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Nourishing Number of Flower-related Graphs

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Abstract

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathcal{P}(\mathbb{N}_0)$ be the power set. If $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$, then its induced map $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ is defined as $f^+(uv) = f(u) + f(v)$ where $f(u) + f(v)$ is the sumset of $f(u)$ and $f(v)$. If f and f^+ are injective, and $|f^+(uv)| = |f(u)|\,|f(v)|$ for all uv in $E(G)$, then f is a strong integer additive set-indexer of G. The nourishing number of G is the least order of the maximal complete subgraph of G such that G admits a strong IASI. In this work, we compute the nourishing number of powers of flower-related graphs and graphs formed by duplicating each vertex in flower-related graphs by an edge.

Keywords: Flower-related graphs; strong integer additive set-indexers; nourishing number of a graph; sumset.
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1. Introduction

Let G be a simple, finite, connected, and undirected graph. The vertex and edge sets of G are represented by $V(G)$ and $E(G)$ respectively. We refer [6] for graph terminologies and notations. We use [3] and [8] for concepts in graph labeling and sumset respectively. Acharya [1] introduced set-valuation of G and termed set-indexer of G. Later, Germina and Anandavally [4] used the concept of sumsets to introduce the notion of integer additive set-labeling(IASL) and integer additive set indexer(IASI) of G. In the following years, a detailed study on the characteristics of such notions was conducted, which can be found in the review paper [11]. Sudev and Germina [12] introduced a special type of IASI termed a strong IASI and initiated research on finding characteristics of strong IASI graphs. They obtained the necessary and sufficient conditions for various graphs to admit strong IASI. They further investigated the admissibility of strong IASI for several graph classes, graph operations, graph products, and associated graphs in [13, 14]. All work based on strong IASI can be found in the review paper [10]. Sudev and Germina [7, 14] introduced the notion of the nourishing number of a graph and obtained it for different graph classes, graph operations, and graph products. Prajapati and

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Vyas [9] extended this work and obtained the nourishing number for various graph classes and graph powers. In this paper, we compute the nourishing number of powers of flower-related graphs and graphs obtained by duplicating each vertex in flower-related graphs by an edge.

2. Preliminaries

In this section, we go through some definitions and results that form an integral part of this work and will be crucial for better understanding. If $A, B \subset \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $A + B = \{a + b\colon a \in A, b \in B\}$ is the sumset of A and B. For $A \subset \mathbb{N}_0$, A is finite and $|A|$ is its cardinality.

Definition 2.1. *[4] An injection $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ is an integer additive set-indexer (IASI) of a graph G if the induced map $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ given by $f^+(uv) = f(u) + f(v)$ is also an injection. If G has such a map f, then G is called an IASI graph.*

Definition 2.2. *[12] If f is a set-indexer of G and satisfies $|f^+(uv)| = |f(u)||f(v)|$ for all vertices u and v of G, then f is called a strong IASI of G. Such a G is called a strong IASI graph.*

If $A, B \subset \mathbb{N}_0$ and $A, B \neq \phi$, then $A < B$ is used in the sense that $A \cap B = \emptyset$ and the sequence $A_1 < A_2 < A_3 < \ldots < A_n$ conveys that the sets are pairwise disjoint. $D_A = \{|a - b| \;:\; a, b \in A, a \neq b\}$ is the difference set of A.

