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Fuzzy Topological Fuzzy Ordered Spaces

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Abstract: In this paper by combining the notions of certain types of fuzzy order and fuzzy topology we introduce fuzzy topological fuzzy ordered spaces. Its various properties are analyzed. We also develop and study order separation axioms called fTi separation axioms for 'fuzzy topological fuzzy ordered spaces'. The relationships between some of these fTi separation axioms are also studied.

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1. Introduction

L.Nachbin in his famous book 'Topology and Order' published in 1965 [6] studied the relationship between topological and ordered structures. Mc Cartan [5] introduced T_i order separation axioms ($i = 1, 2, 3, 4$) in topological ordered spaces. To study of the interdependence between fuzzy topology and order, Katsaras [3] inroduced fuzzy topological ordered spaces in 1981. Here fuzzy topological ordered space is a triplet $(X, \mathcal{T}, \leq)$ where $\mathcal{T}$ is a fuzzy topology on X and $\leq$ is a crisp order on X. We think that we will get better result on relationship between fuzzy topology and order if the order defined on X is a fuzzy order. So, we define a fuzzy topological fuzzy ordered space, as a triple $(X, \mathcal{T}, \rho)$, wher $\mathcal{T}$ is a fuzzy topology on X and ρ is a fuzzy order on X. This space is a generalization of Katsars's fuzzy topological ordered spaces as well as fuzzy topological spaces.

2. Preliminaries

Definition 2.1 ([9]). *A fuzzy relation ρ on X is defined as a map $\rho : X \times X \to I$ where $I = [0, 1]$, ρ is called*

(1). reflexive, if $\rho(x, x) = 1$, for all $x \in X$.

(2). symmetric, if $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$.

(3). antisymmetric, if $\rho(x, y) \wedge \rho(y, x) = 0$ whenever $x \neq y$, for all $x, y \in X$.

(4). transitive, if $\rho(x, z) \wedge \rho(z, y) \leq \rho(x, y)$ for all $x, y, z \in X$.

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A reflexive and transitive fuzzy relation is called a fuzzy preorder. Moreover a preorder which is antisymmetric is called a fuzzy partial order (fuzzy order). A fuzzy symmetric fuzzy preorder is called a fuzzy equivalence (fuzzy similarity).

A set X equipped with fuzzy order relation ρ is called a fuzzy ordered set(foset), we denote it as (X, ρ).

Definition 2.2. *If Y is a subset of a foset (X, ρ), then the fuzzy order ρ is also a fuzzy order on Y, called the induced fuzzy order.*

Definition 2.3. *A fuzzy order ρ is linear(or total)on X if $\rho(x, y) > 0$ or $\rho(y, x) > 0$ for every $x, y \in X$. A fuzzy ordered set (X, ρ) in which ρ is total is called a ρ -fuzzy chain. Conversely, if for $x, y \in X, \rho(x, y) > 0$ iff $x = y$, then (X, ρ) is called a ρ-fuzzy antichain. Given a fuzzy preorder ρ on X , we define $\rho_{op} : X \times X \to I$ by $\rho_{op}(x, y) = \rho(y, x)$. Then, ρ_{op} is also a preorder on X, called the opposite of ρ. ρ is a fuzzy partial order (fuzzy equivalence) iff ρ_{op} is a fuzzy partial order(fuzzy equivalence). Suppose that $(\rho_i)_{i\in\Delta}$ is a collection of fuzzy preorders on X. Then, the pointwise intersection $\rho(x, y) = \bigwedge_{i\in\Delta} \rho_i(x, y)$ is also a fuzzy preorder on X. If ρ is a fuzzy order on X then $\rho \wedge \rho_{op}$ is a fuzzy equivalence on X.*

Definition 2.4. *A fuzzy set $\mu : X \to I$ in a fuzzy preordered set (X, ρ) is called an upper set if $\rho(x, y) > 0 \Rightarrow \mu(x) \leq \mu(y)$ for any $x, y \in X$. Dually, μ is called a lower set if $\rho(x, y) > 0 \Rightarrow \mu(y) \leq \mu(x)$ for any $x, y \in X$.*

A fuzzy set μ is an upper set in (X, ρ) iff it is a lower set in (X, ρ_{op}). In particular, if ρ is a fuzzy equivalence relation then a fuzzy set μ is an upper set in (X, ρ) iff it is an lower set in (X, ρ_{op}).

