International Journal of Mathematics And its Applications
Volume 4, Issue 4 (2016), 17–29.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

M/G/1 Queue with Multiple Optional Services and Deterministic Repair Times

Research Article

S.Vanitha[1]*

1 SSN College of Engineering, Old Mahabalipuram Road, Kalavakkam, Chennai, India.

Abstract: We analyze the steady state behavior of an $M/G/1$ queue with Poisson arrivals subject to multiple optional services and random system breakdowns. Arriving customer has to undergo first essential service and there are j optional services, where $j = 1, 2, \dots, n$. As soon as the essential service of a customer is complete, then with probability $r_j, j = 1 \dots n$, he may opt for any one of the j optional services, in which case his any one of the j services will immediately commence or else with probability $1 - \sum_{j=1}^{n} r_j$, he may opt to leave the system, in which case another customer at the head of the queue is taken up for his essential service.The service times follow arbitrary (general) service distributions. The system is prone to random breakdowns and just after a breakdown the server undergoes repair of a fixed duration. We obtain time dependent as well as steady state probability generating functions for the number in the system. For steady state we obtain explicitly the mean number and the mean waiting time for the system and for the queue. Results for some special cases of interest are derived.

MSC: 60K25, 60K30.

Keywords: Poisson arrivals, general service times, probability generating functions, random breakdowns, idle state, steady state, deterministic repairs, supplementary variable technique.

1. Introduction

Queues and queueing networks occupy a prominent role in the performance analysis of wide range of systems in computer communications, logistics and flexible manufacturing systems. In fact, new results in queueing theory have often been inspired by new technological advances in computer, manufacturing and communication networks. There have been extensive studies in queues with server vacations and breakdowns. Vacation queueing models have been of great interest due to its applicability in real situations. Single server queueing models with vacations have been studied by various authors due to their wide applications in the analysis of processor schedules in computer and switching systems, the analysis of manufacturing system with machine breakdown etc. Levy and Yechailai [10], Takagi [15], Doshi [3], Keilson and Servi [8], Gaver [6], Fuhrman [5], Shantikumar [14], Cramer [2] and Madan [11] are a few among many authors who have studied queues with server vacations with varying vacation policies. In real life situations, a queueing system might suddenly breakdown and hence the server will not be able to continue providing service unless the system is repaired. In recent years, significant work has been done on queues with random breakdowns by several authors which include Kulkarni and Choi [9], Federgruen and So [4], Jayawardene and Kella [7], Aissani and Artalejo [1], Wang, Cao and Li [16], Madan [12,13], Vinck and Bruneel [17].

* *E-mail: vanithas@ssn.edu.in*

In the current work, we consider an M/G/1 queue with optional services subject to random breakdowns and deterministic repair times of fixed length $d(>0)$, using supplementary variable technique due to Cox. Each customer has to undergo first essential service and there are j optional services. As soon as the essential service of a customer is complete, then with probability $r_j, j = 1 \dots n$, he may opt for any one of the j optional services and the service times are assumed to follow general distribution. We assume that the breakdowns are random and time-homogeneous which means that service channel may fail not only while it is working, but it may fail even when it is idle. The rest of the paper is organized as follows. The mathematical description of our model is in section 2 and the equations governing the model are given in section 3. The time dependent solutions have been obtained in section 4 using supplementary variable technique and the corresponding steady state results have been derived explicitly in section 5. Mean number in the system and mean waiting time have been computed in the section 6 and section 7 respectively. some particular cases for this model are discussed in section 8.

