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Certain Combinatorial Properties of Twin Triplets Related to Tchebychev Polynomials

Research Article

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Abstract: In the present paper, Tchebychev polynomials $U_n(x)$, $V_n(x) = U_n(x) - U_{n-1}(x)$ and $W_n(x) = U_n(x) + U_{n-1}(x)$ are extended to two variables. Twin triplets of numbers (y_n, d_n, s_n) and (Y_n, D_n, S_n) are defined and their certain combinatorial properties are described.

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1. Introduction

Many properties of Tchebychev polynomials $T_n(x)$, $U_n(x)$, $V_n(x) = U_n(x) - U_{n-1}(x)$ and $W_n(x) = U_n(x) + U_{n-1}(x)$ [3, 7, 8, 10, 11, 13–15] are more significant in Combinatorial Number theory because they exhibit many continued fractions and Combinatorial Identities [1, 2, 4–6, 8, 12, 16, 17]. In the present paper, Tchebychev polynomials of second kind $U_n(x)$, third kind $V_n(x) = U_n(x) - U_{n-1}(x)$ and fourth kind $W_n(x) = U_n(x) + U_{n-1}(x)$ are extended to two variables in a similar fashion as given in our earlier Work [9]. Twin triplets, namely, (y_n, d_n, s_n) and (Y_n, D_n, S_n) are defined using $(U_n(x), V_n(x), W_n(x))$ and $(U_n(x,y), V_n(x,y), W_n(x,y))$ respectively. Their continued fraction, matrix identity and determinant properties are described with proof. In Section 2, hypergeometric representation, Rodrigue formula, genarating function and determinant formula are derived for $U_n(x,y)$, $V_n(x,y)$ and $W_n(x,y)$. In Section 3, the triplet (y_n, d_n, s_n) of numbers are defined using $U_n(x)$, $V_n(x)$ and $W_n(x)$. Their Combinatorial properties are stated and proved . In the last Section, similar results are obtained for the triplet (Y_n, D_n, S_n) using $U_n(x,y)$, $V_n(x,y)$ and $W_n(x,y)$.

2. A Generalization of Tchebychev Polynomials $(U_n(x),\ V_n(x),\ W_n(x))$ and Extended Results

Definition 2.1. *Generalized Tchebychev Polynomial of Second kind in two variables x and y of degree n, denoted by $U_n(x,y)$ [9] is*

$$U_n(x,y) = \frac{1}{2\sqrt{x^2-y^2}}\left[[x+\sqrt{x^2-y^2}]^{n+1} - [x-\sqrt{x^2-y^2}]^{n+1}\right].$$

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It is a homogeneous polynomial of degree n and hence

$$U_n(x,y) = y^n U_n\left(\frac{x}{y}\right).$$

Generalized Tchebychev Polynomial of Third kind in two variables x and y of degree n, denoted by $V_n(x,y)$ is

$$V_n(x,y) = U_n(x,y) - yU_{n-1}(x,y).$$

It is also a homogeneous polynomial of degree n and hence

$$V_n(x,y) = y^n V_n\left(\frac{x}{y}\right).$$

Generalized Tchebychev Polynomial of Fourth kind in two variables x and y of degree n, denoted by $W_n(x,y)$ is

$$W_n(x,y) = U_n(x,y) + yU_{n-1}(x,y).$$

It is also a homogeneous polynomial of degree n and hence

$$W_n(x,y) = y^n W_n\left(\frac{x}{y}\right).$$

When $y = 1$, $U_n(x,y)$, $W_n(x,y)$ and $V_n(x,y)$ are nothing but $U_n(x)$, $V_n(x)$ and $W_n(x)$ respectively.

