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On The k-Caputo-Fabrizio Fractional Derivative and its Applications

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Abstract: In this paper a generalization of the derivative due to Caputo and Fabrizio in [3] is introduced. We present some useful properties, evaluate its Laplace transform and also obtain the k-fractional integral associated with the new fractional derivative. We will also resolving the k-fractional logistic equation Given by Cerutti [4] with a new fractional operator called on the k-Caputo-Fabrizios fractional derivative with a non-singular kernel. In the same way we will see that when $k = a = 1$ the solution matches with the one given by Camargo and Bruno-Alfonzo [6].

Keywords: Fractional Calculus, Laplace Transform, k-Caputo-Fabrizio fractional derivative.

1. Introduction

As it is well known, in 2015 Caputo and Fabrizio have introduced a new fractional derivative with smooth kernel. Based on the Caputo fractional given by

$$D_{a,t}^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\lambda)^{-\alpha} f^{(1)}(\lambda) d\lambda \tag{1}$$

when $f \in W^{2,1}[a,b]$ and $0 \leq \alpha < 1$. If in (1) by changing the kernel $(t-\lambda)^{-\alpha}$ with the function $e^{-\frac{\alpha}{1-\alpha}t}$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$, we obtain the following new definition of fractional derivative without singular kernel Michele Caputo and Mauro Fabrizio of order α, $0 \leq \alpha < 1$.

$${}^{CF}D_{a,t}^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\lambda)} f^{(1)}(\lambda) d\lambda, \tag{2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ (see [3]). In this paper, by using the k-Gamma function introduced by Díaz and Pariguan [5] and the k-Pochhammer symbol, we present a generalization of the so called Caputo-Fabrizio derivative. To do this will start recalling some definitions and properties.

Definition 1.1. *Let z be a complex number that $R_e(z) > 0$. The k-Gamma function is given by the following integral*

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt \tag{3}$$

The relationship between $\Gamma_k(z)$ and the classical $\Gamma(z)$ is expressed by

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right) \tag{4}$$

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It can be seen that $\Gamma_k(z)$ is such that $\Gamma_k(z) \to \Gamma(z)$ as $k \to 1$. The fractional integral associated with Caputo-Frabrizio fractional derivative is given by (see [6]).

$$ {}^{CF}I_t^{(\alpha)} f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_0^t f(\lambda) d\lambda \tag{5} $$

If f is function such that $f^{(s)}(a) = 0$, $s = 1, 2, \ldots, n$ (see [3])

$$ D^{(n)}({}^{CF}D_{a,t}^{(\alpha)} f(t)) = {}^{CF}D_{a,t}^{(\alpha)}(D^{(n)} f(t)) \tag{6} $$

The Laplace transform of the Caputo-Frabrizio fractional derivative is given by (see [3])

$$ \mathcal{L}\{{}^{CF}D_t^{(\alpha)} f(t)\}(s) = \frac{s\mathcal{L}\{f(t)\}_{(s)} - f(0)}{s + (1-s)\alpha} \tag{7} $$

2. Main Result

Definition 2.1. *Let $f \in W^{1,2}[a,b]$ Sobolev spaces, $0 \le \alpha < 1$. Then, the k-Caputo fractional derivative of order α of the function f is given by*

$$ {}^{C}D_{a,t,k}^{(\alpha)} f(t) = I_{a,t,k}^{1-\alpha} f(t) = \frac{1}{k\Gamma_k(1-\alpha)} \int_a^t (t-\lambda)^{\frac{1-\alpha}{k}-1} f^{(1)}(\lambda) d\lambda \tag{8} $$

where $I_k^{(1-\alpha)} f(t)$ is the k-Riemann-Liuoville fractional integral (see [7, 11]).