Lemma 2.3. *[12] If $A, B \subset \mathbb{N}_0$ and $A, B \neq \phi$ then $|A + B| = |A||B| \iff$ the relation $D_A < D_B$ holds.*

Theorem 2.4. *[12] If each vertex v_i of K_n is labeled by the set $A_i \in \mathcal{P}(\mathbb{N}_0)$, then K_n admits a strong IASI $\iff$ for the difference set D_i of the set-label A_i of v_i there exists a finite sequence $D_1 < D_2 < D_3 < \ldots < D_n$.*

Theorem 2.5. *[12] A connected graph G (on n vertices) admits strong IASI if and only if each vertex v_i of G is labeled by a set A_i in $\mathcal{P}(\mathbb{N}_0)$ and there exists a finite sequence $D_1 < D_2 < D_3 < \ldots < D_m$, where $m \leq n$ is a positive integer and D_i is the difference set of A_i.*

Definition 2.6. *[14] The nourishing number of a graph G is the least order of the maximal complete subgraph of G so that G admits a strong IASI. It is represented by $\kappa(G)$.*

Theorem 2.7. *[14]*

(a) $\kappa(G) = n$, if $G = K_n$;

(b) $\kappa(G) = 2$, if G is bipartite or triangle-free.

Definition 2.8. *[2] If $r \in \mathbb{N}$ then r^{th} power of G, represented by G^r, is the graph with $V(G^r) = V(G)$ and $u, v \in V(G^r)$ are adjacent if they are at a distance atmost r in G.*

Theorem 2.9. *[15] If d is the diameter of G, then G^d is complete.*

Definition 2.10. *[3] The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex vertex of the helm.*

Definition 2.11. *[3] A lotus inside a circle Lc_n is the graph obtained from the cycle with consecutive vertices $v_1, v_2, ..., v_n$ and the star graph $K_{1,n}$ with the central vertex v_0 and end vertices $u_1, u_2, ..., u_n$ by joining each u_i to v_i and v_{i+1} ($v_{n+1} = v_1$).*

Definition 2.12. *[3] Let G_n be a simple nontrivial connected cubic graph with $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \leq i \leq n-1\}$, and $E(G_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_ia_i, d_ib_i, d_ic_i : 0 \leq i \leq n-1\}$, where the edge labels are taken modulo n. Let J_n be a graph obtained from G_n by replacing the edges $b_{n-1}b_0$ and $c_{n-1}c_0$ with $b_{n-1}c_0$ and $c_{n-1}b_0$ respectively. For odd $n \geq 5$, J_n is called a flower snark whereas G_n, J_3 and all J_n with even $n \geq 4$, are called the related graphs of a flower snark J_n.*

Definition 2.13. *[3] The sun flower graph SFL_n is obtained by taking a wheel with the apex vertex v_0 and the consecutive rim vertices $v_1, v_2, ..., v_n$ and additional vertices $w_1, w_2, ..., w_n$ such that each w_i is adjacent to v_i and v_{i+1}, where $i+1$ is taken modulo n.*

Definition 2.14. *[3] Duplication of a vertex v_k by a new edge $e = v'_kv''_k$ in a graph G produces a new graph G' such that $N_{G'}(v'_k) = \{v_k, v''_k\}$ and $N_{G'}(v''_k) = \{v_k, v'_k\}$.*

3. Main Results

In this section, we obtain the nourishing number of r^{th} power of following graphs and their duplicated graphs: flower, lotus inside a circle, flower snark, sun flower.

Theorem 3.1.

$$\kappa(Fl_n^r) = \begin{cases} 3, & \textit{if } r = 1 \\ 2n+1, & \textit{if } r \geq 2. \end{cases} \tag{1}$$

Proof. Let $V(Fl_n) = \{u_i : 0 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq n\}$ where u_0 is the apex vertex, and v_j's are the pendant vertices. Because Fl_n has diameter 2, it is complete when $r \geq 2$. Therefore, $\kappa(Fl_n^r) = 2n+1$. If $r = 1$, then the maximal complete subgraph of Fl_n is of order 3 and has the vertex set $\{u_0, u_k, u_{k+1}\}$ (in the sense $u_{n+1} = u_1$) or $\{u_0, u_k, v_k\}$, for fixed k; $1 \leq k \leq n$. So, $\kappa(Fl_n) = 3$. □