Initial Polynomials: The initial polynomials of generalized Tchebychev polynomials of Second, Third and Fourth kinds in two variables are

$$\begin{aligned}
U_n(x,y):&\quad 1,\ 2x\ ,\ 4x^2 - y^2\ ,\ 8x^3 - 4xy^2\ ,\ldots;\\
V_n(x,y):&\quad 1, 2x - y, 4x^2 - 2xy - y^2, 8x^3 - 4x^2y - 4xy^2 + y^3, \ldots;\\
W_n(x,y):&\quad 1, 2x + y, 4x^2 + 2xy - y^2, 8x^3 + 4x^2y - 4xy^2 - y^3, \ldots.
\end{aligned}$$

Three Term Recurrence Relations: The Recurrence relation satisfied by the generalized Tchebychev Polynomials of Second kind in two variables [9] is

$$\begin{aligned}
U_{n+1}(x,y) &= 2x\ U_n(x,y) - y^2 U_{n-1}(x,y),\\
U_0(x,y) &= 1, U_1(x,y) = 2x,\ n = 1,2,3,\ldots.
\end{aligned}$$

By direct verification using the definition, one can show that the following recurrence relation is satisfied by the generalized Tchebychev Polynomials of Third kind in two variables is

$$\begin{aligned}
V_{n+1}(x,y) &= 2xV_n(x,y) - y^2 V_{n-1}(x,y),\\
V_0(x,y) &= 1, V_1(x,y) = 2x - y,\ n = 1,2,3,\ldots.
\end{aligned}$$

Similarly, the Recurrence relation satisfied by the generalized Tchebychev Polynomials of Fourth kind in two variables is

$$\begin{aligned}
W_{n+1}(x,y) &= 2xW_n(x,y) - y^2 W_{n-1}(x,y),\\
W_0(x,y) &= 1, W_1(x,y) = 2x + y,\ n = 1,2,3,\ldots\ .
\end{aligned}$$

Also the Recurrence relation of Third and Fourth kinds of the generalized Tchebychev Polynomials in two variables in terms of the Second kind are

$$\begin{aligned} V_n(x,y) &= (2x-y)U_{n-1}(x,y) - y^2 U_{n-2}(x,y), \\ U_0(x,y) &= 1, U_1(x,y) = 2x,\ n = 1,2,3,\dots \end{aligned}$$

and

$$\begin{aligned} W_n(x,y) &= (2x+y)U_{n-1}(x,y) - y^2 U_{n-2}(x,y), \\ U_0(x,y) &= 1, U_1(x,y) = 2x,\ n = 1,2,3,\dots. \end{aligned}$$

Hypergeometric Series Representation: Tchebychev Polynomials of Second, Third and Fourth kinds in one variable can be represented in the form of hypergeometric series as follows [11]:

$$\begin{aligned} U_n(x) &= (n+1)\ {}_2F_1\left(-n,\ n+2;\ \frac{3}{2};\ \frac{1-x}{2}\right); \\ V_n(x) &= {}_2F_1\left(-n,\ n+1;\ \frac{1}{2};\ \frac{1-x}{2}\right); \\ W_n(x) &= (2n+1)\ {}_2F_1\left(-n,\ n+1;\ \frac{3}{2};\ \frac{1-x}{2}\right). \end{aligned}$$

The generalized Tchebychev polynomials of Second, Third and Fourth kinds in two variables also have the following extended result and the proof will be similar to that of one variable case.

Theorem 2.2. *The hypergeometric representation for generalized Tchebychev polynomials are*

$$\begin{aligned} U_n(x,y) &= y^n(n+1)\ {}_2F_1\left(-n,\ n+1;\ \frac{3}{2};\ \frac{y-x}{2y}\right), \\ V_n(x,y) &= y^n\ {}_2F_1\left(-n,\ n+1;\ \frac{1}{2};\ \frac{y-x}{2y}\right), \end{aligned}$$

and

$$W_n(x,y) = y^n(2n+1)\ {}_2F_1\left(-n,\ n+1;\ \frac{3}{2};\ \frac{y-x}{2y}\right).$$

Rodrigue Formula: The Rodrigue formula for Tchebychev Polynomials of Second kind in two variables is [9]

$$U_n(x,y) = \frac{1}{y^n}\frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(y^2-x^2)^{-\frac{1}{2}}\frac{\partial^n}{\partial x^n}(y^2-x^2)^{n+\frac{1}{2}}.$$