2. Assumptions Underlying the Model

The following assumptions describe the mathematical model

- Customers arrive at the system one by one in according to a Poisson stream with arrival rate $\lambda(>0)$.
- There is a single server which provides the essential service to all arriving customers. Let $B(v)$ and $b(v)$ respectively be the distribution function and the density function of the essential service times and let $\mu(x)\mathrm{d}x$ be the conditional probability density of completion of the essential service given that the elapsed time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)}, \tag{1}$$

  and therefore

$$b(v) = \mu(v) e^{-\int_0^v \mu(x)\mathrm{d}x}. \tag{2}$$

- There are j optional services where j take values from 1 to n. As soon as the essential service of a customer is complete, then with probability $r_j, j = 1 \dots n$, he may opt for any one of the j optional services, in which case his any one of the j services will immediately commence or else with probability 1-$\sum_{j=1}^{n} r_j$, he may opt to leave the system, in which case another customer at the head of of the queue is taken up for his essential service.
- The service times of j optional services, $j = 1, \dots n$ have different general (arbitrary) distributions with distribution function $B_j(v)$ and the density function $b_j(v)$, $j = 1, \dots n$.
- The service channel is subject to random breakdowns and the failure time distribution is exponential with mean $\dfrac{1}{\alpha}$. Consequently the service channel may fail any time during the interval $(t, t + \mathrm{d}t]$ with the probability αdt. Further we have assumed that the breakdowns are time homogeneous which implies that the service channel may fail any time even including the period of time when it is idle.
- We assume that whenever service channel breaks down, it instantly undergoes a repair process and the repair times are deterministic of a constant (fixed) duration $d(>0)$.
- Various Stochastic Processes involved in the system are independent of each other.

3. Definitions, Notations and the Time-Dependant Equations Governing the System

We define

$P_n(x,t)$: Probability that at time t, there are $n \geq 0$ customers in the queue excluding the one being provided the first essential service and the lapsed service time of this customer is x. Accordingly, $P_n(t) = \int_0^\infty P_n(x,t)\mathrm{d}x$ denotes probability that at time t, there are n customers in the queue excluding one customer in the first essential service irrespective of the value of x.

$P_n^{(j)}(x,t)$: Probability that at time t, there are $n \geq 0$ customers in the queue excluding the one being provided the j^{th} optional service, $j = 1, 2, \ldots n$ and the lapsed service time of this customer is x. Accordingly, $P_n^{(j)}(t) = \int_0^\infty P_n(x,t)^{(j)}\mathrm{d}x$ denotes probability that at time t, there are n customers in the queue excluding one customer in the j^{th} optional service irrespective of the value of x.

$V_n(t)$: Probability that at time t, there are $n \geq 0$ customers in the queue and the server is under repair.

$Q_n(t)$: Probability that at time t, there is no customer in the system and the server is idle but available in the system. Finally, we assume that k_r is the probability of r arrivals during a repair period of duration d so that,

$$K_r = \frac{e^{-\lambda d}(\lambda d)^r}{r!}, r = 0, 1, 2, \ldots. \tag{3}$$

The system has then the following set of differential - difference equations

$$\frac{\partial}{\partial x}P_n(x,t) + \frac{\partial}{\partial t}P_n(x,t) + (\lambda + \mu(x) + \alpha)P_n(x,t) = \lambda P_{n-1}(x,t),\ n = 1, 2, \ldots \tag{4}$$

$$\frac{\partial}{\partial x}P_0(x,t) + \frac{\partial}{\partial t}P_0(x,t) + (\lambda + \mu(x) + \alpha)P_0(x,t) = 0, \tag{5}$$

$$\frac{\partial}{\partial x}P_n^{(j)}(x,t) + \frac{\partial}{\partial t}P_n^{(j)}(x,t) + (\lambda + \mu_j(x) + \alpha)P_n^{(j)}(x,t) = \lambda P_{n-1}^{(j)}(x,t),\ n, j = 1, 2, \ldots, \tag{6}$$

$$\frac{\partial}{\partial x}P_0^{(j)}(x,t) + \frac{\partial}{\partial t}P_0^{(j)}(x,t) + (\lambda + \mu_j(x) + \alpha)P_0^{(j)}(x,t) = 0,\ j = 1, 2, \ldots, n, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0(t) = \alpha Q(t) + V_0(t)[\,-K_0 - K_1 - K_2\cdots], \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) = \alpha\int_0^\infty P_{n-1}(x,t)\mu_1(x)\mathrm{d}x + \alpha\sum_{j=1}^{n}\int_0^\infty P_{n-1}^{(j)}(x,t)\mu_j(x)\mathrm{d}x + V_n(t)[\,-K_0 - K_1 - K_2\ldots],\ n, j = 1, 2, \ldots, \tag{9}$$