Theorem 3.2. *If G is a graph obtained by duplication of every vertex by an edge in Fl_n, then*

$$\kappa(G^r) = \begin{cases} 3, & \textit{if } r = 1 \\ 2n+3, & \textit{if } r = 2 \\ 2n+5, & \textit{if } r = 3 \\ 6n+3, & \textit{if } r \geq 4. \end{cases} \tag{2}$$

Proof. Let $V(G) = \{u_i, u'_i, u''_i : 0 \leq i \leq n\} \cup \left\{v_j, v'_j, v''_j : 1 \leq j \leq n\right\}$, where u'_i, u''_i and v'_i, v''_i are the end vertices of the duplicated edges corresponding to u_i and v_j respectively. Because the diameter of G is 4, G^r is complete when $r \geq 4$. Hence, $\kappa(G^r) = 6n+3$. If $r = 1$, then the complete subgraph induced by $V_1 = \{v_i, u_i, v_{i+1}\}$ (in the sense $v_{n+1} = v_1$), for fixed i, is maximal. So, $\kappa(G) = 3$. If $r = 2$, then u_i's are pairwise adjacent, and v_j's are also pairwise adjacent in G^2. Furthermore, u_i's and v_j's have a distance of two and are thus adjacent. u_0', u_0'' are also adjacent to u_i's and v_j's. Therefore, the subgraph induced by $V_2 = \{u_i : 0 \le i \le n\} \cup \{v_j : 1 \le j \le n\} \cup \{u_0', u_0''\}$ is complete in G^2. There is no complete subgraph of higher order in G^2 because each vertex in $V(G) \setminus V_2$ is at a distance atleast 3 from u_0'. So, the subgraph induced by V_2 is maximal. Hence, $\kappa(G^2) = 2n + 3$. If $r = 3$, then v_j' and v_j'' are adjacent to elements of V_2. Therefore, the subgraph induced by $V_3 = V_2 \cup \left\{v_j', v_j''\right\}$, for fixed j, is complete in G^3. There is no complete subgraph of higher order in G^3 because no vertex in $V(G) \setminus V_3$ is adjacent to all vertices of V_3. So, the subgraph induced by V_3 is maximal. Hence, $\kappa(G^3) = 2n + 5$. □

Theorem 3.3. *If $n \ge 8$,*

$$\kappa(Lc_n^r) = \begin{cases} 3, & \text{if } r = 1 \\ n+1, & \text{if } r = 2 \\ n+5, & \text{if } r = 3 \\ 2n+1, & \text{if } r \ge 4. \end{cases} \tag{3}$$

Proof. Let $V(Lc_n) = \{u_i : 0 \le i \le n\} \cup \{v_j : 1 \le j \le n\}$ where u_0 is the apex vertex, u_i's are the pendant vertices, and v_j's are the vertices of the cycle C_n. Because Lc_n has diameter 4, Lc_n^r is complete when $r \ge 4$. Therefore, $\kappa(Lc_n^r) = 2n + 1$. If $r = 1$, then the maximal complete subgraph of Lc_n is of order 3 and has the vertex set $\{v_k, v_{k+1}, u_k\}$ (in the sense $v_{n+1} = v_1$) , for fixed k; $1 \le k \le n$. So, $\kappa(Lc_n) = 3$. When $r \ge 2$, u_k is adjacent to $u_{k'}$, where $1 \le k, k' \le n$ and $k \ne k'$. If $r = 2$, then $V_2 = \{u_i : 0 \le i \le n\}$ induces a complete subgraph of Lc_n^2. As no v_j is adjacent to all u_i's, V_2 is maximal. Therefore, $\kappa(Lc_n^2) = n + 1$. If $r = 3$, then consider $V_3 = \{u_i : 0 \le i \le n\} \cup \{v_p, v_{p+1}, v_{p+2}, v_{p+3}\}$ (in the sense $v_{n+1} = v_1$), for fixed p; $1 \le p \le n$. This set induces a complete subgraph of Lc_n^3 on $n + 5$ vertices. Moreover, v_p is not adjacent to v_{p+4}. So, a complete subgraph of higher order does not exist. Therefore, V_2 is the maximal complete subgraph of Lc_n^3 with order $n + 5$. Hence, $\kappa(Lc_n^3) = n + 5$. □