If in (8) we replaces $\Gamma_k(1-\alpha) = k^{\frac{1-\alpha}{k}-1}\Gamma(\frac{1-\alpha}{k})$ we obtain

$$ {}^{CF}D_{a,t,k}^{(\alpha)} f(t) = \frac{1}{k^{\frac{1-\alpha}{k}}\Gamma(\frac{1-\alpha}{k})} \int_a^t (t-\lambda)^{-(\frac{\alpha-1}{k}+1)} f^{(1)}(\lambda) d\lambda \tag{9} $$

now if we consider $\frac{\frac{\alpha-1}{k}+1}{1-(\frac{\alpha-1}{k}+1)} = \frac{k}{1-\alpha} - 1$ then, by changing the kernel $(t-\lambda)^{-(\frac{\alpha-1}{k}+1)}$ with the function $e^{-(\frac{k}{1-\alpha}-1)}$ and $\frac{1}{\Gamma(\frac{1-\alpha}{k})}$ with $\frac{M_k(\frac{\alpha}{k})}{\frac{1-\alpha}{k}}$ in (9), we obtain the following

Definition 2.2. *Let $f \in W^{1,2}[a,b]$, $0 \le \alpha < 1$. Then the k-Caputo-Frabrizio fractional derivative of order α of the function f is given by*

$$ {}^{CF}D_{a,t,k}^{(\alpha)} f(t) = \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}} \frac{1-\alpha}{k}} \int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)} f^{(1)}(\lambda) d\lambda \tag{10} $$

where $M_k(\frac{\alpha}{k})$ is normalization function such that $M_k(0) = M_k(1) = 1$.

Note that if $f(t) = c$, $c \in \mathbb{R}$, ${}^{CF}D_{a,t,k}^{(\alpha)} f(t) = 0$ and ${}^{CF}D_{a,t,k}^{(\alpha)} f(t) \to D_{a,t}^{(\alpha)} f(t)$ as $k \to 1$.

Lemma 2.3. *If f is a function such that $f^{(s)}(a) = 0$ con $s = 1, \ldots, n$. Then*

$$ {}^{CF}D_{a,t,k}^{(\alpha)}(D^{(n)} f(t)) = D^{(n)}({}^{CF}D_{a,t,k}^{(\alpha)} f(t)). \tag{11} $$

Proof. We begin considering $n = 1$, then from definition (10), we have

$$ {}^{CF}D_{a,t,k}^{(\alpha)}(D^{(1)} f(t)) = {}^{CF}D_{a,t,k}^{\alpha}(f^{(1)}(t)) = \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}} \frac{1-\alpha}{k}} \int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)} f^{(2)}(\lambda) d\lambda $$

hence, after an integration by part and assuming $f^{(1)}(a) = 0$, we have

$$ \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}} \frac{1-\alpha}{k}} \int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)} f^{(2)}(\lambda) d\lambda = f^(1)(t) \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}} \frac{1-\alpha}{k}} - \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}} \frac{1-\alpha}{k}} (\frac{k}{1-\alpha} - 1) \int_a^t (t-\lambda)^{\frac{1-\alpha}{k}-1} f^{(1)}(\lambda) d\lambda $$

$$ {}^{CF}D^{(\alpha)}_{a,t,k}(D^{(1)}f(t)) = \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\left[f^{(1)}(t) - (\frac{k}{1-\alpha} - 1)\int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)}f^{(1)}(\lambda)d\lambda\right] \tag{12} $$

otherwise

$$ \begin{aligned} D^{(1)}({}^{CF}D^{(\alpha)}_{a,t,k}f(t)) &= \frac{d}{dt}\left[\frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)}f^{(1)}(\lambda)d\lambda\right] \\ &= \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\frac{d}{dt}\left[\int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)}f^{(1)}(\lambda)d\lambda\right] \end{aligned} $$

By applying the Leibniz rule

$$ D^{(1)}({}^{CF}D^{(\alpha)}_{a,t,k}f(t)) = \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\left[f^{(1)}(t) - (\frac{k}{1-\alpha} - 1)\int_a^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)}f^{(1)}(\lambda)d\lambda\right] \tag{13} $$

thus, from (12) and (13) it results

$$ {}^{CF}D^{(\alpha)}_{a,t,k}(D^{(1)}f(t)) = D^{(1)}({}^{CF}D^{(\alpha)}_{a,t,k}f(t)). $$

□

It is easy to generalize the proof for any $n \geq 2$.