Definition 2.5. *Let (X, ρ) be a fuzzy preordered set and $z \in X$ then the fuzzy set $u(z)(x) = \rho(z, x)$ is an upper set, called the principal upper set generated by z. Similarly, the fuzzy set $l(z)(x) = \rho(x, z)$ is a down set, called the principal down set generated by z.*

Definition 2.6. *Let (X, ρ) and (Y, σ) be preordered fuzzy sets.We say that $h : X \to Y$*

(1). order preserving, if $\rho(x, y) \leq \sigma(h(x), h(y))$, for all $x, y \in X$.

(2). order homomorphism, if $\rho(x, y) = \sigma(h(x), h(y))$, for all $x, y \in X$.

(3). order isomorphism, if h is an injective order homomorphism.

3. Fuzzy Topological Fuzzy Ordered Space

Definition 3.1. *A fuzzy set μ on a fuzzy preordered space X is called*

(1). f-increasing if $\rho(x, y) > 0 \Rightarrow \mu(x) \leq \mu(y)$ for $x, y \in X$.

(2). f-decreasing if $\rho(x, y) > 0 \Rightarrow \mu(x) \geq \mu(y)$ for $x, y \in X$.

(3). f-order convex if $\rho(x, z) > 0$ and $\rho(z, y) > 0$ imply $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ for $x, y, z \in X$.

Definition 3.2. *Let μ be a fuzzy preordered set in X then f-increasing hull of μ is the set $fi(\mu)(x) = \sup\{\mu(y) \mid \rho(x, y) > 0\}$ f-decreasing hull of μ is the set $fd(\mu)(x) = \sup \mu(y) \mid \rho(y, x) > 0$ f-convex hull of μ is the set $fc(\mu)(x) = \sup\{\min\{\mu(x_1), \mu(x_2)\} \mid \rho(x_1, x) > 0, \rho(x, x_2) > 0\}$.*

Note 3.3.

(1). $fi(\mu), fd(\mu), fc(\mu)$ are respectively the smallest increasing, decreasing and convex fuzzy preordered sets containing μ.

(2). $fc(\mu) = fi(\mu) \wedge fd(\mu)$.

Definition 3.4. *A fuzzy set μ in a fuzzy topological space $(X, \mathcal{T})$ is called a neighborhood of a fuzzy point x_α if there exists a $\mathcal{T}$-open fuzzy set δ such that $x_\alpha \in \delta$ and $\delta \leq \mu$ i.e. $\alpha \leq \delta(x)$ and $\delta(x) \leq \mu(x)$.*

Definition 3.5. *A fuzzy set μ in $(X, \mathcal{T})$ is $\mathcal{T}$-open if μ is a neighborhood of each fuzzy point x_α for which $x_\alpha \in \mu$.*

Proposition 3.6. *A function $g : (X, \rho) \to (Y, r)$ is f-increasing iff $g^{-1}(\mu)$ is f-increasing (f-decreasing) in X for every f-increasing (f-decreasing) set μ in Y.*

Proof. Suppose $g : X \to Y$ is f-increasing and μ is an f-increasing set in Y. We want to show $g^{-1}(\mu)$ is f-increasing set in X. Let $\rho(x, y) > 0$ Since g is f-increasing, $r(g(x), g(y)) > 0$ in Y. As μ is increasing set in Y, we have $\mu(f(x)) \leq \mu(f(y))$. So $g^{-1}(\mu)(x) \leq g^{-1}(\mu)(y)$.Hence $g^{-1}(\mu)$ is f-increasing set in X. Converse is straightforward.
Similarly, if f is f-decreasing and μ is f-decreasing (resp. f-increasing) then $f^{-1}(\mu)$ is f-increasing(resp. f-decreasing). □

Definition 3.7. *A fuzzy topological fuzzy ordered space is a triple $(X, \mathcal{T}, \rho)$ where X is a nonempty set, $\mathcal{T}$ is a fuzzy topology X and ρ is a fuzzy order on X.*

