Int. J. Math. And Appl., 6(1–B)(2018), 403–406
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Nesbitt Type Inequalities

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Abstract: In 2009 Wei and Wu introduced an n-variable version of Nesbitt's inequality. In this paper we provide a different proof using Radon's inequality. We also use the same technique to prove several of its variations.

MSC: 26D15

Keywords: Nesbitt's Inequality, Radon's Inequality.

1. Introduction

In 1903, Nesbitt introduced in [4] his famous inequality: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ for all positive real numbers a,b,c, with equality holds when $a=b=c$. This inequality was then applied to prove many other mathematical inequalities with sums of fractions. In 2009 Wei and Wu introduced the following generalizations in [5]: $\frac{x}{ky+z}+\frac{y}{kz+x}+\frac{z}{kx+y}\geq\frac{3}{1+k}$ for positive real numbers x,y,z,k; and $\frac{x_1}{x_2+x_3+\cdots+x_n}+\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}+\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}\geq\frac{n}{n-1}$ for positive real numbers $x_1,x_2,\cdots,x_n$ when $n\geq 2$. In that paper Wei and Wu used Cauchy-Schwarz inequality to prove the first generalization, and used Chebyshev's inequality to prove the second generalization. In this paper we provide a different proof of these results, and extend the second result even further to the case of more than one element in the numerator. Before we start our main results, we shall introduce the inequalities we applied in our proofs.

Theorem 1.1 (Radon's Inequality)**.** *Let $a_1,a_2,\cdots,a_n$ and $b_1,b_2,\cdots,b_n$ be positive real numbers. If p is also a positive real number, then*

$$\frac{a_1^{p+1}}{b_1^p}+\frac{a_2^{p+1}}{b_2^p}+\cdots+\frac{a_n^{p+1}}{b_n^p}\geq\frac{(a_1+a_2+\cdots+a_n)^{p+1}}{(b_1+b_2+\cdots+b_n)^p}.$$

The equality occurs when $n=1$ or when $a_i=b_i$ for all i.

Though useful, the proof of this inequality, together with its other applications are not related to our results hence are omitted here. Interested readers may check [1] for those information.

Theorem 1.2 (Rearrangement Inequality)**.** *Let $a_1\leq a_2\leq\cdots\leq a_n$ and $b_1\leq b_2\leq\cdots\leq b_n$ be real numbers. For any permutation $(x_1,x_2,\cdots,x_n)$ of $(a_1,a_2,\cdots,a_n)$ we have the following inequalities:*

$$a_1b_1+a_2b_2+\cdots+a_nb_n\geq x_1b_1+x_2b_2+\cdots+x_nb_n\geq a_nb_1+a_{n-1}b_2+\cdots+a_1b_n.$$

The equality occurs when $n=1$ or when $a_i=b_i$ for all i.

Similarly, interested readers may check [2] or [3] for its proof and other applications.

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2. Main Results

Our first theorem is a slight generalization of Theorem 1 in [5]. Instead of just one coefficient in the denominator, we proved the case of two coefficients.

Theorem 2.1. *Let a, b, x, y, z be positive real numbers. Then*

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

Proof. Applying Radon's inequality we have

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} = \frac{x^2}{axy+bxz} + \frac{y^2}{ayz+bxy} + \frac{z^2}{axz+byz}$$

$$\geq \frac{(x+y+z)^2}{(a+b)(xy+yz+zx)} \geq \frac{3}{a+b}.$$

The last inequality is true due to rearrangement inequality, $x^2+y^2+z^2 \geq xy+yz+zx$. The equality occurs when $x=y=z$. □

The next theorem was introduced by Wei and Wu in [5] as Theorem 2. We apply Radon's inequality and provide a different proof.

Theorem 2.2 (Wei and Wu)**.** *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, where $n \geq 2$. Then*

$$\frac{x_1}{x_2+x_3+\cdots+x_n} + \frac{x_2}{x_1+x_3+x_4+\cdots+x_n} + \cdots + \frac{x_n}{x_1+x_2+\cdots+x_{n-1}} \geq \frac{n}{n-1}.$$