Theorem 3.4. *If G is a graph obtained by duplication of every vertex by an edge in $Lc_n, n \ge 8$, then*

$$\kappa(G^r) = \begin{cases} 3, & \text{if } r = 1 \\ n+3, & \text{if } r = 2 \\ n+6, & \text{if } r = 3 \\ 4n+3, & \text{if } r = 4 \\ 4n+11, & \text{if } r = 5 \\ 6n+3, & \text{if } r \ge 6. \end{cases} \tag{4}$$

Proof. Let $V(G) = \{u_i, u_i', u_i'' : 0 \le i \le n\} \cup \left\{v_j, v_j', v_j'' : 1 \le j \le n\right\}$, where u_i', u_i'' and v_j', v_j'' are the end vertices of the duplicated edges corresponding to u_i and v_j respectively. If $r \ge 6$, G^r is complete because the diameter of G is 6. Hence, $\kappa(G^r) = 6n + 3$. If $r = 1$, then the complete subgraph induced by $V_1 = \{v_i, u_i, v_{i+1}\}$ (in the sense $v_{n+1} = v_1$), for fixed i, is maximal. So, $\kappa(G) = 3$. If $r = 2$, then u_k is adjacent to $u_{k'}$ and u_0', u_0'' are at a distance atmost two from all u_i's. Therefore, the subgraph induced by $V_2 = \{u_i : 0 \leq i \leq n\} \cup \{u'_0, u''_0\}$ is complete in G^2. Since all vertices in $V(G) \setminus V_2$ is at a distance atleast 3 from u'_0, there is no complete subgraph of higher order in G^2. So, the subgraph induced by V_2 is maximal. Hence, $\kappa(G^2) = n + 3$. If $r = 3$, then v_j, v_{j+1} and v_{j+2} are pairwise adjacent and each of it is adjacent to u_i, u'_0 and u''_0. Therefore, the subgraph induced by $V_3 = V_2 \cup \{v_j, v_{j+1}, v_{j+2}\}$ (in the sense $v_{n+k} = v_k$), for fixed j, is complete in G^3. Since no vertex in $V(G) \setminus V_3$ is adjacent to all vertices of V_3, there is no complete subgraph of higher order in G^3. So, the subgraph induced by V_3 is maximal. Hence, $\kappa(G^3) = n + 6$. If $r = 4$, then u'_i and u''_i are adjacent to u'_k, u''_k $(i \neq k)$ and every vertex in V_3. Moreover, v_j's are pairwise adjacent, and each is adjacent to u'_i, u''_i, v, where $v \in V_3$. Therefore, $V_4 = \{u_i : 0 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq n\} \cup \{u'_i, u''_i : 0 \leq i \leq n\}$ induces a complete subgraph in G^4. For $v'_j \in V(G) \setminus V_4$, v'_j is not adjacent to $v_{\lceil \frac{j+n}{2} \rceil}$. So, there is no complete subgraph of higher order in G^4. Therefore, the subgraph induced by V_4 is maximal. Hence, $\kappa(G^4) = 4n + 3$. If $r = 5$, then $v'_j, v''_j, v'_{j+1}, v''_{j+1}, v'_{j+2}, v''_{j+2}, v'_{j+3}$, and v''_{j+3} are pairwise adjacent and each is adjacent to every vertex of V_4. Therefore, the subgraph induced by $V_5 = V_4 \cup \left\{v'_j, v''_j, v'_{j+1}, v''_{j+1}, v'_{j+2}, v''_{j+2}, v'_{j+3}, v''_{j+3}\right\}$ is complete in G^5. If $v'_q \in V(G) \setminus V_5$, v'_q is not adjacent to either v'_j or v'_{j+3} or both. So, there is no complete subgraph of higher order in G^5. Therefore, the subgraph induced by V_5 is maximal. Hence, $\kappa(G^5) = 4n + 11$. □

