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On the Solution of General Family of Fractional Differential Equation Involving Hilfer Derivative Operator and $\overline{H}$-function

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Abstract: In this paper, we first give solution to a general family of fractional differential equation involving Hilfer derivative operator and the fractional integral operator whose kernel is the $\overline{H}$-function. Next, we record here solutions of two fractional differential equations involving the function associated with Gaussian Model free energy and Polylogarithm function of order g as special cases of our main result. These special cases are believed to be new. On account of the general nature of $\overline{H}$-function in our main findings, the results derived earlier by Srivastava et al. [15], Srivastava and Tomovski [16] and Tomovski et al. [17] follow as special cases.

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1. Introduction

Fractional differential equations involving known integral operators have been studied earlier by a large number of authors (see, for details, [15], [16] and [17]) and have diverse applications. Motivated by above mentioned work and the references cited therein, we make use of the following functions and fractional integral operators: The $\overline{H}$-function occurring in the present paper was introduced by Inayat Hussain [9] and studied by Bushman and Srivastava [1] and others, it is defined and represented in the following manner:

$$\overline{H}^{m,n}_{p,q}\left[z\left|\begin{array}{cc}(e_j,E_j;\in_j)_{1,n}, & (e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m}, & (f_j,F_j;\Im_j)_{m+1,q}\end{array}\right.\right]=\frac{1}{2\pi\omega}\int_{\mathfrak{L}}\overline{\Theta}(\xi)z^{\xi}d\xi \tag{1}$$

where, $\omega=\sqrt{-1}$, $z\in\mathbb{C}\setminus\{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi)=\frac{\prod\limits_{j=1}^{m}\Gamma(f_j-F_j\xi)\prod\limits_{j=1}^{n}\{\Gamma(1-e_j+E_j\xi)\}^{\in_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-f_j+F_j\xi)\}^{\Im_j}\prod\limits_{j=n+1}^{p}\Gamma(e_j-E_j\xi)} \tag{2}$$

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and

$$1 \leqq m \leqq q \qquad \text{and} \qquad 0 \leqq n \leqq p \qquad (m, q \in \mathbb{N} = \{1, 2, 3, \cdots\};\ n, p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{3}$$

The nature of contour $\mathfrak{L}$ in (1) and various conditions on its parameters can be seen in the paper by Gupta, Jain and Agarwal [4]. In this paper we make use of the Riemann-Liouville fractional integral operator I^{p}_{a+} and the Riemann-Liouville fractional derivative operator D^{p}_{a+}, which are defined by (see, for details, [10], [11] and [13]):

$$(I^{\mu}_{a+}f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt \qquad \big(\Re(\mu) > 0\big) \tag{4}$$

and

$$(D^{\mu}_{a+}f)(x) = \left(\frac{d}{dx}\right)^n (I^{n-\mu}_{a+}f)(x) \qquad \big(\Re(\mu) > 0;\ n = [\Re(\mu)] + 1\big), \tag{5}$$

where $[x]$ denotes the greatest integer in the real number x. Hilfer [7] generalized the operator in (5) and defined a general fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 \leqq \nu \leqq 1$ with respect to x as follows:

$$(D^{\mu,\nu}_{a+}f)(x) = \left(I^{\nu(1-\mu)}_{a+}\, \frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+}f\right)\right)(x). \tag{6}$$

Equation (6) yields the classical Riemann-Liouville fractional derivative operator D^{μ}_{a+} when $\nu = 0$ and for $\nu = 1$ it reduces to the fractional derivative operator introduced by Joseph Liouville (1809-1882) in 1832, which is called the Liouville-Caputo fractional derivative operator (see [3], [10] and [17]). Now, the Laplace transform $\mathcal{L}[f(x)](s)$ of the function $f(x)$ is defined as follows:

$$\mathcal{L}[f(x)](s) = \int_0^\infty e^{-sx}\, f(x)dx \qquad \big(\Re(s) > 0\big), \tag{7}$$

provided that the integral exists, we recall the following known result (see, for details, [16] and [17]):