The Rodrigue formula for Tchebychev Polynomials of Third kind in one variable is [7]

$$V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\frac{d^n}{dx^n}\left((1-x^2)^n\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right).$$

Using the above formula, we can extend

$$\begin{aligned} V_n(x,y) &= y^n V_n\left(\frac{x}{y}\right) \\ &= y^n\frac{(-1)^n 2^n n!}{(2n)!}\left(\frac{y-x}{y+x}\right)^{\frac{1}{2}}\frac{\partial^n}{\partial x^n}\left(\frac{(y^2-x^2)^n}{(y^2)^n}\left(\frac{y+x}{y-x}\right)^{\frac{1}{2}}\right) \\ &= \frac{1}{y^n}\frac{(-1)^n 2^n n!}{(2n)!}\left(\frac{y-x}{y+x}\right)^{\frac{1}{2}}\frac{\partial^n}{\partial x^n}\left((y^2-x^2)^n\left(\frac{y+x}{y-x}\right)^{\frac{1}{2}}\right). \end{aligned}$$

Similarly the Rodrigue formula for genarlised Tchebychev polynomials of Fourth kind in two variables can be derived. The extended result is stated in the following theorem.

Theorem 2.3. *The Rodrigue formula for generalized Tchebychev polynomials are*

$$U_n(x,y) = \frac{1}{y^n}\frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(y^2-x^2)^{-\frac{1}{2}}\frac{\partial^n}{\partial x^n}(y^2-x^2)^{n+\frac{1}{2}} \ ,$$

$$V_n(x,y) = \frac{1}{y^n}\frac{(-1)^n 2^n n!}{(2n)!}\left(\frac{y-x}{y+x}\right)^{\frac{1}{2}}\frac{\partial^n}{\partial x^n}\left((y^2-x^2)^n\left(\frac{y+x}{y-x}\right)^{\frac{1}{2}}\right)$$

and

$$W_n(x,y) = \frac{1}{y^n}\frac{(-1)^n 2^n n!}{(2n)!}\left(\frac{y+x}{y-x}\right)^{\frac{1}{2}}\frac{\partial^n}{\partial x^n}\left((y^2-x^2)^n\left(\frac{y-x}{y+x}\right)^{\frac{1}{2}}\right) \ .$$

Generating Functions: The Generating function for Tchebychev Polynomials of Second kind in two variables [9] is

$$\sum_{n=0}^{\infty} U_n(x,y)t^n = \frac{1}{1-2xt+t^2y^2}.$$

Keeping in mind the three term recurrence relation for $V_n(x,y)$, we proceed with the derivation. Put $f(x,y,t) = \sum_{n=0}^{\infty} V_n(x,y)t^n$. For the purpose of manipulations we write

$$\begin{aligned}
f(x,y,t) &= V_0(x,y) + V_1(x,y)t + \cdots + V_{n+1}(x,y)t^{n+1} + \cdots \\
-2xtf(x,y,t) &= -2xV_0(x,y)t - 2xV_1(x,y)t^2 - \cdots - 2xV_n(x,y)t^{n+1} - \cdots \\
y^2t^2f(x,y,t) &= V_0(x,y)y^2t^2 + V_1(x,y)y^2t^3 + \cdots + V_{n-1}(x,y)y^2t^{n+1} + \cdots \ .
\end{aligned}$$

Summing all the above three expressions on both sides, we get

$$(1-2xt+t^2y^2)f(x,y,t) = 1 + (2x-y)t - 2xt.$$

Hence,

$$f(x,y,t) = \frac{1-yt}{1-2xt+t^2y^2}.$$

Similarly the generalized Tchebychev polynomials of Fourth kind in two variables can be shown to have a genarating function stated in the following theorem combined with that of $U_n(x,y)$ and $V_n(x,y)$.