$$\frac{d}{dt}Q(t) = -(\lambda + \alpha)Q(t) + V_0(t)K_0 + \left[1 - \sum_{j=1}^{n} r_j\right]\int_0^\infty P_0(x,t)\mu(x)\mathrm{d}x + \alpha\sum_{j=1}^{n}\int_0^\infty P_0^{(j)}(x,t)\mu_j(x)\mathrm{d}x. \tag{10}$$

Equations (4)-(10) are to be solved subject to the following boundary conditions

$$P_0(0,t) = Q(t)\lambda + V_0(t)K_1 + V_1(t)K_0 + \left[1 - \sum_{j=1}^{n} r_j\right]\int_0^\infty P_1(x,t)\mu(x)\mathrm{d}x, + \sum_{j=1}^{n}\int_0^\infty P_1^{(j)}(x,t)\mu_j(x)\mathrm{d}x \tag{11}$$

$$\begin{aligned} P_n(0,t) &= V_0(t)K_{n+1} + V_1(t)K_n + \cdots + V_n(t)K_1 + V_{n+1}(t)K_0 + \\ &+ \left[1 - \sum_{j=1}^{n} r_j\right]\int_0^\infty P_{n+1}(x,t)\mu(x)\mathrm{d}x + \sum_{j=1}^{n}\int_0^\infty P_{n+1}^{(j)}(x,t)\mu_j(x)\mathrm{d}x,\quad n = 1, 2, \ldots, \end{aligned} \tag{12}$$

$$P_n^{(j)}(0,t) = r_j\int_0^\infty P_n(x,t)\mu_1(x)\mathrm{d}x,\ n = 0, 1, \ldots, j = 1, 2, \ldots, n, \tag{13}$$

We assume that initially there is no customer in the system and the server is idle so that the initial conditions are

$$Q(0) = 1, P_n(0) = 0,\ P_n^{(j)}(0) = 0,\ V_0(0) = 0 = V_n(0), n \geq 0, j = 1, 2, \ldots, n. \tag{14}$$

4. Generating Functions of the Queue Length: The Time-dependent Solution

We define the probability generating functions,

$$\left.\begin{aligned} P(x,z,t) &= \sum_{n=0}^{\infty} z^n P(x,t),\\ P(z,t) &= \sum_{n=0}^{\infty} z^n P(t),\\ P^{(j)}(x,z,t) &= \sum_{n=0}^{\infty} z^n P^{(j)}(x,t),\\ P^{(j)}(z,t) &= \sum_{n=0}^{\infty} z^n P^{(j)}(t), j=1,2,\ldots,n,\\ V(z,t) &= \sum_{n=0}^{\infty} z^n V_n(t). \end{aligned}\right\} \tag{15}$$

which are convergent inside the circle given by $|z| \le 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t,\ \Re(s) > 0. \tag{16}$$

Taking the Laplace transforms of equations (4) to (13) and using (14), we obtain

$$\frac{\partial}{\partial x}\overline{P}_n(x,s) + (s+\lambda+\mu(x)+\alpha)\overline{P}_n(x,s) = \lambda\overline{P}_{n-1}(x,s),\ \ n=1,2,\ldots \tag{17}$$

$$\frac{\partial}{\partial x}\overline{P}_0(x,s) + (s+\lambda+\mu(x)+\alpha)\overline{P}_0(x,s) = 0, \tag{18}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(j)}(x,s) + (s+\lambda+\mu_j(x)+\alpha)\overline{P}_n^{(j)}(x,s) = \lambda\overline{P}_{n-1}^{(j)}(x,s),\ \ n=1,2,\ldots, \tag{19}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(j)}(x,s) + (s+\lambda+\mu_j(x)+\alpha)\overline{P}_0^{(j)}(x,s) = 0,\ j=1,2,\ldots,n, \tag{20}$$