Theorem 3.5. *If $n \geq 5$,*

$$\kappa(J_n^r) = \begin{cases} 2, & \text{if } r = 1 \\ 4(r-1), & \text{if } 2 \leq r < \left\lceil \frac{n}{2} \right\rceil + 1 \\ 4n, & \text{if } r \geq \left\lceil \frac{n}{2} \right\rceil + 1. \end{cases} \tag{5}$$

Proof. Let $V(J_n^r) = \{x_{k'} : 1 \leq k' \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq n\} \cup \{w_k : 1 \leq k \leq n\}$ where $x_{k'}$ is the central vertex and u_i, v_j, w_k are outer vertices of $K_{1,3}$. Because J_n has diameter $\left\lceil \frac{n}{2} \right\rceil + 1$, J_n is complete when $r \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Therefore, $\kappa(J_n^r) = 4n$. As J_n is triangle free, $\kappa(J_n) = 2$. If $2 \leq r < \left\lceil \frac{n}{2} \right\rceil + 1$, then $V_1 = \{u_i, x_i, w_i, v_i \mid 1 \leq i \leq r-1\}$ induces a complete subgraph of J_n^r. This subgraph is also maximal since any $v \in V(J_n^r) \setminus V_1$ is not adjacent to all vertices of V_1. Therefore, $\kappa(J_n^r) = 4(r-1)$. □

Theorem 3.6. *If G is a graph obtained by duplication of every vertex by an edge in $J_n, n \geq 5$, then*

$$\kappa(G^r) = \begin{cases} 3r, & \text{if } r = 1, 2 \\ 10, & \text{if } r = 3 \\ 12r - 34, & \text{if } 3 < r < \left\lceil \frac{n}{2} \right\rceil + 3 \text{ and } r \text{ is odd} \\ 12r - 28, & \text{if } 3 < r < \left\lceil \frac{n}{2} \right\rceil + 3 \text{ and } r \text{ is even} \\ 12n, & \text{if } r \geq \left\lceil \frac{n}{2} \right\rceil + 3. \end{cases} \tag{6}$$

Proof. Consider $V(G^r) = \left\{x_p, x'_p, x''_p : 1 \leq p \leq n\right\} \cup \{u_i, u'_i, u''_i : 1 \leq i \leq n\} \cup \left\{v_j, v'_j, v''_j : 1 \leq j \leq n\right\}$ $\cup \left\{w_k, w'_k, w''_k : 1 \leq k \leq n\right\}$, where x_p is the central vertex and u_i, v_j, w_k are outer vertices of $K_{1,3}$. Because the diameter of G is $\left\lceil \frac{n}{2} \right\rceil + 3$, G^r is complete for $r \geq \left\lceil \frac{n}{2} \right\rceil + 3$. Therefore, $\kappa(G^r) = 12n$. If $r = 1$, then the complete subgraph induced by $V_1 = \{v_i, v'_i, v''_i\}$ is maximal. So, $\kappa(G) = 3$. If $r = 2$, then x_p, u_p, w_p and v_p are adjacent to each other and they are adjacent to x'_p, x''_p. Therefore, the subgraph induced by $V_2 = \left\{u_p, x_p, w_p, v_p, x'_p, x''_p\right\}$ is complete in G^2. There is no complete subgraph of higher order in G^2 because no vertex in $V(G) \setminus V_2$ is adjacent to all the vertices in V_2. So, this subgraph is maximal. Hence, $\kappa(G^2) = 6$. If $r = 3$, then $u'_i, u''_i, x'_i, x''_i, u_{i-1}, u_{i+1}$ are pairwise adjacent and each is adjacent to u_i, x_i, w_i, v_i. So, $V_3 = \{u_i, x_i, w_i, v_i, u'_i, u''_i, x'_i, x''_i, u_{i-1}, u_{i+1}\}$ induces the complete subgraph. This subgraph is also maximal because each vertex in $V(G) \setminus V_3$ is at a distance atleast 4 from one or more vertices of V_3. Hence, $\kappa(G^3) = 10$. If $r > 3$ and r is even, then consider $V_4 = \{u_i, x_i, w_i, v_i \mid 1 \le i \le r-1\} \cup \{u'_i, u''_i, x'_i, x''_i, w'_i, w''_i, v'_i, v''_i \mid 2 \le i \le r-2\}$. The subgraph induced by V_4 is a maximal complete subgraph of G^r. So, $\kappa(G^r) = 12r - 28$. If $r > 3$ and r is odd, then consider $V_5 = \{u_i, x_i, w_i, v_i \mid 1 \le i \le r-2\} \cup \{u'_i, u''_i, x'_i, x''_i, w'_i, w''_i, v'_i, v''_i \mid 2 \le i \le r-3\} \cup \left\{x'_1, x''_1, u'_1, u''_1, u'_{r-2}, u''_{r-2}\right\}$. The subgraph induced by V_5 is a maximal complete subgraph of G^r. So, $\kappa(G^r) = 12r - 34$. □