4. Fuzzy T_1 Ordered space

Definition 4.1. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is said to be an lower fT_1 ordered space if for each pair of elements $x, y \in X$ with $\rho(x, y) = 0$ there exists a decreasing $\mathcal{T}$-open neighborhood μ of y such that $\mu(x) = 0$. A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is said to be an upper fT_1 ordered space if for each pair of elements $x, y \in X$ with $\rho(x, y) = 0$ there exists a increasing $\mathcal{T}$-open neighborhood μ of x such that $\mu(y) = 0$. A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is said to be fT_1 ordered space if it is both upper and lower fT_1 ordered.*

Theorem 4.2. *Let $(X, \mathcal{T}, \rho)$ be a fuzzy topological fuzzy ordered space. Then $(X, \mathcal{T}, \rho)$ is lower (resp. upper) fT_1 ordered space iff for each pair of points $x, y \in X$ with $\rho(x, y) = 0$ there exists a $\mathcal{T}$-open neighborhood μ of y (resp. of x) such that $d(\mu(x)) = 0$ (resp. $i(\mu(y)) = 0$).*

Proof. Follows from the definitions of $d(\mu)$ and $i(\mu)$ respectively. □

Definition 4.3. *For $a \in X$, we define the fuzzy sets u_a, l_a w.r.to ρ as, $u_a(x) = \rho(a, x)$; $l_a(x) = \rho(x, a)$ for every $x \in X$.*

Definition 4.4. *A fuzzy topological ordered space $(X, \mathcal{T}, \rho)$ is upper semiclosed ordered (resp. lower semiclosed ordered) if the fuzzy set u_a is closed(resp. l_a is closed). $(X, \mathcal{T}, \rho)$ is semiclosed ordered iff both u_a and l_a are closed w.r.to $\mathcal{T}$.*

Proposition 4.5. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fT_1 ordered iff order ρ on X is semiclosed.*

Proof. Suppose $(X, \mathcal{T}, \rho)$ is fT_1 ordered. Then it is lower fT_1 ordered. So, for each pair of points $x, y \in X$ with $\rho(x, y) = 0$ there exists a $\mathcal{T}$-open neighborhood μ of y such that $d(\mu(x)) = 0$. Then, for $a, b \in X$ such that $\rho(a, b) = 0$, we have $[1 - u_a](b) > 0$. So, $1 - u_a$ is a $\mathcal{T}$ -open neighborhood of each b. Therefore, u_a is $\mathcal{T}$ -closed. Similarly, by using $(X, \mathcal{T}, \rho)$ is upper fT_1 ordered, we get, l_a is $\mathcal{T}$-closed.
Conversely, suppose for each $a \in X$, u_a and l_a are $\mathcal{T}$-closed. Let $a, b \in X$ with $\rho(a, b) = 0$. By hypothesis, $\lambda = 1 - u_b$ is $\mathcal{T}$-open and $\lambda(a) > 0, \lambda(b) = 0$. Now let $\rho(x, y) > 0$ We want to show $\lambda(x) \leq \lambda(y)$. If $\lambda(x) = 0$ then the result is obvious. If $\lambda(x) > 0$ then $x \in 1 - u_b$. Therefore $\rho(b, x) = 0$ which imply $\rho(b, y) = 0$ (because if $\rho(b, y) > 0$ then $\rho(x, y) > 0 \Rightarrow \rho(b, x) > 0$, which is a contradiction). Hence, $y \in 1 - u_b$ that is $\lambda(y) \geq \lambda(x)$. So, λ is an increasing neighborhood of a such that $\lambda(a) > 0, \lambda(b) = 0$. Similarly, taking $\mu = 1 - l_a$, we get a decreasing $\mathcal{T}$-open neighborhood μ of b which does not contain a. Therefore $(X, \mathcal{T}, \rho)$ is fT_1 ordered. □

Proposition 4.6. *Let $(x, \mathcal{T}, \rho)$ be a fuzzy fT_1 ordered and $Y \subset X$ then $(Y, \mathcal{T}_Y, \rho_Y)$ where $\rho_Y = \rho \cap (X \times Y)$ and $\mathcal{T}_Y = \{\alpha|_Y \mid \alpha \in \mathcal{T}\}$ is fuzzy fT_1 ordered.*