Theorem 2.4. *The genarating function for generalized Tchebychev polynomials are*

$$\begin{aligned}
\sum_{n=0}^{\infty} U_n(x,y)t^n &= \frac{1}{1-2xt+t^2y^2} \ , \\
\sum_{n=0}^{\infty} V_n(x,y)t^n &= \frac{1-yt}{1-2xt+t^2y^2} \quad \textit{and} \\
\sum_{n=0}^{\infty} W_n(x,y)t^n &= \frac{1+yt}{1-2xt+t^2y^2} \ .
\end{aligned}$$

Determinant Formulas: We state the following theorem for generalized Tchebychev polynomials of Second, Third and Fourth kinds without proof because they follow directly from their three term recurrence relations.

Theorem 2.5. *The determinants formulas for generalized Tchebychev polynomials are*

$$U_n(x,y) = \begin{vmatrix} 2x & -y & 0 & \cdots & 0 & 0 \\ -y & 2x & -y & 0 & \cdots & 0 \\ 0 & -y & \ddots & -y & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -y & 2x & -y \\ 0 & 0 & \cdots & 0 & -y & 2x \end{vmatrix},$$

$$V_n(x,y) = \begin{vmatrix} 2x-y & -y & 0 & \cdots & 0 & 0 \\ -y & 2x & -y & 0 & \cdots & 0 \\ 0 & -y & \ddots & -y & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -y & 2x & -y \\ 0 & 0 & \cdots & 0 & -y & 2x \end{vmatrix}$$

$$W_n(x,y) = \begin{vmatrix} 2x+y & -y & 0 & \cdots & 0 & 0 \\ -y & 2x & -y & 0 & \cdots & 0 \\ 0 & -y & \ddots & -y & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -y & 2x & -y \\ 0 & 0 & \cdots & 0 & -y & 2x \end{vmatrix}.$$

3. Certain combinatorial Identities of the triplet (y_n, d_n, s_n)

Let $y_n := U_n(N)$, $d_n := V_n(N) = U_n(N) - U_{n-1}(N)$ and $s_n := W_n(N) = U_n(N) + U_{n-1}(N)$, where $N = 2, 3, \ldots$. Then they satisfy the following three term recurrence relations:

$$y_{n+1} = 2Ny_n - y_{n-1}, \quad y_0 = 1, \quad y_1 = 2N \tag{1}$$

$$d_{n+1} = 2Nd_n - d_{n-1}, \quad d_0 = 1, \quad d_1 = 2N-1 \tag{2}$$

$$s_{n+1} = 2Ns_n - s_{n-1}, \quad s_0 = 1, \quad s_1 = 2N+1 \tag{3}$$

$$d_{n+1} = (2N-1)y_n - y_{n-1}, \quad y_0 = 1, \quad y_1 = 2N \tag{4}$$

$$s_{n+1} = (2N+1)y_n - y_{n-1}, \quad y_0 = 1, \quad y_1 = 2N\ , \tag{5}$$

which are needed to derive some of their combinatorial properties.

Theorem 3.1. *The pair (d_n, y_n) satisfies the following identities:*

(i) $\begin{bmatrix} d_n & y_n \\ d_{n+1} & y_{n+1} \end{bmatrix} = \begin{bmatrix} d_1 & y_1 \\ d_2 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ (2N-2) & (2N-1) \end{bmatrix}^{n-1}.$

(ii) $d_n y_{n+1} - y_n d_{n+1} = 1.$

(iii) $\frac{d_{n+1}}{y_n} = (2N-1) - \frac{1}{2N} - \frac{1}{2N} - \cdots - \frac{1}{2N}.$

Proof.

(i) The result is proved by using Mathematical Induction on n. For $n = 1$, the result is obvious.

Suppose for $n = k$, the result is true:

$$\begin{bmatrix} d_k & y_k \\ d_{k+1} & y_{k+1} \end{bmatrix} = \begin{bmatrix} d_1 & y_1 \\ d_2 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ (2N-2) & (2N-1) \end{bmatrix}^{k-1}.$$

The result for $n = k + 1$, directly follows once we apply the following relations:

$$d_{k+1} = d_k + (2N-2)y_k \tag{6}$$

$$y_{k+1} = d_k + (2N-1)y_k \tag{7}$$

Relation (4) and the definition $d_k = y_k - y_{k-1}$ will directly yield (6). The relation (1) and the definition $d_k = y_k - y_{k-1}$ will directly yield the relation (7).