Proof. Applying Radon's inequality again, we have

$$\begin{aligned}
&\frac{x_1}{x_2+x_3+\cdots+x_n} + \frac{x_2}{x_1+x_3+x_4+\cdots+x_n} + \cdots + \frac{x_n}{x_1+x_2+\cdots+x_{n-1}} \\
&= \frac{x_1^2}{x_1x_2+x_1x_3+\cdots+x_1x_n} + \frac{x_2^2}{x_1x_2+x_2x_3+x_2x_4+\cdots+x_2x_n} + \cdots + \frac{x_n^2}{x_1x_n+x_2x_n+\cdots+x_{n-1}x_n} \\
&\geq \frac{(x_1+\cdots+x_n)^2}{2\left[(x_1x_2+x_1x_3+\cdots+x_1x_n)+(x_2x_3+x_2x_4+\cdots+x_2x_n)+\cdots+(x_{n-1}x_n)\right]} \\
&= 1 + \frac{x_1^2+x_2^2+\cdots+x_n^2}{2\left[(x_1x_2+x_1x_3+\cdots+x_1x_n)+(x_2x_3+x_2x_4+\cdots+x_2x_n)+\cdots+(x_{n-1}x_n)\right]}.
\end{aligned}$$

According to arrangement inequality, we know that

$$(n-1)\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i x_{i+1} + \sum_{i=1}^{n} x_i x_{(i+2)} + \cdots + \sum_{i=1}^{n} x_i x_{i+(n-1)},$$

in which we understand $x_{(n+k)}$ as x_k for any k. Therefore,

$$(n-1)\sum_{i=1}^{n} x_i^2 \geq 2\left[(x_1x_2+x_1x_3+\cdots+x_1x_n)+(x_2x_3+x_2x_4+\cdots+x_2x_n)+\cdots+(x_{n-1}x_n)\right].$$

That means

$$\frac{x_1^2+x_2^2+\cdots+x_n^2}{2\left[(x_1x_2+x_1x_3+\cdots+x_1x_n)+(x_2x_3+x_2x_4+\cdots+x_2x_n)+\cdots+(x_{n-1}x_n)\right]} \geq \frac{1}{n-1},$$

which completes the proof. The equality occurs when all x_i's are equal. □

For the next step, it is very natural to consider the case when the coefficients are added to the denominators like Theorem 2.1. That inequality indeed is still true.

Theorem 2.3. *Let $a_1, a_2, \cdots, a_{n-1}$ and $x_1, x_2, \cdots, x_n$ be positive real numbers, where $n \geq 2$. Then*

$$\frac{x_1}{a_1x_2 + a_2x_3 + \cdots + a_{n-1}x_n} + \frac{x_2}{a_1x_1 + a_2x_3 + a_3x_4 + \cdots + a_{n-1}x_n} + \cdots + \frac{x_n}{a_1x_1 + a_1x_2 + \cdots + a_{n-1}x_{n-1}} \geq \frac{n}{a_1 + \cdots + a_{n-1}}.$$

The proof of the above inequality requires lots of calculation. For an obvious reason, we only show the proof of the 4-variable case here, namely $\frac{x_1}{ax_2+bx_3+cx_4} + \frac{x_2}{ax_3+bx_4+cx_1} + \frac{x_3}{ax_4+bx_1+cx_2} + \frac{x_4}{ax_1+bx_2+cx_3} \geq \frac{4}{a+b+c}$.

Proof. *(4-variable case)* Since the inequality is symmetric, we may assume that $x_1 \geq x_2 \geq x_3 \geq x_4$. Applying Radon's inequality we have

$$\begin{aligned}
&\frac{x_1}{ax_2 + bx_3 + cx_4} + \frac{x_2}{ax_3 + bx_4 + cx_1} + \frac{x_3}{ax_4 + bx_1 + cx_2} + \frac{x_4}{ax_1 + bx_2 + cx_3} \\
&\quad = \frac{x_1^2}{ax_1x_2 + bx_1x_3 + cx_1x_4} + \frac{x_2^2}{ax_2x_3 + bx_2x_4 + cx_1x_2} + \frac{x_3^2}{ax_3x_4 + bx_1x_3 + cx_2x_3} + \frac{x_4^2}{ax_1x_4 + bx_2x_4 + cx_3x_4} \\
&\quad \geq \frac{(x_1 + x_2 + x_3 + x_4)^2}{(a+c)(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) + 2b(x_1x_3 + x_2x_4)}.
\end{aligned}$$

We therefore only need to prove that

$$\frac{(x_1 + x_2 + x_3 + x_4)^2}{(a+c)(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) + 2b(x_1x_3 + x_2x_4)} \geq \frac{4}{a+b+c},$$

or equivalently,

$$(a+b+c)(x_1 + x_2 + x_3 + x_4)^2 - 4(a+c)(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) - 8b(x_1x_3 + x_2x_4) \geq 0.$$

After we simplify the left side of the above inequality we have

$$(a+c)(x_1 - x_2 + x_3 - x_4)^2 + b\left[(x_1 - x_2 - x_3 + x_4)^2 + 4(x_2 - x_3)(x_1 - x_4)\right],$$

which is obviously non-negative. □

In the next result, we generalize Theorem 2.2 to the case of two elements rotated to the numerator. The notation $C(n,2)$ is the 2-combination of a set of n elements, or the binomial coefficient $\begin{pmatrix} n \\ 2 \end{pmatrix}$.