Proof. Let $(Y, \mathcal{T}_Y, \leq_Y)$ be a subspace of $(X, \mathcal{T}, \rho)$. Let $a, b \in Y$ such that $\rho(a, b) = 0$. So, $a, b \in X$ such that $\rho(a, b) = 0$. As X is T_1-ordered there exists an increasing neighborhood λ^* of a in X such that $\lambda^*(b) = 0$ and a decreasing neighborhood μ^* of b in X such that $\mu^*(a) = 0$. Then, $\lambda = \lambda^*|_Y$ is an increasing neighborhood of a in Y such that $\lambda(b) = 0$ and $\mu = \mu^*|_Y$ is a decreasing neighborhood of b in Y such that $\mu(a) = 0$. Hence $(Y, \mathcal{T}_Y, \rho_Y)$ is fuzzy T_1 ordered. $\square$

Proposition 4.7. *If $(X, \mathcal{T}, \rho)$ is a fuzzy topological T_1 ordered space and (X, δ, ρ) is a fuzzy topological ordered space with $\mathcal{T} \leq \delta$ then (X, δ, ρ) is also fuzzy T_1 ordered.*

Proof. Fuzzy topological ordered space $(X, \mathcal{T}, \rho)$ is fuzzy T_1 ordered, so for each $x \in X$, u_x and l_x are fuzzy closed sets in $(X, \mathcal{T})$. But $\mathcal{T} \leq \delta$, hence u_x and l_x are fuzzy closed sets in (X, δ). Therefore, (X, δ, ρ) is fuzzy fT_1 ordered. $\square$

Proposition 4.8. *Let f be a order preserving continuous function from $(X, \mathcal{T}, \rho)$ to an fuzzy topological fuzzy ordered space (Y, δ, r). If (Y, δ, r) is fT_1 ordered then $(X, \mathcal{T}, \rho)$ is also fT_1 ordered.*

Proof. Let $\rho(x, y) > 0$ in X. Since f is order preserving $r * (f(x), f(y)) > 0$ in Y. Hence, there exists an increasing(decreasing) neighborhood λ^* such that $\lambda^*(f(x)) > 0$ $(\lambda^*(f(y)) > 0)$ and $\lambda^*(f(y)) = 0(\lambda^*(f(x)) = 0)$. Let $\lambda = f^{-1}(\lambda^*)$. As f is order preserving and fuzzy continuous λ is an increasing (decreasing) fuzzy $\mathcal{T}$-open neighborhood of x in X such that $\lambda(x) > 0$ (resp. $\lambda(y) > 0$) and $\lambda(y) = 0$ (resp $\lambda(x) = 0$). Thus, X is fuzzy fT_1 ordered. $\square$

Note 4.9. *In this paper we use Δ as an indexing set.*

Definition 4.10. *Let $\{(X_t, \mathcal{T}_t, \rho_t) \mid t \in \Delta\}$ be a family of ordered fuzzy topological spaces. Let $X = \prod\{X_t \mid t \in \Delta\}$ and let $\mathcal{T}$ be the product fuzzy topology on X. Let ρ be a binary relation on X defined as, $\rho(x, y) = \bigwedge_{t \in \Delta} \rho_t(x_t, y_t)$ for $x = (x_t)$ and $y = (y_t) \in X$. Then, ρ is a fuzzy partial order on X. The ordered fuzzy topological space $(X, \mathcal{T}, \rho)$ is called the fuzzy ordered fuzzy topological product of the family $\{(X_t, \mathcal{T}_t, \rho_t) \mid t \in \Delta\}$.*

Theorem 4.11. *The product of a family of fuzzy fT_1 ordered spaces is also fuzzy fT_1 ordered.*

