, Sz)}{1 + d(z, Sz)} \right| + \gamma \left| \frac{d(z, Qz)d(z, Sz)}{1 + d(z, Sz)} \right| \\
&\quad + \eta \left| [\frac{d(z, z)d(z, Sz) + d(Qz, Sz)d(Qz, z)}{d(z, Sz) + d(Qz, z)}] \right| + |d(z, z)| \\
\frac{1}{s}|d(Qz, z)| &\leq \alpha |d(z, Sz)| + \gamma \left| \frac{d(z, Sz)}{1 + d(z, Sz)} \right| |d(z, Sz)| + \eta \left| \frac{d(Qz, Sz)}{d(z, Sz) + d(Qz, z)} \right| |d(Qz, z)| \\
\frac{1}{s}|d(Qz, z)| &\leq (\alpha + \gamma + \eta)|d(Qz, z)| \\
\Rightarrow |d(Qz, z)| &\leq s(\alpha + \gamma + \eta)|d(z, Qz)|
\end{aligned}$$

Since $s(\alpha+\gamma+\eta)+\beta<1 \Rightarrow |d(Qz,z)|=0 \Rightarrow Qz=z$. There fore $Sz=Qz=z$. Thus $Pz=Qz=Rz=Sz=z$. Consequently z is a common fixed point of P,Q,R and S.

Uniqueness: Let $t(\neq z)$ be another fixed point of P,Q,R and S, then $Pt=Qt=Rt=St=t$.

$$\begin{aligned} d(z,t) &= d(Pz,Qt) \\ &\preceq \alpha d(Rz,St)+\beta\frac{d(Rz,St)d(Qt,St)}{1+d(Rz,St)}+\gamma\frac{d(Rz,Qt)d(Pz,St)}{1+d(Rz,St)}+\eta\frac{d(Pz,Rz)d(Pz,St)+d(Qt,St)d(Qt,Rz)}{d(Pz,St)+d(Qt,Rz)} \\ |d(z,t)| &\leq \alpha\,|d(z,t)|+\gamma\left|\frac{d(z,t)}{1+d(z,t)}\right||d(z,t)| \\ |d(z,t)| &\leq (\alpha+\gamma)\,|d(z,t)| \end{aligned}$$

Since $s(\alpha+\gamma)+\beta<1$, $s\geq 1$ therefore $\alpha+\gamma<1$. $|d(z,t)|=0 \Rightarrow z=t$. Hence z is a unique common fixed point of P, Q, R and S. $\square$

Example 3.2. *Let $X=[0,1]$ be endowed with complex valued b-metric space and $d: X\times X\to C$ with $d(x,y)=|x-y|^2+i\,|x-y|^2$. Now to find s, we have*

$$\begin{aligned} d(x,y) &= |x-y|^2+i\,|x-y|^2 \\ &\preceq |(x-z)+(z-y)|^2+i\,|(x-z)+(z-y)|^2 \\ &\preceq \left[|x-z|^2+|z-y|^2+2\,|x-z|\,|z-y|\right]+i\left[|x-z|^2+|z-y|^2+2\,|x-z|\,.\,|z-y|\right] \\ &\preceq \left[|x-z|^2+|z-y|^2+|x-z|^2+|z-y|^2\right]+i\left[|x-z|^2+|z-y|^2+|x-z|^2+|z-y|^2\right] \\ &= 2\left\{\left[|x-z|^2+i\,|x-z|^2\right]+\left[|z-y|^2+i\,|z-y|^2\right]\right\} \end{aligned}$$

that is $d(x,y)\preceq 2[d(x,z)+d(z,y)]$ where $s=2$. Define P, Q, R and $S: X\to X$ by

$$Px=\left(\frac{x}{2}\right)^8,\ Qx=\left(\frac{x}{2}\right)^4,\ Rx=\left(\frac{x}{2}\right)^4,\ Sx=\left(\frac{x}{2}\right)^2,$$

It has to be seen that

$$0\leq d(Px,Qy),d(Rx,Sy),\frac{d(Rx,Px)d(Qy,Sy)}{1+d(Rx,Sy)},\frac{d(Rx,Qy)d(Px,Sy)}{1+d(Rx,Sy)},\frac{d(Px,Rx)d(Px,Sy)+d(Qy,Sy)(Qy,Rx)}{d(Px,Sy)+d(Qy,Rx)}$$

in all aspects. It is sufficient to show that $d(Px,Qy)\leq d(Rx,Sy)$, $\forall\ x,y\in[0,1]$ and $s(\alpha+\gamma+\eta)+\beta<1$, $\alpha,\beta,\gamma,\eta\geq 0$.