$$\mathcal{L}[(D^{\mu,\nu}_{0+}f)(x)](s) = s^{\mu}\mathcal{L}[f(x)](s) - s^{-\nu(1-\mu)}\left(I^{(1-\nu)(1-\mu)}_{0+}f\right)(0+) \qquad \big(\Re(s) > 0;\ 0 < \mu < 1\big), \tag{8}$$

where the initial-value term:

$$\left(I^{(1-\nu)(1-\mu)}_{0+}f\right)(0+)$$

involves the Riemann-Liouville fractional integral (4) $(\textit{with } a = 0)$ of the function $f(t)$ of order

$$\mu \mapsto (1-\nu)(1-\mu) \tag{9}$$

evaluated in the limit as $x \to 0+$. The familiar Mittag-Leffler functions $E_{\mu}(z)$ and $E_{\mu,\nu}(z)$ are defined by the following series:

$$E_{\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)} =: E_{\mu,1}(Z) \qquad (z \in \mathbb{C};\ \Re(\mu) > 0) \tag{10}$$

and

$$E_{\mu,\nu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)} \qquad (z, \nu \in \mathbb{C};\ \Re(\mu) > 0) \tag{11}$$

respectively. By means of the series representation, a generalization of the Mittag-Leffler function $E_{\mu,\nu}(z)$ of (11) was introduced by Prabhakar [12] as follows:

$$E_{\mu,\nu}^{\lambda}(z) := \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\Gamma(\mu n+\nu)} \frac{z^n}{n!} \qquad (z,\nu,\lambda \in \mathbb{C};\ \Re(\mu)>0) \tag{12}$$

The following Laplace transform formula for the generalized Mittag-Leffler function $E_{\mu,\nu}^{\lambda}(z)$ was given by Prabhakar [12]:

$$\mathcal{L}[x^{\nu-1}E_{\mu,\nu}^{\lambda}(\omega x^{\mu})](s) = \frac{s^{\lambda\mu-\nu}}{(s^{\mu}-\omega)^{\lambda}} \tag{13}$$

$$(\mu,\omega,\lambda \in \mathbb{C}; \Re(\nu)>0; \Re(s)>0; \tfrac{\omega}{s^{\mu}}<1)$$

Prabhakar [12] also introduced the following fractional integral operator:

$$(\mathbf{E}_{\mu,\nu,\omega;a+}^{\lambda}\phi)(x) := \int_a^x (x-t)^{\nu-1}\, E_{\mu,\nu}^{\lambda}[w(x-t)^{\mu}]\phi(t)dt \qquad (x>a) \tag{14}$$

in the space $L(a,b)$ of Lebesgue integrable functions on a finite closed interval $[a,b]$ $(b>a)$ of the real line $\Re$ given by

$$L(a,b) = \{f : \|f\|_1 = \int_b^a f(x)dx < \infty\} \tag{15}$$

A Fractional Integral Operator Involving $\overline{H}$-function

In our present investigation we make use of a fractional integral operator with $\overline{H}$-function in its kernel defined as follows:

$$\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\,\varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\,\overline{H}_{p,q}^{m,n}[w(x-t)^{\gamma}]\varphi(t)dt \tag{16}$$

$$\left(\Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\};\ 1\leqq m\leqq q;\ 0\leqq n\leqq p;\ \Re(\beta)+\min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\}>0\right).$$

If we take $w=1$ and $m=1$ in (16) we obtain a fractional integral operator introduced by Harjule(see for details [6]). Now, by using the *Convolution Theorem* for the Laplace Transform in (7), we find from the definition (16) that

$$\begin{aligned}\mathcal{L}\left[\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\,\varphi\right)(x)\right](s) &= \mathcal{L}\left[x^{\beta-1}\,\overline{H}_{p,q}^{m,n}[wx^{\gamma}]\right](s)\cdot\mathcal{L}[\varphi(x)](s)\\ &= s^{-\beta}\overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),\ \ (e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right.\right]\Phi(s)\end{aligned} \tag{17}$$

$$\left(\Re(s)>0;\ \gamma>0;\ \Re(\beta)+\min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\}>0\right)$$

where,

$$\Phi(s) := \mathcal{L}[\varphi(x)](s) \qquad \big(\Re(s)>0\big).$$

In its special case when $\varphi(x)\equiv 1$, (17) immediately yields

$$\mathcal{L}\left[\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\,1\right)(x)\right](s) = s^{-\beta-1}\overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),\ \ (e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right.\right] \tag{18}$$

$$\left(\Re(s)>0;\ \gamma>0;\ \Re(\beta)+\min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\}>0\right)$$