Proof. Let $\{(X_t, \mathcal{T}_t, \leq_t) \mid t \in \Delta\}$ be a family of fuzzy fT_1 ordered spaces and $(X, \mathcal{T}, \leq)$ be the fuzzy product fuzzy ordered space. Let $x = (x_t), y = (y_t) \in X$ be such that $\rho(x, y) = 0$. Then, there exists $\alpha \in \Delta$ such that $\rho_\alpha(x_\alpha, y_\alpha) = 0$. Since $(X_\alpha, \mathcal{T}_\alpha, \rho_\alpha)$ is fuzzy fT_1 ordered, there exists an increasing open set λ_α in $\mathcal{T}_\alpha$ such that $\lambda_\alpha(x_\alpha) > 0$ and $\lambda_\alpha(y_\alpha) = 0$ and an decreasing open set μ_α in $\mathcal{T}_\alpha$ such that $\mu_\alpha(x_\alpha) = 0$ and $\mu_\alpha(y_\alpha) > 0$. Define, $\lambda = \prod\{\lambda_t \mid t \in \Delta\}$ where $\lambda_t = 1_{X_t}$ if $t \neq \alpha$ and $\lambda_t = \lambda_\alpha$ if $t = \alpha$. $\mu = \prod\{\mu_t \mid t \in \Delta\}$ where $\mu_t = 1_{X_t}$ if $t \neq \alpha$ and $\mu_t = \mu_\alpha$ if $t = \alpha$. Then λ is an increasing fuzzy open set such that $\lambda(x) > 0, \lambda(y) = 0$ while μ is a decreasing fuzzy open set such that $\mu(x) = 0, \mu(y) > 0$.

$$\begin{aligned}
\lambda(y) &= \prod\{\lambda_t \mid t \in \Delta\}(y) \\
&= \min\{\lambda_t(y_t) \mid t \in \Delta\} \\
&= \min\{\{\lambda_t(y_t) \mid t \neq \alpha\}, \lambda_\alpha(y_\alpha)\} \\
&= \min\{1, 0\} \\
&= 0
\end{aligned}$$

Hence, $(X, \mathcal{T}, \rho)$ is fT_1 ordered. $\square$

5. Fuzzy T_2 Ordered Space

Definition 5.1. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is said to be a fT_2 ordered space if for each pair of elements $x, y \in X$ with $\rho(x, y) = 0$ there exists a increasing $\mathcal{T}$-open neighborhood μ of x and a decreasing $\mathcal{T}$-open neighborhood λ of y such that $\mu \wedge \lambda = 0$.*

Proposition 5.2. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is said to be a fT_2 ordered space iff for each pair of elements $x, y \in X$ with $\rho(x, y) = 0$ there exists $\mathcal{T}$-open neighborhoods μ and λ of x and y respectively such that $i(\mu) \wedge d(\lambda) = 0$.*

Proof. Follows from the definitions of $i(\mu)$ and $d(\lambda)$. □

Proposition 5.3. *Every fT_2 ordered space is a fT_1 ordered space.*

Definition 5.4. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is called a f-Hausdorff space iff for points $x, y \in X$ with $x \neq y$ there exists fuzzy neighborhoods λ and μ of x and y respectively such that $\lambda \wedge \mu = 0$.*

Remark 5.5. *Every fT_2 ordered space is f-Hausdorff space, but f-Hausdorff space need not be fT_2 ordered space.*

Definition 5.6. *A fuzzy order relation ρ on a fuzzy topological space $(X, \mathcal{T})$ is closed if ρ is a closed fuzzy set in the product space $X \times X$.*

Proposition 5.7. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is a fT_2 ordered space if and only if the order ρ on a fuzzy topological space $(X, \mathcal{T})$ is closed.*