$$\begin{aligned} d(Px,Qy) &= \left[|Px-Qy|^2+i\,|Px-Qy|^2\right] \\ &= \left[\left|\left(\frac{x}{2}\right)^8-\left(\frac{y}{2}\right)^4\right|^2+i\left|\left(\frac{x}{2}\right)^8-\left(\frac{y}{2}\right)^4\right|^2\right] \\ &= \frac{1}{2^8}\left[\left|\frac{x^8}{2^4}-\frac{y^4}{1}\right|^2+i\left|\frac{x^8}{2^4}-y^4\right|^2\right] \\ &= \frac{1}{256}\left[\left|\left(\frac{x^2}{2}\right)^4-y^4\right|^2+i\left|\left(\frac{x^2}{2}\right)^4-y^4\right|^2\right] \qquad (3) \\ d(Rx,Sy) &= \left[|Rx-Sy|^2+i\,|Rx-Sy|^2\right] \\ &= \left[\left|\left(\frac{x}{2}\right)^4-\left(\frac{y}{2}\right)^2\right|^2+i\left|\left(\frac{x}{2}\right)^4-\left(\frac{y}{2}\right)^2\right|^2\right] \\ &= \frac{1}{2^4}\left[\left|\frac{x^4}{2^2}-y^2\right|^2+i\left|\frac{x^4}{2^2}-y^2\right|^2\right] \end{aligned}$$

$$= \frac{1}{16}\left[\left|\left(\frac{x^2}{2}\right)^2 - y^2\right|^2 + i\left|\left(\frac{x^2}{2}\right)^2 - y^2\right|^2\right] \tag{4}$$

From equations (3) and (4) we have $d(Px, Qy) \leq d(Rx, Sy)$, $\forall\ x, y \in [0, 1]$. Therefore each and every conditions of Theorem 3.1 are satisfied. Observe that the point $0 \in X$ remains fixed under mappings P, Q, R and S is indeed unique.

If we put $R = S = I$ (Identity mapping) in Theorem 3.1 we get the following corollary.

Corollary 3.3. *Let P and Q be two self-mappings of complete complex valued b-metric spaces (X, d) satisfying*

$$d(Px, Qy) \leq \alpha d(x, y) + \frac{\beta d(x, Px)d(y, Qy)}{1 + d(x, y)} + \gamma\frac{d(x, Qy)d(y, Px)}{1 + d(x, y)} + \eta\frac{d(Px, x)d(Px, y)d(Qy, y)d(Qy, x)}{d(Px, y) + d(Qy, x)}\ \forall\ x, y \in X$$

where α, β, γ and η are non-negative reals such that $s(\alpha + \gamma + \eta) + \beta < 1$. Then P and Q have a unique common fixed point.

Remark 3.4. *If we put $\gamma = \eta = 0$ in Corollary 3.3 then we get Theorem 3.1 of Mukheimer [6].*

Remark 3.5. *If we put $s = 1$ and $\gamma = \eta = 0$ in Corollary 3.3 then we get Theorem 3.1 of Azam et al., [1].*

Remark 3.6. *If we put $\eta = 0$ in Corollary 3.3 then we get Theorem 3.1 of Singh et al., [12].*

If we set $R = S$ in Theorem 3.1 then we get another corollary.

Corollary 3.7. *Let P, Q and R be three self-mappings of a complete complex valued b-metric spaces (X, d) satisfying.*

(1). $P(X) \subseteq R(X)$ and $Q(X) \subseteq R(X)$,

(2). $d(Px, Qy) \leq \alpha d(Rx, Ry) + \beta\frac{d(Rx, Px)d(BQy, Ry)}{1+d(Rx, Ry)} + \gamma\frac{d(Rx, Qy)d(Px, Ry)}{1+d(Rx, Ry)} + \eta\left[\frac{d(Px, Rx)d(Px, Ry)+d(Qy, Ry)d(Qy, Rx)}{d(Px, Ry)+d(Qy, Rx)}\right]\ \forall\ x, y \in X$ where α, β, γ and η are non-negative reals such that $s(\alpha + \gamma + \eta) + \beta < 1$.

(3). If pairs (P, R) and (Q, R) are weakly compatible and B(X) is a closed subspace of X.

Then P, Q and R have a unique common fixed point.

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