(ii) The result directly follows by taking determinant on both sides of **(i)** because

$$\begin{vmatrix} d_1 & y_1 \\ d_2 & y_2 \end{vmatrix} = (2N-1)\cdot(4N^2-1) - 2N\cdot(4N^2-2N-1) = 1.$$

(iii) The three term Recurrence relations (4) and (1) can be rewritten as follows:

$$\frac{d_{n+1}}{y_n} = (2N-1) - \frac{1}{\frac{y_n}{y_{n-1}}} \quad \text{and}$$

$$\frac{y_n}{y_{n-1}} = 2N - \frac{1}{\frac{y_{n-1}}{y_{n-2}}}.$$

By combining them and using recursion, finally we arrive at

$$\frac{d_{n+1}}{y_n} = (2N-1) - \frac{1}{2N\,-}\,\frac{1}{2N\,-}\,\frac{1}{2N\,-\cdots-}\,\frac{1}{2N}, \quad \text{because } \frac{y_1}{y_0} = 2N.$$

□

Theorem 3.2. *The pair (s_n, y_n) satisfies the following identities:*

(i) $\begin{bmatrix} s_n & y_n \\ s_{n+1} & y_{n+1} \end{bmatrix} = \begin{bmatrix} s_1 & y_1 \\ s_2 & y_2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2N+2 & 2N+1 \end{bmatrix}^{n-1}.$

(ii) $s_n y_{n+1} - y_n s_{n+1} = -1.$

(iii) $\frac{s_{n+1}}{y_n} = (2N+1) - \frac{1}{2N\,-}\,\frac{1}{2N\,-\cdots-}\,\frac{1}{2N\,-}\,\frac{1}{2N}.$

Proof.

(i) The result is proved by using Mathematical Induction on n. For $n = 1$, the result is obvious. Suppose for $n = k$, the result is true:

$$\begin{bmatrix} s_k & y_k \\ s_{k+1} & y_{k+1} \end{bmatrix} = \begin{bmatrix} s_1 & y_1 \\ s_2 & y_2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ (2N+2) & (2N+1) \end{bmatrix}^{k-1}.$$

The result for $n = k+1$, directly follows once we apply the following relations:

$$s_{k+1} = -s_k + (2N+2)y_k \quad (8)$$

$$y_{k+1} = -s_k + (2N+1)y_k \quad (9)$$

The derivation of (8) and (9) is similar to (6) and (7) respectively.

(ii) The identity can be directly deduced by applying determinant on both sides of **(i)** because

$$\begin{vmatrix} s_1 & y_1 \\ s_2 & y_2 \end{vmatrix} = (2N+1)\cdot(4N^2-1) - 2N\cdot(4N^2+2N-1) = -1.$$

(iii) The three term Recurrence relations (4) and (1) yields

$$\frac{s_{n+1}}{y_n} = (2N+1) - \frac{1}{2N} - \frac{1}{2N} - \cdots - \frac{1}{2N} - \frac{1}{2N}, \quad \text{because } \frac{y_1}{y_0} = 2N.$$

□

Theorem 3.3. *The pair (s_n, d_n) satisfies the following identities:*

(i) $\begin{bmatrix} s_n & d_n \\ s_{n+1} & d_{n+1} \end{bmatrix} = \begin{bmatrix} s_1 & d_1 \\ s_2 & d_2 \end{bmatrix} \begin{bmatrix} N & N-1 \\ N+1 & N \end{bmatrix}^{n-1}.$

(ii) $s_n d_{n+1} - d_n s_{n+1} = -2.$

(iii) $\frac{s_{n+1}}{d_n} = \frac{1}{1} - \frac{2N}{1+2N} - \frac{1}{2N} - \frac{1}{2N} - \frac{1}{2N} - \cdots - \frac{1}{2N}.$

Proof.