2. Required Results

The following formulae to be used in the main theorem was given by Tomovski el al. [17]:

$$\begin{aligned}\frac{s^{\beta_i(\alpha_i-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c} &= \frac{1}{b}\left(\frac{s^{\beta_i(\alpha_i-1)}}{s^{\alpha_2}+\frac{c}{b}}\right)\left(\frac{1}{1+\frac{a}{b}(\frac{s^{\alpha_1}}{s^{\alpha_2}+\frac{c}{b}})}\right) = \frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\frac{s^{\alpha_1 r+\beta_i\alpha_i-\beta_i}}{(s^{\alpha_2}+\frac{c}{b})^{r+1}}\\ &= \mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_i(1-\alpha_i)-1}\\ &\quad\cdot E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_i(1-\alpha_i)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\Bigg](s) \qquad (i=1,2)\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{F(s)}{as^{\alpha_1}+bs^{\alpha_2}+c} &= \frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\left(\frac{s^{\alpha_1 r}}{(s^{\alpha_2}+\frac{c}{b})^{r+1}}F(s)\right) = \mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\\ &\quad\cdot\left(x^{(\alpha_2-\alpha_1)r+\alpha_2-1}E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2}\left(-\frac{c}{b}x^{\alpha_2}\right)*f(x)\right)\Bigg](s)\\ &= \mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\left(E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f\right)(x)\Bigg](s).\end{aligned} \tag{20}$$

Further, we obtain

$$\begin{aligned}&\frac{\lambda}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{-\beta-1}\,\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right.\right]\\ &= \mathcal{L}\Bigg[\frac{\lambda}{b}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\left(-\frac{a}{b}\right)^r(r+1)_j\,x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\ &\cdot\overline{H}^{m,n+1}_{p+1,q+1}\left[wx^{\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta,\gamma;1)\end{matrix}\right.\right]\Bigg]\end{aligned} \tag{21}$$

Proof. We first express $\overline{H}$-function in the form of contour integral and then interchange the order of summation and integration(which is permissible under the conditions stated) in the left hand side of (21)

$$= \frac{\lambda}{2\pi ib}\int_{\mathfrak{L}}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\frac{s^{\alpha_1 r-\gamma\xi-\beta-1}}{(s^{\alpha_2}+\frac{c}{b})^{r+1}}\;w^{\xi}\;\Gamma(\beta+\gamma\xi)\;\overline{\Theta}(\xi)\,d\xi \tag{22}$$

using (19) we get

$$= \frac{\lambda}{2\pi ib}\int_{\mathfrak{L}}\mathcal{L}\Bigg[\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\Bigg(x^{(\alpha_2-\alpha_1)r+\alpha_2+\gamma\xi+\beta}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\gamma\xi+\beta+1}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\Bigg](s)\Gamma(\beta+\gamma\xi)\overline{\Theta}(\xi)\,d\xi \tag{23}$$

Further, we express generalization of the Mittag-Leffler function in the series form and reinterpret $\overline{H}$-function in order to obtain the right hand side of (21). □

3. A General Family of Fractional Differential Equations

In this section, a general family of fractional differential equations [17, p.803, Eq.(3.7)] given by (24), was introduced in [8] for dielectric relaxation in glasses but its general solution was not given, though the laplace transformed relaxation function and the corresponding dielectric susceptibility were calculated. Therefore, in this section we proceed to find its general solution. Consider the following fractional differential equation:

$$a\left(D_{0+}^{\alpha_1,\beta_1}\,y\right)(x)+b\left(D_{0+}^{\alpha_2,\beta_2}\,y\right)(x)+cy(x)=g(x) \tag{24}$$