Lemma 2.4. *Let f be a sufficiently well-behaved function and let α be a real number, $0 \leq \alpha < 1$. The Laplace transform of the k-Caputo-Frabrizio fractional derivative of the function f is given by*

$$ \mathcal{L}\left\{{}^{CF}D^{(\alpha)}_{t,k}f(t)\right\}(s) = \frac{M_k(\frac{\alpha}{k})(sF(s) - f(0))}{k^{\frac{1-\alpha}{k}-1}(s + (1-s)\alpha + k - 1)}. \tag{14} $$

Proof. By using the definition (10) and taking into account some of their basic properties

$$ \begin{aligned} \mathcal{L}\left\{{}^{CF}D^{(\alpha)}_{t,k}f(t)\right\}(s) &= \mathcal{L}\left\{\frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\int_0^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)}f^{(1)}(\lambda)d\lambda\right\}(s) \\ &= \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\mathcal{L}\left\{\int_0^t e^{-(\frac{k}{1-\alpha}-1)(t-\lambda)}f^{(1)}(\lambda)d\lambda\right\}(s) \\ &= \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\mathcal{L}\left\{e^{-(\frac{k}{1-\alpha}-1)t} * f^{(1)}(t)\right\}(s) \\ &= \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}\mathcal{L}\left\{f^{(1)}(t)\right\}(s)\mathcal{L}\left\{e^{-(\frac{k}{1-\alpha}-1)t}\right\}(s) \\ &= \frac{M_k(\frac{\alpha}{k})}{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}(sF(s) - f(0))\frac{1}{s + (\frac{k}{1-\alpha}) - 1} \\ \mathcal{L}\left\{{}^{CF}D^{(\alpha)}_{t,k}f(t)\right\}(s) &= \frac{M_k(\frac{\alpha}{k})(sF(s) - f(0))}{k^{\frac{1-\alpha}{k}-1}(s + (1-s)\alpha + k - 1)} \end{aligned} \tag{15} $$

where $\mathcal{L}\{f(t)\}(s) = F(s)$. □

Note that if in (15) $k \to 1$ we have (7).

3. The Associated k-Fractional Integral

Consider now the following fractional differential equation

$$ {}^{CF}D^{(\alpha)}_{t,k}f(t) = u(t), \quad t \geq 0 \tag{16} $$

By applying the Laplace transform to the equation (16) we obtain

$$\mathcal{L}\left\{{}^{CF}D_{t,k}^{(\alpha)}f(t)\right\}(s) = \mathcal{L}\left\{u(t)\right\}(s)$$

that is, taking into account (14), we have

$$\frac{M_k(\frac{\alpha}{k})(sF(s)-f(0))}{k^{\frac{1-\alpha}{k}-1}(s+(1-s)\alpha+k-1)} = \mathcal{L}\left\{u(t)\right\}(s) \tag{17}$$

$$M_k\left(\frac{\alpha}{k}\right)(sF(s)-f(0)) = \mathcal{L}\left\{u(t)\right\}(s)k^{\frac{1-\alpha}{k}-1}(s+(1-s)\alpha+k-1) \tag{18}$$