Theorem 2.4. *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, $n \geq 3$. Then*

$$\frac{x_1 + x_2}{x_3 + x_4 + \cdots + x_n} + \frac{x_1 + x_3}{x_2 + x_4 + \cdots + x_n} + \frac{x_2 + x_3}{x_1 + x_4 + \cdots + x_n} + \cdots + \frac{x_{n-1} + x_n}{x_1 + x_2 + \cdots + x_{n-2}} \geq \frac{2C(n,2)}{n-2},$$

in which the numerators of the left side fractions consist of all the combinations of x_i, x_j from $x_1, \cdots, x_n$.

Proof. To prove this inequality, we split each fraction at the left side to two fractions.

$$\begin{aligned}
\frac{x_1 + x_2}{x_3 + x_4 + \cdots + x_n} &= \frac{x_1^2}{x_1x_3 + x_1x_4 + \cdots + x_1x_n} + \frac{x_2^2}{x_2x_3 + x_2x_4 + \cdots + x_2x_n}, \\
\frac{x_1 + x_3}{x_2 + x_4 + \cdots + x_n} &= \frac{x_1^2}{x_1x_2 + x_1x_4 + \cdots + x_1x_n} + \frac{x_3^2}{x_2x_3 + x_3x_4 + \cdots + x_3x_n},
\end{aligned}$$

$$\vdots$$

$$\frac{x_{n-1}+x_n}{x_1+x_2+\cdots+x_{n-2}}=\frac{x_{n-1}^2}{x_1x_{n-1}+x_2x_{n-1}+\cdots+x_{n-2}x_{n-1}}+\frac{x_n^2}{x_1x_n+x_2x_n+\cdots+x_{n-2}x_n}.$$

Summing the above and apply the Radon's inequality again, we have

$$\frac{x_1+x_2}{x_3+x_4+\cdots+x_n}+\frac{x_1+x_3}{x_2+x_4+\cdots+x_n}+\frac{x_2+x_3}{x_1+x_4+\cdots+x_n}+\cdots+\frac{x_{n-1}+x_n}{x_1+x_2+\cdots+x_{n-2}}$$
$$\geq\frac{(n-1)^2\left(x_1+\cdots+x_n\right)^2}{2\left(n-2\right)\left[\left(x_1x_2+x_1x_3+\cdots+x_1x_n\right)+\left(x_2x_3+x_2x_4+\cdots+x_2x_n\right)+\cdots+\left(x_{n-1}x_n\right)\right]}.$$

In our proof of Theorem 2.2, we have already shown that

$$\frac{\left(x_1+\cdots+x_n\right)^2}{2\left[\left(x_1x_2+x_1x_3+\cdots+x_1x_n\right)+\left(x_2x_3+x_2x_4+\cdots+x_2x_n\right)+\cdots+\left(x_{n-1}x_n\right)\right]}\geq\frac{n}{n-1}.$$

Combining the above two inequalities, we conclude that

$$\frac{x_1+x_2}{x_3+x_4+\cdots+x_n}+\frac{x_1+x_3}{x_2+x_4+\cdots+x_n}+\frac{x_2+x_3}{x_1+x_4+\cdots+x_n}+\cdots+\frac{x_{n-1}+x_n}{x_1+x_2+\cdots+x_{n-2}}\geq\frac{n(n-1)}{n-2}=\frac{2C(n,2)}{n-2}.$$

□

The above generalization can be understood this way. In each fraction of the left side, there are two elements at the numerator and $(n-2)$ elements at the denominator, so we have the factor $\frac{2}{n-2}$ at the right side. Totally, there are $C(n,2)$ fractions in the sum of the left side, so that provides the factor $C(n,2)$ at the right side. Following the same logic, we have the next generalization.

Theorem 2.5. *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, let $k<n$ be a positive integer, and let $S(k)_1, S(k)_2, \cdots, S(k)_{C(n,k)}$ be the sums of k elements in $x_1, x_2, \cdots, x_n$ for all $C(n,k)$ combinations respectively. Then*

$$\frac{S(k)_1}{S(n)-S(k)_1}+\frac{S(k)_2}{S(n)-S(k)_2}+\cdots+\frac{S(k)_{C(n,k)}}{S(n)-S(k)_{C(n,k)}}\geq\frac{kC(n,k)}{n-k},$$

where $S(n)=x_1+\cdots+x_n$.

The proof of Theorem 2.5 can be done using the same technique used in the proof of Theorem 2.4, though one needs to count terms very carefully, hence is omitted here.

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