$$\left(0<\alpha_1\leqq\alpha_2<1;\ 0\leqq\beta_1\leqq 1;\ 0\leqq\beta_2\leqq 1\,\text{and}\,a,b,c\in\mathbb{R}\right)$$

in the space of Lebesgue integrable functions (see [3, 16]) $y\in L(0,\infty)$ with the initial conditions:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}\,y\right)(0+)=C_i \qquad (i=1,2), \tag{25}$$

where, without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1)\leqq(1-\beta_2)(1-\alpha_2).$$

if $C_1<\infty$, then $C_2=0$ unless $(1-\beta_1)(1-\alpha_1)=(1-\beta_2)(1-\alpha_2)$.

3.1. Main Theorem

Theorem 3.1. *The following fractional differential equation*:

$$a\left(D_{0+}^{\alpha_1,\beta_1}\,y\right)(x)+b\left(D_{0+}^{\alpha_2,\beta_2}\,y\right)(x)+cy(x)=\lambda\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\,1\right)(x)+f(x) \tag{26}$$

$$\left(0<\alpha_1\leqq\alpha_2<1;0\leqq\beta_1\leqq 1;\ 0\leqq\beta_2\leqq 1;\ \Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\};\ 1\leqq m\leqq q;0\leqq n\leqq p;\ \Re(\beta)+\min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\}>0\right)$$

with the initial condition:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}\,y\right)(0+)=C_i \qquad (i=1,2), \tag{27}$$

has its solution in the space $L(0,\infty)$ *given by*

$$\begin{aligned} y(x)=&\frac{1}{b}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a}{b}\right)^r\Bigg[aC_1x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\ &+b\,C_2x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)+E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f(x)\\ &+\lambda\sum_{j=0}^{\infty}(r+1)_j\;x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\,\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\ &\cdot\overline{H}^{m,n+1}_{p+1,q+1}\left[wx^{\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta,\gamma;1)\end{matrix}\right.\right]\Bigg] \end{aligned} \tag{28}$$

where C_1,C_2 *and* λ *are arbitrary constants and the function* f *is suitably prescribed.*

Proof. We denote by $Y(s)$ the Laplace transform of the function $y(x)$, which is given as in (7). Then, by applying the Laplace transform operator $\mathcal{L}$ to each side of (26), and using the formulas (8) and (18) and the initial condition (27), we find that

$$a(s^{\alpha_1}Y(s)-C_1\,s^{\beta_1(\alpha_1-1)})+b(s^{\alpha_2}Y(s)-C_2\,s^{\beta_2(\alpha_2-1)})+c\,Y(s)=$$

$$F(s)+\lambda s^{-\beta-1}\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right.\right] \tag{29}$$

which readily yields

$$\begin{aligned}Y(s)=&\frac{a\,C_1}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{\beta_1(\alpha_1-1)}+\frac{b\,C_2}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{\beta_2(\alpha_2-1)}+\frac{F(s)}{(as^{\alpha_1}+bs^{\alpha_2}+c)}\\ &+\frac{\lambda}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{-\beta-1}\,\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right.\right].\end{aligned} \tag{30}$$

Using (19), (20) and (21) we obtain

$$\begin{aligned}Y(s)=&\mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\Bigg[aC_1\,x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\ &+bC_2\,x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)+\left(E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f\right)(x)\\ &+\lambda\sum_{j=0}^{\infty}(r+1)_j\,x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\ &\cdot\overline{H}^{m,n+1}_{p+1,q+1}\left[wx^{\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta,\gamma;1)\end{matrix}\right.\right]\Bigg]\Bigg](s)\end{aligned} \tag{31}$$

Finally, by applying the inverse of Laplace transform, we get the solution (28) asserted by the main theorem. □