thus, from (17) and (18) it results

$$\begin{aligned} F(s) &= \frac{\mathcal{L}\left\{u(t)\right\}(s)k^{\frac{1-\alpha}{k}-1}(s+(1-s)\alpha+k-1)}{sM_k(\frac{\alpha}{k})} + \frac{f(0)}{s} \\ &= \frac{f(0)}{s} + \mathcal{L}\left\{u(t)\right\}(s)\left(\frac{k^{\frac{1-\alpha}{k}-1}s(1-\alpha)}{sM_k(\frac{\alpha}{k})} + \frac{k^{\frac{1-\alpha}{k}-1}(k+\alpha-1)}{sM_k(\frac{\alpha}{k})}\right) \\ &= \frac{f(0)}{s} + \mathcal{L}\left\{u(t)\right\}(s)\left(\frac{k^{\frac{1-\alpha}{k}-1}(1-\alpha)}{M_k(\frac{\alpha}{k})} + \frac{k^{\frac{1-\alpha}{k}-1}(k+\alpha-1)}{sM_k(\frac{\alpha}{k})}\right) \end{aligned} \tag{19}$$

By applying the inverse Laplace transform to the equation (19) and taking into account some of their basic properties, it results

$$\begin{aligned} f(t) &= f(0)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + \frac{k^{\frac{1-\alpha}{k}-1}(1-\alpha)}{M_k(\frac{\alpha}{k})}u(t) + \frac{k^{\frac{1-\alpha}{k}-1}(k+\alpha-1)}{M_k(\frac{\alpha}{k})}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{u(t)\right\}\right\}(t) \\ &= f(0) + \frac{k^{\frac{1-\alpha}{k}-1}(1-\alpha)}{M_k(\frac{\alpha}{k})}u(t) + \frac{k+\alpha-1}{M_k(\frac{\alpha}{k})}\int_0^t u(s)ds \end{aligned}$$

In other words, the function defined as

$$f(t) = c + \frac{k^{\frac{1-\alpha}{k}-1}(1-\alpha)}{M_k(\frac{\alpha}{k})}u(t) + \frac{k^{\frac{1-\alpha}{k}-1}(k+\alpha-1)}{M_k(\frac{\alpha}{k})}\int_0^t u(s)ds \tag{20}$$

where $c \in \mathbb{R}$ is a constant, is also a solution of (16). Thus, we consequence, we expect that the k-Caputo-Frabrizio fractional integral type must be defined as follows

Definition 3.1. *Let $0 \leq \alpha < 1$. Then the k-Caputo-Frabrizio fractional integral of order α of a function f is given by*

$${}^{CF}I_k^{(\alpha)}f(t) = \frac{k^{\frac{1-\alpha}{k}-1}k-1+\alpha}{M_k\left(\frac{\alpha}{k}\right)}\int_0^t u(\lambda)d\lambda + \frac{k^{\frac{1-\alpha}{k}}\frac{1-\alpha}{k}}{M_k\left(\frac{\alpha}{k}\right)}u(t). \tag{21}$$

Note that ${}^{CF}I_k^{(\alpha)}f(t) \to {}^{CF}I^{(\alpha)}$ as $k \to 1$.

4. Logistic Equation with Caputo-Frabrizio Derivative

The logistic equation was first published in 1838 by Pierre Franois Verhulst to exemplify the increasing world population based on the available statistics, this exemplify is closely related to the exponential growing, studied afterwards by Thomas Robert Multhus. The logistic equation can be applied to models dependent on time and covers a vast area of application, such as, the population increase, epidemic diseases spreading and social networks broadcasting among others. In this paper we are solving the k-fractional logistic equation given by Cerruti with a new fractional operator with a non-singular nucleus defined by Caputo-Frabrizio and generalized by Pablo I. Pucheta. After the considerations made by Camargo and Bruno-Alfonso in ([6]), we present the equation.

$$^{CF}D^{(\alpha)}f(t) = \lambda[1 - f(t)] \tag{22}$$

Where $^{CF}D^{(\alpha)}$ denote the Caputo-Fabrizio fractional derivative introduced in (1). By applying the Laplace transform to the equation (22) taking into account some of their basic properties and $M(\alpha) = 1$, we have

$$\mathcal{L}\left\{^{CF}D_t^{(\alpha)}f(t)\right\}(s) = \frac{sF(s) - f(0)}{s + (1-s)\alpha} \tag{23}$$

where $\mathcal{L}(f) = F(s)$

$$\mathcal{L}\left\{\lambda[1 - f(t)]\right\}(s) = \lambda(\frac{1}{s} - F(s)) \tag{24}$$