Proof. Suppose $(X, \mathcal{T}, \rho)$ is a fuzzy topological ordered space where the order ρ is closed. Let $\rho(x, y) = 0$ for $x, y \in X$. Then, $(x, y) \notin \rho$. Since, ρ is a fuzzy closed set in $(X \times X, \mathcal{T}')$, where $\mathcal{T}'$ is the product topology on $X \times X$. We have, $1 - \rho$ is fuzzy open set in $(X \times X, \mathcal{T}')$. Now, $\rho(x, y) = 0$. So, $1 - \rho(x, y) = 1 > 0$. $\therefore, 1 - \rho$ is a fuzzy open neighborhood of $(x, y) \in X \times X$. Hence, we can find a fuzzy open set $\lambda \times \mu$ such that $\lambda \times \mu \leq (1 - \rho)$ where λ is a fuzzy open set such that $\lambda(x) > 0$ and μ is a fuzzy open set such that $\mu(y) > 0$. Now we show $i(\lambda) \wedge d(\mu) = 0$. For if there is $z \in X$ such that $(i(\lambda) \wedge d(\mu))(z) > 0$ then $i(\lambda)(z) \wedge d(\mu)(z) > 0$ If $\rho(y, z) \leq \rho(z, x)$ then $\rho(z, x) > 0 \Rightarrow i(\lambda)(x) > d(\mu)(z) > 0$ and $\rho(y, z) > 0 \Rightarrow d(\mu)(y) > d(\mu)(z) > 0$. Therefore $i(\lambda)(x) > 0, d(\mu)(y) > 0$. Hence, $\rho(x, y) > 0$, which a contradiction.

Conversely, suppose fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fT_2 ordered space, to show ρ is fuzzy closed set in $(X \times X, \mathcal{T}')$. Let $(x, y) \in 1 - \rho$ then $(1 - \rho)(x, y) > 0$ So, $\rho(x, y) = 0$. By hypothesis, there exists fuzzy open set λ and μ such that λ is increasing fuzzy open neighborhood of x and μ is a decreasing fuzzy open neighborhood of y and $\lambda \wedge \mu = 0$. Clearly, $\lambda \times \mu$ is a fuzzy open neighborhood of (x, y) such that $\lambda \times \mu(x, y) > 0$. It is easy to verify that $\lambda \times \mu < 1 - \rho$. So $1 - \rho$ is a fuzzy open set. ρ is a fuzzy closed set in $X \times X$. Hence, ρ is a closed order. □

Proposition 5.8. *Let $(X, \mathcal{T}, \rho)$ be a fT_2 -ordered space and $Y \subset X$, then $(Y, \mathcal{T}_Y, \rho_Y)$ is also fT_2 ordered.*

Proof. Let $(Y, \mathcal{T}_Y, \leq_Y)$ be a subspace of $(X, \mathcal{T}, \leq)$. Let $a, b \in Y$ such that $\rho(a, b) = 0$. So, $a, b \in X$ such that $\rho(a, b) = 0$. As X is fT_2-ordered there exists an increasing neighborhood λ^* of a in X and a decreasing neighborhood μ^* of b in X such that $\lambda^* \wedge \mu^* = 0$. Then, $\lambda = \lambda^*|_Y$ is an increasing neighborhood of a in Y and $\mu = \mu^*|_Y$ is a decreasing neighborhood of b in Y such that $\lambda \wedge \mu = 0$. Hence $(Y, \mathcal{T}_Y, \rho_Y)$ is fuzzy T_2 ordered. □

Proposition 5.9. *Let $(X, \mathcal{T}, \rho)$ be a fT_2-ordered and (X, δ, ρ) is an fuzzy topological ordered space with $\mathcal{T} \leq \delta$ then (X, δ, ρ) is also fT_2 ordered.*

Proof. Fuzzy topological ordered space $(X, \mathcal{T}, \rho)$ is fuzzy fT_2 ordered, so ρ is a fuzzy closed sets in $(X, \mathcal{T})$. But $\mathcal{T} \leq \delta$, hence ρ is a fuzzy closed sets in (X, δ). Therefore, (X, δ, ρ) is fT_2 ordered. □

Proposition 5.10. *If f is an order preserving fuzzy continuous mapping from $(X, \mathcal{T}, \rho)$ to a fT_2 ordered space (Y, δ, r) then $(X, \mathcal{T}, \rho)$ is also fuzzy T_2 ordered.*

Proof. Suppose $f : (X, \mathcal{T}, \rho) \to (Y, \delta, r)$ is an increasing fuzzy continuous map. Let $\rho(x, y) = 0$ in X. Hence $r(f(x), f(y)) = 0$ in Y. But (Y, δ, r) is a fT_2 ordered space, so there exists an increasing fuzzy open set λ and a decreasing fuzzy open set μ such that λ is a fuzzy open neighborhood of $f(x)$ and μ is a fuzzy open neighborhood of $f(y)$ such that $\lambda \wedge \mu = 0$. Since, f is increasing, λ is increasing it follows that $f^{-1}(\lambda)$ is increasing. Also, since f is increasing, μ is decreasing it follows that $f^{-1}(\mu)$ is decreasing. Also, f is continuous, implies $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy open sets containing x and y respectively. $f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = f^{-1}(0) = 0$. Hence, X is fT_2 ordered. □