(i) Using the definitions $d_n = y_n - y_{n-1}$ and $s_n = y_n + y_{n-1}$, we have

$$\begin{aligned} Ns_n + (N+1)d_n &= N(y_n + y_{n-1}) + (N+1)(y_n - y_{n-1}) \\ &= (2N+1)y_n - y_{n-1} = s_{n+1}. \end{aligned}$$

Hence

$$s_{n+1} = Ns_n + (N+1)d_n. \quad (10)$$

Similarly

$$d_{n+1} = (N-1)s_n + Nd_n. \quad (11)$$

The proof of (i) by induction is exactly similar to Theorems 3.1 and 3.2 part (i).

(ii) The identity can be directly deduced by applying determinant on both sides of **(i)** because

$$\begin{vmatrix} s_1 & d_1 \\ s_2 & d_2 \end{vmatrix} = (2N+1)\cdot(4N^2-2N-1) - (2N-1)\cdot(4N^2+2N-1) = -2.$$

(iii) Using definitions of s_n, d_n and the relation (1), we can proceed as follows:

$$\begin{aligned}\frac{s_{n+1}}{d_n} &= \frac{y_{n+1}+y_n}{y_n-y_{n-1}}\\ &= \frac{y_{n+1}+y_n}{(y_{n+1}+y_n)-2Ny_n}\\ &= \frac{1}{1_-}\frac{2N}{1+2N_-}\frac{1}{2N_-}\frac{1}{2N_-}\frac{1}{2N}\,{}_{-\cdots-}\frac{1}{2N},\quad \text{because } \frac{y_1}{y_0}=2N.\end{aligned}$$

□

Theorem 3.4. *The sequence* (s_n) *satisfies the following identities:*

(i) $\begin{bmatrix} s_{n-1} & s_n \\ s_n & s_{n+1}\end{bmatrix} = \begin{bmatrix} s_0 & s_1 \\ s_1 & s_2\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2N\end{bmatrix}^{n-1}.$

(ii) $s_{n-1}s_{n+1}-s_n^2=-2(N+1).$

(iii) $\frac{s_{n+1}}{s_n}=2N-\frac{1}{2N}\,{}_-\frac{1}{2N}\,{}_-\frac{1}{2N}\,{}_{-\cdots-}\frac{1}{(2N+1)}.$

Theorem 3.5. *The sequence* (d_n) *satisfies the following identities:*

(i) $\begin{bmatrix} d_{n-1} & d_n \\ d_n & d_{n+1}\end{bmatrix} = \begin{bmatrix} d_0 & d_1 \\ d_1 & d_2\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2N\end{bmatrix}^{n-1}.$

(ii) $d_{n-1}d_{n+1}-d_n^2=2(N-1).$

(iii) $\frac{d_{n+1}}{d_n}=2N-\frac{1}{2N}\,{}_-\frac{1}{2N}\,{}_-\frac{1}{2N}\,{}_{-\cdots-}\frac{1}{(2N-1)}.$

Theorem 3.6. *The sequence* (y_n) *satisfies the following identities [9]:*

(i) $\begin{bmatrix} y_{n-1} & y_n \\ y_n & y_{n+1}\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & N\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2N\end{bmatrix}^{n-1}.$

(ii) $y_{n-1}y_{n+1}-y_n^2=-1.$

(iii) $\frac{y_{n+1}}{y_n}=2N-\frac{1}{2N}\,{}_-\frac{1}{2N}\,{}_-\frac{1}{2N}\,{}_{-\cdots-}\frac{1}{2N}.$

The proofs of Theorems 3.4 and 3.5 part (i) directly follow by (3) and (2) respectively. Also by taking determinant on both sides of Theorems 3.4 and 3.5 part (i) the results of Theorems 3.4 and 3.5 part (ii) follow, respectively. The results of Theorems 3.4 and 3.5 part (iii) directly follow by the recurrence relations (3) and (2) respectively.

