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Initial Coefficients Bounds for an Unified Class of Meromorphic Bi-univalent Functions

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Abstract: In this paper, we introduce and investigate a new unified class of meromorphic bi-univalent functions defined on the domain $\{z \in \mathbb{C} : 1 < |z| < \infty\}$, which are associated with meromorphic functions. We find estimates on the initial Taylor-MacLaurin coefficients for functions in these subclasses. Several new consequences of these results are also pointed out in the form of corollaries.

MSC: 30C45.

Keywords: Analytic function, Meromorphic function, Bi-Univalent functions, Coefficient estimates.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all normalized functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \tag{1}$$

which are analytic in the open unit disk, $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$. Then clearly, every $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{2}$$

A function $f \in \mathcal{A}$ given by (1.1) is said to be bi-univalent in $\mathbb{U}$ if both $f$and f^{-1} are univalent in $\mathbb{U}$. The class of these bi-univalent functions is denoted by Σ. Followed by Brannan and Taha [1] (see also [2, 18]), Srivastava et al. [17] and many

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other researchers (viz [3–5, 7, 12, 19]) introduced certain sub classes of bi-univalent function class Σ and obtained the bounds on initial Taylor-MacLaurin coefficients. Let $\mathcal{S}'$ denote the class of meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{3}$$

defined on the domain $\mathbb{U}^* = \{z : z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $g \in \mathcal{S}'$ is univalent, it has an inverse say g^{-1}, which satisfies

$$g^{-1}(g(z)) = z \quad (z \in \mathbb{U}^*)$$

and

$$g(g^{-1}(w)) = w, \quad (M < |w| < \infty; M > 0).$$

Moreover, the inverse function $g^{-1} = h$ has a series expansion of the form

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{c_n}{w^n}, \quad (M < |w| < \infty). \tag{4}$$

A function $g \in \mathcal{S}'$ is said to be meromorphic bi-univalent if both g and g^{-1} are meromorphic univalent in $\mathbb{U}^*$. Let $\Sigma^{'}$ denote the class of all meromorphic bi-univalent functions in $\mathbb{U}^*$ given by (3). A simple calculation shows that,

$$g^{-1}(w) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + ... \tag{5}$$

Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [14] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma^{'}$ with $b_0 = 0$ and Duren [6] proved that $|b_n| \leq \frac{2}{(n+1)}$ for $g \in \Sigma^{'}$ with $b_k = 0, 1 \leq k \leq \frac{n}{2}$. For the coefficients of inverse meromorphic univalent function $g^{-1}(w)$, Springer [16], proved that

$$|c_3| \leq 1 \; and \; |c_3 + \frac{c_1^2}{2}| \leq \frac{1}{2}$$

and conjectured that

$$|c_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!}, \qquad n = 1, 2, 3...$$

In 1997, Kubota[10] proved that springer conjecture is true for n= 3,4,5 and subsequently schober[15] obtained a sharp bounds for the coefficients $c_{2n-1}, 1 \leq n \leq 7$. Recently Panigrahi [11], Bulut [4], Hamidi et al. [8] , Jannani and murugusundaramoorthy [9] and many others have introduced and investigated subclasses of meromorphically bi-univalent functions class $\Sigma^{'}$ and obtained bounds for their initial coefficients.In the present paper certain new subclass $R^{p,q}_{\Sigma'}(\tau, \alpha, \gamma)$ of the function class $\Sigma^{'}$ is introduced and by using the technique of Xu et al. [20] estimates on the coefficients $|b_0|$, $|b_1|$ and $|b_2|$ are obtained for function $g(z)$ belonging to the class $R^{p,q}_{\Sigma'}(\tau, \alpha, \gamma)$.

Definition 1.1. *Let the functions p, q : $\mathbb{U}^* \to \mathbb{C}$ be analytic functions defined as*

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + ...$$
$$q(z) = 1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \frac{q_3}{z^3} + ...$$

such that $\min\{Re(p(z)), Re(q(z))\} > 0$, $z \in \mathbb{U}^*$. *For* $0 \leq \alpha < 1$ *and* $\gamma \geq 0$, *a function* $g \in \Sigma^{'}$ *given by* (3) *is said to be in the class* $R^{p,q}_{\Sigma'}(\tau, \alpha, \gamma)$, *if the following conditions are satisfied:*

$$1 + \frac{1}{\tau}[(1-\alpha)\frac{g(z)}{z} + \alpha g^{'}(z) + \gamma z g^{''}(z) - 1] \in p(\mathbb{U}^*) \quad (z \in \mathbb{U}^*) \tag{6}$$

and

$$1+\frac{1}{\tau}[(1-\alpha)\frac{h(w)}{w}+\alpha h^{'}(w)+\gamma wh^{''}(w)-1]\in q(\mathbb{U}^{*})\quad (w\in\mathbb{U}^{*}) \tag{7}$$

where $\tau\in\mathbb{C}\backslash\{0\}$ *and the function* $h=g^{-1}$ *is given by* (5).