Theorem 5.11. *The product of a family of fuzzy T_2 ordered spaces is also fuzzy T_2 ordered.*

Proof. Let $\{(X_t, \mathcal{T}_t, \rho_t) \mid t \in \Delta\}$ be a family of fT_2 ordered spaces and $(X, \mathcal{T}, \rho)$ be the product of ordered fuzzy topological spaces. If $(x, y) \in X$ such that $\rho(x, y) = 0$ then there exists $t_0 \in \Delta$ such that $\rho_{t_0}(x_{t_0}, y_{t_0}) = 0$. Then there exists fuzzy open sets λ_{t_0} and μ_{t_0} in X_{t_0} such that λ_{t_0} is increasing and μ_{t_0} is decreasing, λ_{t_0} is a fuzzy open neighborhood of x_{t_0}, μ_{t_0} is fuzzy open neighborhood of y_{t_0} and $\lambda_{t_0} \wedge \mu_{t_0} = 0$. Define $\lambda = \prod_{t \in \Delta} \lambda_t$, where $\lambda_t = 1_{X_t}$ if $t \neq t_0$ and $\lambda_t = \lambda_{t_0}$ if $t \neq t_0$. $\mu = \prod_{t \in \Delta} \mu_t$ where $\mu_t = 1_{X_t}$ if $t \neq t_0$ and $\mu_t = \mu_{t_0}$ if $t \neq t_0$. Then λ is an increasing fuzzy open set of X and mu is a decreasing fuzzy open set of X such that λ is a fuzzy open neighborhood of x and μ is a fuzzy open neighborhood of y and $\lambda \wedge \mu = 0$. Hence $(X, \mathcal{T}, \leq)$ is fT_2 ordered. □

6. Fuzzy Regular Fuzzy Ordered Space

Definition 6.1. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fuzzy lower regular fuzzy ordered if for all decreasing closed sets λ and for all $x \in X$ such that $\lambda(x) = 0$ there exists an increasing open set μ and a decreasing open set ν such that $\mu(x) > 0, \lambda \leq \nu$ and $\mu \wedge \nu = 0$. Similarly, a fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fuzzy upper regular fuzzy ordered if for all increasing closed sets λ and for all $x \in X$ such that $\lambda(x) = 0$ there exists an decreasing open set μ and a increasing open set ν such that $\mu(x) > 0, \lambda \leq \nu$ and $\mu \wedge \nu = 0$. A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fuzzy regular fuzzy ordered space if it is both fuzzy upper and lower fuzzy regular ordered.*

Remark 6.2. *Here we define order on I^X as, for $\lambda, \mu \in I^X$ we have $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$ for all $x \in X$.*

Proposition 6.3. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fuzzy regular fuzzy ordered if the following condition is satisfied: For each $x \in X$ and an increasing (resp.decreasing) $\mathcal{T}$-open fuzzy neighborhood μ of x, there exists an increasing (resp. decreasing) $\mathcal{T}$ open set ν such that $\mu(x) > 0$ and $\nu \leq I(\nu) \leq \mu(\nu \leq D(\nu) \leq \mu)$, where $D(\mu) = \inf\{\rho \mid \rho \geq \mu, \rho, \rho$ is closed and decreasing$\}$ is the smallest decreasing closed set containing μ.*

$$D(\mu)(x) = \vee\{\mu(y) \mid \rho(x, y) > 0\}$$

$$I(\mu) = \inf\{\rho \mid \rho \geq \mu, \rho \text{ is closed and increasing}\}$$

is the smallest increasing closed set containing μ.