Thus, from (23) and (24) we have

$$\begin{aligned}
\frac{sF(s) - f(0)}{s + (1-s)\alpha} &= \lambda\left(\frac{1}{s} - F(s)\right) \\
sF(s) - f(0) &= (s + (1-s)\alpha)\lambda s^{-1} - (s + (1-s)\alpha)\lambda F(s) \\
sF(s) + (s + (1-s)\alpha)\lambda F(s) &= (s + (1-s)\alpha)\lambda s^{-1} + f(0) \\
F(s)[s + \lambda s + \alpha\lambda - \alpha\lambda s] &= \lambda + (1-s)\alpha\lambda s^{-1} + f(0) \\
F(s)[s(1 + \lambda - \alpha\lambda) + \alpha\lambda] &= \lambda + (1-s)\alpha\lambda s^{-1} + f(0) \\
F(s) &= \frac{\lambda + (1-s)\alpha\lambda s^{-1} + f(0)}{s(1 + \lambda - \alpha\lambda) + \alpha\lambda} \\
F(s) &= \frac{\frac{\lambda + (1-s)\alpha\lambda s^{-1} + f(0)}{1 + \lambda - \alpha\lambda}}{s + \frac{\alpha\lambda}{1+\lambda-\alpha\lambda}}
\end{aligned} \tag{25}$$

Distributing in (25) and considering that

$$\frac{1}{s + \frac{\alpha\lambda}{1+\lambda-\alpha\lambda}} = \mathcal{L}\left\{e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}}\right\}(s) \tag{26}$$

$$\frac{\frac{-\alpha\lambda}{1+\lambda-\alpha\lambda}}{(s-0)(s - \frac{-\alpha\lambda}{1+\lambda-\alpha\lambda})} = \mathcal{L}\left\{e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}} - 1\right\}(s) \tag{27}$$

Thus, from (26) and (27), we have

$$F(s) = \frac{\lambda}{1+\lambda-\alpha\lambda}\mathcal{L}\left\{e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}}\right\}(s) - \mathcal{L}\left\{e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}} - 1\right\}(s) + \frac{\alpha\lambda}{1+\lambda-\alpha\lambda}\mathcal{L}\left\{e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}}\right\}(s) + \frac{f(0)}{1+\lambda-\alpha\lambda}\mathcal{L}\left\{e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}}\right\}(s) \tag{28}$$

If in (28) applying the inverse Laplace transform it results

$$f(t) = e^{\frac{-\alpha\lambda t}{1+\lambda-\alpha\lambda}}\left(\frac{\lambda}{1+\lambda-\alpha\lambda} - 1 - \frac{\alpha\lambda}{1+\lambda-\alpha\lambda} + \frac{f(0)}{1+\lambda-\alpha\lambda}\right) + 1 \tag{29}$$

Note that if $\alpha = 1$, then (29) coincides with classical logistic equation.

5. Logistic Equation with the k-Caputo-Frabrizio Derivative

After the considerations made by Cerutti in ([4]), we presented the equation.

$$^{CF}D_k^{(\alpha)}f(t) = k^{\frac{\alpha-1}{k}}\lambda[1 - f(t)] \tag{30}$$

By applying the Laplace transform to the equation (30) taking into account some of their basic properties and $M(\frac{\alpha}{k}) = 1$, we have

$$\mathcal{L}\left\{{}^{CF}D_k^{(\alpha)} f(t)\right\}(s) = \frac{sF(s) - f(0)}{k^{\frac{1-\alpha}{k}} k^{-1}(s + (1-s)\alpha + k - 1)} \tag{31}$$

where $\mathcal{L}(f) = F(s)$

$$\mathcal{L}\left\{k^{\frac{\alpha-1}{k}} \lambda[1 - f(t)]\right\}(s) = k^{\frac{\alpha-1}{k}} \lambda\left(\frac{1}{s} - F(s)\right) \tag{32}$$