On different selecting of the functions $p(z)$, $q(z)$ and involved parameters α,τ,γ one can state the various new subclasses of $\Sigma^{'}$ some of them are illustrated in the following examples. Thus, selecting

$$p(z)=q(z)=\left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\lambda}=1+\frac{2\lambda}{z}+\frac{2\lambda^2}{z^2}+\frac{2\lambda^3}{z^3}+...\ (0<\lambda\leq 1, z\in\mathbb{U}^{*}),$$

in the above definition, we obtain

Example 1.2. *Suppose that* $0\leq\alpha<1$, $\gamma\geq 0$, $0<\lambda\leq 1$ *and* $\tau\in\mathbb{C}\backslash\{0\}$. *A function* $g(z)\in\Sigma^{'}$ *is said to be in the class* $R^{\lambda}_{\Sigma'}(\tau,\alpha,\gamma)$ *if the following conditions are satisfied:*

$$\left|arg\left\{1+\frac{1}{\tau}[(1-\alpha)\frac{g(z)}{z}+\alpha g^{'}(z)+\gamma zg^{''}(z)-1]\right\}\right|<\frac{\lambda\pi}{2}\quad (z\in\mathbb{U}^{*})$$

and

$$\left|arg\left\{1+\frac{1}{\tau}[(1-\alpha)\frac{h(w)}{w}+\alpha h^{'}(w)+\gamma wh^{''}(w)-1]\right\}\right|<\frac{\lambda\pi}{2}\quad (w\in\mathbb{U}^{*}).$$

Selecting $p(z)=q(z)=\frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}}=1+\frac{2(1-\beta)}{z}+\frac{2(1-\beta)}{z^2}+\frac{2(1-\beta)}{z^3}+...\ (0\leq\beta<1, z\in\mathbb{U}^{*})$, in the Definition 1.1, we obtain

Example 1.3. *Suppose that* $0\leq\alpha,\beta<1$, $\gamma\geq 0$, *and* $\tau\in\mathbb{C}\backslash\{0\}$. *A function* $g(z)\in\Sigma^{'}$ *is said to be in the class* $T^{\beta}_{\Sigma'}(\tau,\alpha,\gamma)$ *if the following conditions are satisfied:*

$$Re\left\{1+\frac{1}{\tau}[(1-\alpha)\frac{g(z)}{z}+\alpha g^{'}(z)+\gamma zg^{''}(z)-1]\right\}>\beta\quad (z\in\mathbb{U}^{*})$$

and

$$Re\left\{1+\frac{1}{\tau}[(1-\alpha)\frac{h(w)}{w}+\alpha h^{'}(w)+\gamma wh^{''}(w)-1]\right\}>\beta\quad (w\in\mathbb{U}^{*}).$$

In the following section, we find estimates of the coefficients $|b_0|$, $|b_1|$ and $|b_2|$ for the function $g(z)$ belonging to the class $R^{p;q}_{\Sigma'}(\tau,\alpha,\gamma)$ by employing the techniques of [13, 20].

2. Coefficient Bounds for the Function Class $R^{p;q}_{\Sigma'}(\tau,\alpha,\gamma)$

Theorem 2.1. *Let the function* $g(z)\in\Sigma^{'}$ *given by (1.3) be in the class* $R^{p;q}_{\Sigma'}(\tau,\alpha,\gamma)$. *Then*

$$|b_0|\leq\min\left\{\frac{|\tau|(|p_1|+|q_1|)}{2(1-\alpha)},\frac{|\tau|}{1-\alpha}\sqrt{\frac{|p_1|^2+|q_1|^2}{2}}\right\}, \tag{8}$$

$$|b_1|\leq\min\left\{\frac{|\tau|(|p_2|+|q_2|)}{2|1-2\alpha+2\gamma|},\frac{|\tau|}{|1-2\alpha+2\gamma|}\sqrt{\frac{|p_2|^2+|q_2|^2}{2}}\right\} \tag{9}$$

$$|b_2|\leq\frac{|\tau||p_3|}{|1-3\alpha+6\gamma|} \tag{10}$$

Proof. First of all, we write the argument inequalities in (1.6) and (1.7) in their equivalent form as follows:

$$1+\frac{1}{\tau}[(1-\alpha)\frac{g(z)}{z}+\alpha g^{'}(z)+\gamma zg^{''}(z)-1]=p(z)\quad (z\in\mathbb{U}^{*}) \tag{11}$$

and

$$1+\frac{1}{\tau}[(1-\alpha)\frac{h(w)}{w}+\alpha h^{'}(w)+\gamma wh^{''}(w)-1]=q(w) \quad (w\in\mathbb{U}^*), \tag{12}$$

respectively, where functions $p(z)$ and $q(w)$ satisfy the conditions of Definition 1.1. Now,upon equating the coefficients of

$$1+\frac{1}{\tau}[(1-\alpha)\frac{g(z)}{z}+\alpha g^{'}(z)+\gamma zg^{''}(z)-1]=1+\frac{1}{\tau}[(1-\alpha)\frac{b_0}{z}+(1-2\alpha+2\gamma)\frac{b_1}{z^2}+(1-3\alpha+6\gamma)\frac{b_2}{z^3}+(1-4\alpha+12\gamma)\frac{b_3}{z^4}+...] \tag{13}$$