Theorem 3.7. *If $n \ge 6$,*

$$\kappa(SFL_n^r) = \begin{cases} 3, & \text{if } r = 1 \\ n+1, & \text{if } r = 2 \\ n+4, & \text{if } r = 3 \\ 2n+1, & \text{if } r \ge 4. \end{cases} \tag{7}$$

Proof. Let $V(SFL_n^r) = \{v_i : 0 \le i \le n\} \cup \left\{w_j : 1 \le j \le n\right\}$, where v_0 is the apex vertex and v_i's are the rim vertices. Because SFL_n has diameter 4, SFL_n^r is complete when $r \ge 4$. Hence, $\kappa(SFL_n^r) = 2n+1$. If $r = 1$, then the complete subgraph induced by $V_1 = \{v_0, v_i, v_{i+1}\}$(in the sense $v_{n+1} = v_1$), for fixed i; $1 \le i \le n$, is maximal. So, $\kappa(SFL_n) = 3$. If $r = 2$, then v_i is adjacent to v_j for $1 \le i, j \le n$, $i \ne j$ and $i, j \ne 0$. Since w_i is at a distance 3 from v_{i+4}(take $i+4$ modulo n), they are not adjacent in SFL_n^2. Therefore, the complete subgraph induced by $V_2 = \{v_i : 0 \le i \le n\}$ is maximal. So, $\kappa(SFL_n^2) = n+1$. If $r = 3$, then w_j, w_{j+1} and w_{j+2} are pairwise adjacent and each is adjacent to $v_i, 0 \le i \le n$. So, the complete subgraph induced by $V_3 = \{v_i : 0 \le i \le n\} \cup \left\{w_j, w_{j+1}, w_{j+2}\right\}$ (in the sense $w_{n+1} = w_1$), where j is fixed; $1 \le j \le n$, is maximal. So, $\kappa(SFL_n^3) = n+4$. □

Theorem 3.8. *If G is a graph obtained by duplication of every vertex by an edge in $SFL_n, n \ge 6$, then*

$$\kappa(G^r) = \begin{cases} 3, & \text{if } r = 1 \\ n+3, & \text{if } r = 2 \\ n+10, & \text{if } r = 3 \\ 4n+3, & \text{if } r = 4 \\ 4n+2\left\lceil \frac{n+1}{2} \right\rceil + 1, & \text{if } r = 5. \\ 6n+3, & \text{if } r \ge 6. \end{cases} \tag{8}$$

Proof. Let $V(G) = \{v_i, v'_i, v''_i : 0 \le i \le n\} \cup \left\{w_j, w'_j, w''_j : 1 \le j \le n\right\}$. Because the diameter of G is 6,