$$I(\mu)(x) = \vee\{\mu(y) \mid \rho(x, y) > 0\}$$

Proof. Suppose $(X, \mathcal{T}, \rho)$ is a fuzzy lower (resp. upper) regularly fuzzy ordered space. Let $x \in X$ and let μ be an increasing (resp. decreasing) $\mathcal{T}$-open neighborhood of x, then $1 - \mu$ is $\mathcal{T}$-closed, decreasing(increasing)in X and $(1 - \mu)(x) = 0$. By hypothesis, there exists increasing (decreasing) fuzzy open set ν and a decreasing(increasing) fuzzy open set λ such that $\nu(x) > 0, 1 - \mu \leq \lambda, \lambda \wedge \nu = 0$. Hence, $\nu \leq 1 - \lambda \leq \mu$. So, $I(\nu) \leq I(1 - \lambda) = 1 - \lambda$, since $1 - \lambda$ is $\mathcal{T}$ -closed. Therefore $\nu \leq I(\nu) \leq \mu(\nu \leq D(\nu) \leq \mu)$. Converse, is straightforward. □

Proposition 6.4. *If $(X, \mathcal{T}, \rho)$ is fuzzy regular fuzzy ordered space then every fuzzy ordered subspace $(Y, \mathcal{T}_Y, \leq_Y)$ is also fuzzy regularly fuzzy ordered space.*

Proof. Let $(Y, \mathcal{T}_Y, \rho_Y)$ be ordered subspace of the fuzzy upper regularly ordered space $(X, \mathcal{T}, \rho)$ and let $x \in Y$ and μ be any $\mathcal{T}_Y$ open decreasing fuzzy neighborhood of x in Y. Then there exists a $\mathcal{T}$ open decreasing fuzzy set λ^* such that $\lambda = \lambda^*|_Y$ with $\lambda^*(x) > 0$. Since $(X, \mathcal{T}, \rho)$ is upper fuzzy regular fuzzy ordered set, there exists a $\mathcal{T}$ open decreasing fuzzy set μ^* such that $\mu^*(x) > 0$ and $\mu^* \leq D(\mu^*) \leq \lambda^*$. By restriction of μ^* and $D(\mu^*)$ to Y, we have $\mu \leq D(\mu) \leq \lambda$. Therefore $(Y, \mathcal{T}_Y, \rho_Y)$ is upper fuzzy regularly ordered. □

Definition 6.5. *A fuzzy topological fuzzy ordered space $(X, \mathcal{T}, \rho)$ is fuzzy lower (resp. upper) fT_3 ordered iff it is fuzzy lower (resp. upper) fT_1 ordered and lower (resp.upper) fuzzy regular fuzzy ordered.*

Definition 6.6. *$(X, \mathcal{T}, \rho)$ is fuzzy fT_3 ordered space if $(X, \mathcal{T}, \rho)$ is fuzzy fT_1 ordered and fuzzy regularly fuzzy ordered.*

Proposition 6.7. *If $(X, \mathcal{T}, \rho)$ is fT_3 ordered space then $(X, \mathcal{T}, \rho)$ is fT_2 ordered space.*

Theorem 6.8. *The product of a family of fuzzy regular ordered spaces is also fuzzy regular ordered space.*

Proof. Let $\{(X_t, \mathcal{T}_t, \rho_t) \mid t \in \Delta\}$ be a family of fuzzy regular ordered spaces and $(X, \mathcal{T}, \rho)$ be the product of ordered fuzzy topological spaces. Let $x \in X$ in the product topology. Let μ be a decreasing fuzzy $\mathcal{T}$-open set containing x.Since, the projection $P_t : X \to X_t$ is order preserving continuous function, the point x_t is contained in a decreasing $\mathcal{T}_t$ -open set for each $t \in \Delta$ such that $\mu = \{P_t^{-1}(\lambda_t) \mid t \in \Delta\}$. As $(X_t, \mathcal{T}_t, \rho_t)$ is regular, there exists a decreasing $\mathcal{T}_t$ -open set ν_t such that $x_t \in \nu_t \leq D(\nu_t) \leq \mu_t$. So, $x \in P_t^{-1}(\nu_t) \leq P_t^{-1}(D(\nu_t)) \leq \mu$. Hence $(X, \mathcal{T}, \rho)$ is fuzzy regular ordered. □

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