Thus, from (31) and (22) we have

$$\frac{sF(s) - f(0)}{k^{\frac{1-\alpha}{k}} k^{-1}(s + (1-s)\alpha + k - 1)} = k^{\frac{\alpha-1}{k}} \lambda\left(\frac{1}{s} - F(s)\right) = k^{\frac{-(1-\alpha)}{k}} \lambda(\frac{1}{s} - F(s)) \tag{33}$$

$$\frac{k(sF(s) - f(0))}{(s + (1-s)\alpha + k - 1)} = \lambda\left(\frac{1}{s} - F(s)\right) \tag{34}$$

If in (34) we clear $F(s)$, we obtain

$$F(s) = \frac{\frac{\left((s+(1-s)\alpha+(k-1))\lambda s^{-1} + kf(0)\right)}{k+\lambda-\alpha\lambda}}{s + \frac{\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}} \tag{35}$$

$$\begin{aligned} F(s) = {} & \frac{\lambda}{k+\lambda-\alpha\lambda} \frac{1}{s + \frac{\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}} - \frac{\frac{-\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}}{(s-0)\left(s - \frac{-\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}\right)} \\ & - \frac{\alpha\lambda}{k+\lambda-\alpha\lambda} \frac{1}{s + \frac{\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}} + \frac{f(0)}{k+\lambda-\alpha\lambda} \frac{1}{s + \frac{\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}} \end{aligned} \tag{36}$$

Considering that

$$\frac{1}{s + \frac{\alpha\lambda+(k-1)\lambda}{k+\lambda-\alpha\lambda}} = \mathcal{L}\left\{e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}}\right\}(s) \tag{37}$$

$$\frac{\frac{-(\alpha\lambda+(k-1)\lambda)}{k+\lambda-\alpha\lambda}}{(s-0)\left(s - \frac{-(\alpha\lambda+(k-1)\lambda)}{k+\lambda-\alpha\lambda}\right)} = \mathcal{L}\left\{e^{\frac{-(\alpha\lambda+(k-1)\lambda)t}{k+\lambda\alpha\lambda}} - 1\right\}(s) \tag{38}$$

and if replaced (37), (38) in (36), it result

$$\begin{aligned} F(s) \; = \; & \frac{\lambda}{k+\lambda-\alpha\lambda}\mathcal{L}\left\{e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}}\right\}(s) - \mathcal{L}\left\{e^{\frac{-(\alpha\lambda+(k-1)\lambda)t}{k+\lambda\alpha\lambda}} - 1\right\}(s) \\ & - \frac{\alpha\lambda}{k+\lambda-\alpha\lambda}\mathcal{L}\left\{e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}}\right\}(s) + \frac{f(0)}{k+\lambda-\alpha\lambda}\mathcal{L}\left\{e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}}\right\}(s) \end{aligned} \tag{39}$$

Applying inverse Laplace transform in (39), we have

$$f(t) = \frac{\lambda}{k+\lambda-\alpha\lambda} e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}} - e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}} + 1 - \frac{\alpha\lambda}{k+\lambda-\alpha\lambda} e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}} + \frac{f(0)}{k+\lambda-\alpha\lambda} e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}} \tag{40}$$

$$f(t) = e^{\frac{-(\alpha\lambda(k-1)\lambda)t}{k+\lambda-\alpha\lambda}} \left(\frac{\lambda}{k+\lambda-\alpha\lambda} - 1\frac{\alpha\lambda}{k+\lambda-\alpha\lambda} + \frac{f(0)}{k+\lambda-\alpha\lambda}\right) + 1 \tag{41}$$

Note that if $k = 1$ (41) coincide with (29).

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