Corollary 3.2. *If we reduce $\overline{H}$-function occurring in the right hand side of (26) to the function associated with Gaussian Model free energy([5, p.4126, 4127, Eq.(23),(28)] and [9, p.98, Eq.(1.4)]), we observe that the following fractional differential equation:*

$$a\left(D^{\alpha_1,\beta_1}_{0+}\,y\right)(x)+b\left(D^{\alpha_2,\beta_2}_{0+}\,y\right)(x)=\lambda\left(\mathcal{F}^{w;1,1;\gamma}_{0+;1,2;\beta}\,1\right)(x)+f(x) \tag{32}$$

$$\left(0<\alpha_1\leqq\alpha_2<1;0\leqq\beta_1\leqq1;\;0\leqq\beta_2\leqq1;\;\Re(\beta)>0;\;w\in\mathbb{C}\setminus\{0\}\right)$$

with the initial condition (27) has its solution in the space $L(0,\infty)$ given by

$$\begin{aligned}y(x)=&\frac{1}{b}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a}{b}\right)^r\Bigg[aC_1x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\ &+b\,C_2x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)+E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f(x)\\ &-\frac{\lambda}{4\pi^{\frac{d}{2}}}\sum_{j=0}^{\infty}(r+1)_j\;x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\,\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\ &\cdot\overline{H}^{1,3}_{3,3}\left[-x^{\gamma}\left|\begin{matrix}(1-\beta-\gamma,\gamma;1),(0,1;2),(-\frac{1}{2},1;d)\\ \\ (0,1),(-1,1;1+d),(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta-\gamma,\gamma;1)\end{matrix}\right.\right]\Bigg]\end{aligned} \tag{33}$$

where C_1,C_2 and λ are arbitrary constants and the function f is suitably prescribed.

Corollary 3.3. *If we reduce $\overline{H}$-function to the Polylogarithm function of order p [2, p.30] in the integral operator on the right-hand side of (26), we obtain the following fractional differential equation:*

$$a\left(D_{0+}^{\alpha_1,\beta_1}\, y\right)(x)+b\left(D_{0+}^{\alpha_2,\beta_2}\, y\right)(x)=\lambda\left(\mathbb{F}_{0+;1,2;\beta}^{w;1,1;\gamma}\, 1\right)(x)+f(x) \tag{34}$$

$$\left(0<\alpha_1\leqq\alpha_2<1;\ 0\leqq\beta_1\leqq 1;\ 0\leqq\beta_2\leqq 1;\ \Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\};\ p\leqq q+1\right)$$

with the initial condition (27) has its solution in the space $L(0,\infty)$ given by

$$\begin{aligned} y(x)=&\frac{1}{b}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a}{b}\right)^r\Bigg[aC_1x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}\,E_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}^{r+1}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\ &+b\,C_2x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}\,E_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}^{r+1}\left(-\frac{c}{b}\,x^{\alpha_2}\right)+E_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}^{r+1}f(x)\\ &-\lambda\sum_{j=0}^{\infty}(r+1)_j\ x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\,\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\ &\cdot\overline{H}_{2,3}^{1,2}\left[-wx^{\gamma}\,\middle|\,\begin{matrix}(1-\beta,\gamma;1),(1,1;p+1)\\ \\ (1,1),(0,1;p),(-\alpha_2 j-(\alpha_2-\alpha_1)r-\beta,\gamma;1)\end{matrix}\right]\Bigg]\end{aligned} \tag{35}$$

where C_1, C_2 and λ are arbitrary constants and the function f is suitably prescribed.

Known Special Cases of Our Main Findings

If we consider $\lambda=0$ in the right hand side of (26), we get the result obtained by Tomovski et al. [17, p.803, theorem 5]. Again, if we take a=1, b=c=0 and reduce $\overline{H}$-function to the Mittag-Leffler function(see[14] and [16]) in the integral operator on the right hand side of (26), we get the result obtained by Srivastava and Tomovski [16, p.207,theorem 8]. Further, if we take a=1, b=c=0 and reduce $\overline{H}$-function to the H-function [6, p.10, Eq.(1.1.42)], we get the result obtained by Srivastava et al.[15, p.115, theorem 2].

4. Conclusion and Observations

In this paper, we have given solution to a general family of fractional differential equation involving Hilfer derivative operator and the fractional integral operator whose kernel is the $\overline{H}$-function. Our main result generalizes the results obtained recently by Srivastava et al.[15], Srivastava and Tomovski[16] and Tomovski et al.[17]. Further, corollaries of the main result have been derived.

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