$$s\overline{V}_0(s) = \alpha\overline{Q}(s) + \overline{V}_0(s)[-K_0-K_1-K_2\ldots] \tag{21}$$

$$s\overline{V}_n(s) = \alpha\int_0^{\infty}\overline{P}_{n-1}(x,s)\mu(x)\mathrm{d}x + \alpha\sum_{j=1}^{n}\int_0^{\infty}\overline{P}_{n-1}^{(j)}(x,s)\mu_j(x)\mathrm{d}x + \overline{V}_n(s)[\,-K_0-K_1-K_2\ldots],\ \ n=1,2,\ldots, \tag{22}$$

$$(s+\lambda+\alpha)\overline{Q}(s) = 1 + \overline{V}_0(s)K_0 + \left[1-\sum_{j=1}^{n} r_j\right]\int_0^{\infty}\overline{P}_0(x,s)\mu(x)\mathrm{d}x + \sum_{j=1}^{n}\int_0^{\infty}\overline{P}_0^{(j)}(x,s)\mu_j(x)\mathrm{d}x, \tag{23}$$

$$\overline{P}_0(0,s) = \overline{Q}(s)\lambda + \overline{V}_0(s)K_1 + \overline{V}_1(s)K_0 + \left[1-\sum_{j=1}^{n} r_j\right]\int_0^{\infty}\overline{P}_1(x,s)\mu(x)\mathrm{d}x, + \sum_{j=1}^{n}\int_0^{\infty}\overline{P}_1^{(j)}(x,s)\mu_j(x)\mathrm{d}x \tag{24}$$

$$\begin{aligned} \overline{P}_n(0,s) &= \overline{V}_0(s)K_{n+1} + \overline{V}_1(s)K_n + \cdots + \overline{V}_n(s)K_1 + \overline{V}_{n+1}(s)K_0 + \\ &\quad + \left[1-\sum_{j=1}^{n} r_j\right]\int_0^{\infty}\overline{P}_{n+1}(x,t)\mu(x)\mathrm{d}x + \sum_{j=1}^{n}\int_0^{\infty}\overline{P}_{n+1}^{(j)}(x,s)\mu_j(x)\mathrm{d}x,\ \ n=1,2,\ldots, \end{aligned} \tag{25}$$

$$\overline{P}_n^{(j)}(0,s) = r_j\int_0^{\infty}\overline{P}_n(x,s)\mu_1(x)\mathrm{d}x,\ n=0,1,\ldots,j=1,2,\ldots,n, \tag{26}$$

Now multiplying equation (17) by z^n and summing over n from 1 to ∞, adding to equation (18) and using the generating functions defined in (15), we get

$$\frac{\partial}{\partial x}\overline{P}(x,z,s) + (s+\lambda-\lambda z+\mu(x)+\alpha)\overline{P}(x,z,s) = 0, \tag{27}$$

Performing similar operations on equations (19) to (22) we obtain

$$\frac{\partial}{\partial x}\overline{P}^{(j)}(x,z,s)+(s+\lambda-\lambda z+\mu_j(x)+\alpha)\overline{P}^{(j)}(x,z,s)=0, \tag{28}$$

$$(s+1)\overline{V}(z,s)=\alpha\overline{Q}(s)+\alpha z\left\{\int_0^\infty \overline{P}(x,z,s)\mathrm{d}x+\int_0^\infty \overline{P}^{(j)}(x,z,s)\mathrm{d}x\right\}. \tag{29}$$