with those of $p(z)$ and coefficients of

$$\begin{aligned}1+\frac{1}{\tau}[(1-\alpha)\frac{h(w)}{w}+\alpha h^{'}(w)+\gamma wh^{''}(w)-1]=1+\frac{1}{\tau}\Big[-(1-\alpha)\frac{b_0}{w}-(1-2\alpha+2\gamma)\frac{b_1}{w^2}-(1-3\alpha+6\gamma)\frac{b_0+b_0b_1}{w^3}\\ -(1-4\alpha+12\gamma)\frac{b_3+2b_0b_1+b_0^2b_1+b_1^2}{w^4}+...\Big]\end{aligned} \tag{14}$$

with those of $q(w)$, we get

$$\frac{1}{\tau}(1-\alpha)b_0=p_1 \tag{15}$$

$$\frac{1}{\tau}(1-2\alpha+2\gamma)b_1=p_2 \tag{16}$$

$$\frac{1}{\tau}(1-3\alpha+6\gamma)b_2=p_3 \tag{17}$$

$$-\frac{1}{\tau}(1-\alpha)b_0=q_1 \tag{18}$$

$$-\frac{1}{\tau}(1-2\alpha+2\gamma)b_1=q_2 \tag{19}$$

$$-\frac{1}{\tau}(1-3\alpha+6\gamma)(b_2+b_0b_1)=q_3 \tag{20}$$

from (15) and (18), we get

$$\frac{2}{\tau^2}(1-\alpha)^2b_0^2=p_1^2+q_1^2 \tag{21}$$

$$|b_0|^2\le\frac{|\tau|^2(|p_1|^2+|q_1|^2)}{2(1-\alpha)^2}$$

$$\frac{2}{\tau}(1-\alpha)b_0=p_1-q_1 \tag{22}$$

$$|b_0|\le\frac{|\tau|(|p_1|+|q_1|)}{2(1-\alpha)}$$

so we get the desired estimates on the coefficients $|b_0|$ as asserted in (8). Next, in order to find the bound on the coefficient $|b_1|$, we subtract (19) from (16), then

$$\frac{2}{\tau}(1-2\alpha+2\gamma)b_1=p_2-q_2 \tag{23}$$

by squaring and adding (16) and (19), computation leads to

$$b_1^2=\frac{\tau^2(p_2^2+q_2^2)}{2(1-2\alpha+2\gamma)^2}. \tag{24}$$

Therefore, we find from the equations (23) and (24) that

$$|b_1|\le\frac{|\tau|(|p_2|+|q_2|)}{2|1-2\alpha+2\gamma|}$$

and

$$|b_1|\le\frac{|\tau|}{|1-2\alpha+2\gamma|}\sqrt{\frac{|p_2|^2+|q_2|^2}{2}}.$$

Finally, to determine the bound on $|b_2|$, consider the sum of (17) and (20)

$$\frac{1}{\tau}(1-3\alpha+6\gamma)b_0b_1 = p_3+q_3 \tag{25}$$

subtracting (20) from (17), we obtain

$$\frac{1}{\tau}(1-3\alpha+6\gamma)(2b_2+b_0b_1) = p_3-q_3 \tag{26}$$

using (25) in (26), we get

$$b_2 = \frac{\tau p_3}{1-3\alpha+6\gamma}.$$

This evidently completes the proof of Theorem 2.1. □

By setting $p(z)=q(z)=\left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\lambda} = 1+\frac{2\lambda}{z}+\frac{2\lambda^2}{z^2}+\frac{2\lambda^3}{z^3}+\ldots$ $(0<\lambda\le 1, z\in\mathbb{U}^*)$, in Theorem 2.1, we conclude the following results:

Corollary 2.2. *Let the function $g(z)$ given by (3) be in the class $R^{\lambda}_{\Sigma'}(\tau,\alpha,\gamma)$, then*

$$|b_0| \le \frac{2|\tau|\lambda}{1-\alpha},$$
$$|b_1| \le \frac{2|\tau|\lambda^2}{|1-2\alpha+2\gamma|},$$
$$|b_2| \le \frac{2|\tau|\lambda^3}{|1-3\alpha+6\gamma|}.$$

Next, by setting $p(z)=q(z)=\frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}} = 1+\frac{2(1-\beta)}{z}+\frac{2(1-\beta)}{z^2}+\frac{2(1-\beta)}{z^3}+\ldots$ $(0\le\beta<1, z\in\mathbb{U}^*)$, in Theorem 2.1, we conclude the following results:

Corollary 2.3. *Let the function $g(z)$ given by (1.3) be in the class $T^{\beta}_{\Sigma'}(\tau,\alpha,\gamma)$, then*

$$|b_0| \le \frac{2|\tau|(1-\beta)}{1-\alpha},$$
$$|b_1| \le \frac{2|\tau|(1-\beta)}{|1-2\alpha+2\gamma|},$$
$$|b_2| \le \frac{2|\tau|(1-\beta)}{|1-3\alpha+6\gamma|}.$$

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