4. Certain combinatorial Identities of the triplet (Y_n, D_n, S_n)

Let $Y_n := U_n(N,K)$, $D_n := V_n(N,K)$, and $S_n := W_n(N,K)$, where $N = 2,3,\ldots,$ $K = 1,2,\ldots,N-1$ and N^2-K^2 is not a square number. When $K=1, Y_n=y_n$, $D_n=d_n$ and $S_n=s_n$. Hence we can expect similar identities for Y_n , V_n and W_n. The three term recurrence relations are as follows:

$$Y_{n+1} = 2NY_n-K^2Y_{n-1},\quad Y_0=1,\quad Y_1=2N. \tag{12}$$

$$D_{n+1} = 2ND_n-K^2D_{n-1},\quad D_0=1,\quad D_1=2N-K. \tag{13}$$

$$S_{n+1} = 2NS_n-K^2S_{n-1},\quad S_0=1,\quad S_1=2N+K. \tag{14}$$

$$D_{n+1} = (2N-K)Y_n-K^2Y_{n-1},\quad Y_0=1,\quad Y_1=2N. \tag{15}$$

$$S_{n+1} = (2N+K)Y_n-K^2Y_{n-1},\quad Y_0=1,\quad Y_1=2N. \tag{16}$$

Theorem 4.1. *The pair (D_n, Y_n) satisfies the following identities:*

(i) $\begin{bmatrix} D_n & Y_n \\ D_{n+1} & Y_{n+1} \end{bmatrix} = \begin{bmatrix} D_1 & Y_1 \\ D_2 & Y_2 \end{bmatrix} \begin{bmatrix} K & K \\ 2N-2K & 2N-K \end{bmatrix}^{n-1}.$

(ii) $D_nY_{n+1} - Y_nD_{n+1} = K^{2n+1}.$

(iii) $\frac{D_{n+1}}{Y_n} = (2N-K) - \frac{K^2}{2N} - \frac{K^2}{2N} - \cdots - \frac{K^2}{2N}.$

Theorem 4.2. *The pair (S_n, Y_n) satisfies the following identities:*

(i) $\begin{bmatrix} S_n & Y_n \\ S_{n+1} & Y_{n+1} \end{bmatrix} = \begin{bmatrix} S_1 & Y_1 \\ S_2 & Y_2 \end{bmatrix} \begin{bmatrix} -K & -K \\ 2N+2K & 2N+K \end{bmatrix}^{n-1}.$

(ii) $S_nY_{n+1} - Y_nS_{n+1} = -K^{2n+1}.$

(iii) $\frac{S_{n+1}}{Y_n} = (2N+K) - \frac{K^2}{2N} - \frac{K^2}{2N} - \cdots - \frac{K^2}{2N}.$

Theorem 4.3. *The pair (S_n, D_n) satisfies the following identities:*

(i) $\begin{bmatrix} S_n & D_n \\ S_{n+1} & D_{n+1} \end{bmatrix} = \begin{bmatrix} S_1 & D_1 \\ S_2 & D_2 \end{bmatrix} \begin{bmatrix} N & N-K \\ N+K & N \end{bmatrix}^{n-1}.$

(ii) $S_nD_{n+1} - D_nS_{n+1} = -2K^{2n+1}.$

(iii) $\frac{S_{n+1}}{D_n} = \frac{K}{1} - \frac{2N}{K+2N} - \frac{K^2}{2N} - \frac{K^2}{2N} - \frac{K^2}{2N} - \cdots - \frac{K^2}{2N}.$

Theorem 4.4. *The sequence (S_n) satisfies the following identities:*

(i) $\begin{bmatrix} S_{n-1} & S_n \\ S_n & S_{n+1} \end{bmatrix} = \begin{bmatrix} S_0 & S_1 \\ S_1 & S_2 \end{bmatrix} \begin{bmatrix} 0 & -K^2 \\ 1 & 2N \end{bmatrix}^{n-1}.$

(ii) $S_{n-1}S_{n+1} - S_n^2 = -2(N+K)K^{2n-1}.$

(iii) $\frac{S_{n+1}}{S_n} = 2N - \frac{K^2}{2N} - \frac{K^2}{2N} - \frac{K^2}{2N} - \cdots - \frac{K^2}{(2N+K)}.$