G^r is complete for $r \geq 6$. Hence, $\kappa(G^r) = 6n+3$. If $r = 1$, then the complete subgraph induced by $V_1 = \{v_0, v_i, v_{i+1}\}$ (in the sense $v_{n+1} = v_1$), for fixed i; $1 \leq i \leq n$, is maximal. So, $\kappa(G) = 3$. If $r = 2$, then v_i is adjacent to v_j for $1 \leq i,j \leq n$, $i \neq j$ and $i,j \neq 0$. Also, v_0, v'_0, v''_0 are pairwise adjacent and each is adjacent to $v_i, 1 \leq i \leq n$. Therefore, the subgraph induced by $V_2 = \{v_i : 0 \leq i \leq n\} \cup \{v'_0, v''_0\}$ is complete in G^2. Since each vertex in $V(G) \setminus V_2$ is at a distance atleast 3 from v'_0, there is no complete subgraph of higher order in G^2. So, the subgraph induced by V_2 is maximal. Hence, $\kappa(G^2) = n+3$. If $r = 3$, then w_j, w_{j+1} and w_{j+2} are pairwise adjacent and each of it is adjacent to $v_i, 0 \leq i \leq n$. In addition, $v'_{j+1}, v''_{j+1}, v'_{j+2}, v''_{j+2}$ are pairwise adjacent and each is adjacent to $v_i, w_j, w_{j+1}, w_{j+2}$. Therefore, the subgraph induced by $V_3 = \{v_i : 0 \leq i \leq n\} \cup \{w_j, w_{j+1}, w_{j+2}\} \cup \left\{v'_0, v''_0, v'_{j+1}, v''_{j+1}, v'_{j+2}, v''_{j+2}\right\}$ (in the sense $v_{n+1} = v_1$), for fixed j, is complete in G^3. Since no vertex in $V(G) \setminus V_3$ is adjacent to all vertices of V_3, there is no complete subgraph of higher order in G^3. So, the subgraph induced by V_3 is maximal. Hence, $\kappa(G^3) = n+10$. If $r = 4$, then w_j and w_k are pairwise adjacent and each is adjacent to $v_i, 0 \leq i \leq n$ and v'_0, v''_0. v'_i, v''_i and v'_k, v''_k, $i \neq k$, are pairwise adjacent and each is adjacent to $v_i, 0 \leq i \leq n$ and v'_0, v''_0. In addition, w_j is adjacent to each v'_i, v''_i. Therefore, the subgraph induced by $V_4 = \{v_i : 0 \leq i \leq n\} \cup \left\{w_j : 1 \leq j \leq n\right\} \cup \{v'_i, v''_i : 0 \leq i \leq n\}$ is complete in G^4. For $w'_j \in V(G) \setminus V_4$, w'_j is not adjacent to $w_{\lceil \frac{j+n}{2} \rceil}$. So, there is no complete subgraph of higher order in G^4. Therefore, the subgraph induced by V_4 is maximal. Hence, $\kappa(G^4) = 4n+3$. Consider $r = 5$. For $1 \leq p < \lceil \frac{n+1}{2} \rceil$, w'_p and w''_p are pairwise adjacent and each is adjacent to every vertex of V_4 in G^5. Therefore, the subgraph induced by $V_5 = V_4 \cup \left\{w'_p, w''_p : 1 \leq p < \lceil \frac{n+1}{2} \rceil\right\}$ is complete in G^5. For $w'_q \in V(G) \setminus V_5$, w'_q is not adjacent to $w'_{q - \lceil \frac{n+1}{2} \rceil + 1}$. So, there is no complete subgraph of higher order in G^5. Therefore, the subgraph induced by V_5 is maximal. Hence, $\kappa(G^5) = 4n + 2\left\lceil \frac{n+1}{2} \right\rceil + 1$. □

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