Theorem 4.5. *The Sequence (D_n) satisfies the following identities:*

(i) $\begin{bmatrix} D_{n-1} & D_n \\ D_n & D_{n+1} \end{bmatrix} = \begin{bmatrix} D_0 & D_1 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} 0 & -K^2 \\ 1 & 2N \end{bmatrix}^{n-1}.$

(ii) $D_{n-1}D_{n+1} - D_n^2 = 2(N-K)K^{2n-1}.$

(iii) $\frac{D_{n+1}}{D_n} = 2N - \frac{K^2}{2N} - \frac{K^2}{2N} - \frac{K^2}{2N} - \cdots - \frac{K^2}{(2N-K)}.$

Theorem 4.6. *The sequence (Y_n) satisfies the following identities:*

(i) $\begin{bmatrix} Y_{n-1} & Y_n \\ Y_n & Y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2N \end{bmatrix} \begin{bmatrix} 0 & -K^2 \\ 1 & 2N \end{bmatrix}^{n-1}.$

(ii) $Y_{n-1}Y_{n+1} - Y_n^2 = -K^{2n-2}.$

(iii) $\frac{S_{n+1}}{D_n} = 2N - \frac{K^2}{2N} - \frac{K^2}{2N} - \frac{K^2}{2N} - \cdots - \frac{K^2}{2N}.$

The proofs of the above results will be quite similar to those of one variable case except for small adaptation to incorporate K^2. Hence, we have described the similar identities for $Y_n(N,K)$, $V_n(N,K)$ and $W_n(N,K)$ without giving proofs.

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References

[1] W.S.Anglin, *The Queen of Mathematics, An Introduction to Number Theory*, Kluwer Academy Publishers, (1995).

[2] D.M.Burton, *Elementary Number Theory*, Wm.C.Brown Company Publisher, (1989).

[3] W.Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, New York, (2004).

[4] R.L.Graham, D.E.Kunth and O.Patashnik, *Concrete Mathematics, Second Edition*, Pearson Education Inc., (1994).

[5] G.H.Hardy and E.M.Wright, *An Introduction to Theory of Numbers*, Clarendon Press, (1979).

[6] T.Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley Interscience Publication, New York, (2001).

[7] J.C.Mason and D.C.Handscomb, *Chebyshev Polynomials*, CRC Press LLC, New York, (2003).

[8] J.Morgado, *Note on the Chebyshev Polynomials and Applications to the Fibonacci Number*, Portugaliae Mathematica, 52(1995), 363-378.

[9] R.Rangarajan, P.Shashikala and C.K.Honne gowda, *Certain Combinatorial Identities of Twin Pairs Related to Tchebychev Polynomials*, International Journal of Computer Applications, 154(05)(2016), 01-05.

[10] R.Rangarajan and P.Shashikala, *A Pair of Clasical Orthogonal Polynomials Connected to Catalan Numbers*, Adv Studies Contemp.Math., 23(2013), 323-335.

[11] E.D.Rainville, *Special Functions*, The Macmillan company, New york, (1960).

[12] J.Riordan, *Combinatorial Identities*, Robert E. Krieger Publishing Company, New York, (1979).

[13] T.Rivlin, *Chebyshev Polynomials : From Approximation Theory to Number Theory, Second edition*, Wiley and Sons, New York, (1990).

[14] P.Shashikala, *Studies on mathematical analysis of orthogonal polynomials*, Ph. D. Thesis, University of Mysore, Mysore, (2014).

[15] P.Shashikala and R.Rangarajan, *Tchebychev and Brahmagupta Polynomials and Golden Ratio:Two New Interconections*, International J.Math. Combin., 3(2016), 57-67.

[16] H.M.Stark, *An Introduction to Number Theory*, Cambridge MIT Press, New York, (1994).

[17] S.Vajda, *Fibonacci and Lucas Numbers and Golden Section, Theory and Applications*, Ellis-Horwood, London, (1989).