For the boundary conditions, we multiply both sides of equation (24) by z, multiply both sides of equation (25) by z^{n+1}, sum over n from 1 to ∞, add the two results and use equation (15) to get

$$\begin{aligned} z\overline{P}(0,z,s) &= \lambda z\overline{Q}(s)+\overline{V}(z,s)e^{-\lambda d[1-z]}+\left[1-\sum_{j=1}^{n} r_j\right]\int_0^\infty \overline{P}(x,z,s)\mu(x)\mathrm{d}x \\ &+\sum_{j=1}^{n}\int_0^\infty \overline{P}^{(j)}(x,z,s)\mu_j(x)\mathrm{d}x-\sum_{j=1}^{n}\int_0^\infty \overline{P}_0^{(j)}(x,s)\mu_j(x)\mathrm{d}x-\overline{V}_0(s)K_0-\left[1-\sum_{j=1}^{n} r_j\right]\int_0^\infty \overline{P}_0(x,s)\mu(x)\mathrm{d}x. \end{aligned} \tag{30}$$

Performing similar operation on equation (26), we have

$$\overline{P}^{(j)}(0,z,s)=r_j\int_0^\infty \overline{P}(x,z,s)\mu(x)\mathrm{d}x. \tag{31}$$

Using equation (23), equation (30) become

$$\begin{aligned} z\overline{P}(0,z,s) &= \left[1-\sum_{j=1}^{n} r_j\right]\int_0^\infty \overline{P}(x,z,s)\mu(x)\mathrm{d}x+\sum_{j=1}^{n}\int_0^\infty \overline{P}^{(j)}(x,z,s)\mu_j(x)\mathrm{d}x \\ &+\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]-[-\lambda z+\lambda+\alpha]\overline{Q}(s). \end{aligned} \tag{32}$$

Integrating equation (27) from 0 to x yields

$$\overline{P}(x,z,s)=\overline{P}(0,z,s)\ e^{-(s+\lambda-\lambda z+\alpha)x-\int_0^x \mu(t)dt}, \tag{33}$$

where $\overline{P}(0,z,s)$ is given by equation (32) Again integrating equation (33) by parts with respect to x yields

$$\overline{P}(z,s)=\overline{P}(0,z,s)\left[\frac{1-\bar{b}(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right], \tag{34}$$

where

$$\bar{b}(s+\lambda-\lambda z+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda z+\alpha)x}db(x) \tag{35}$$

is the Laplace-Stieltjes transform of the essential service time $b(x)$. Now multiplying both sides of equation (33) by $\mu(x)$ and integrating over x, we obtain

$$\int_0^\infty \overline{P}(x,z,s)\mu(x)\mathrm{d}x=\overline{P}(0,z,s)\bar{b}(s+\lambda-\lambda z+\alpha). \tag{36}$$

Similarly, on integrating equation (28) from 0 to x, we get

$$\overline{P}^{(j)}(x,z,s)=\overline{P}^{(j)}(0,z,s)\ e^{-(s+\lambda-\lambda z+\alpha)x-\int_0^x \mu_j(t)dt}, \tag{37}$$

where $\overline{P}^{(j)}(0,z,s)$ is given by equation (31). Again integrating equation (37) by parts with respect to x yields

$$\overline{P}^{(j)}(z,s)=\overline{P}^{(j)}(0,z,s)\left[\frac{1-\bar{b}_j(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right], \tag{38}$$

where

$$\bar{b}_j(s+\lambda-\lambda z+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda z+\alpha)x}db_j(x) \tag{39}$$

is the Laplace-Stieltjes transform of n optional service times $b_j(x)$. We see that by virtue of equation (37), we have

$$\int_0^\infty \overline{P}^{(j)}(x,z,s)\mu_j(x)\mathrm{d}x=\overline{P}^{(j)}(0,z,s)\bar{b}_j(s+\lambda-\lambda z+\alpha). \tag{40}$$

By using equation (36), equation (31) reduces to

$$\overline{P}^{j}(0,z,s)=r_j\overline{P}(0,z,s)\bar{b}(s+\lambda-\lambda z+\alpha). \tag{41}$$

Using equation (41), equation (40) becomes

$$\int_0^\infty \overline{P}^{(j)}(x,z,s)\mu_j(x)\mathrm{d}x=r_j\overline{P}(0,z,s)\bar{b}(s+\lambda-\lambda z+\alpha)\bar{b}_j(s+\lambda-\lambda z+\alpha). \tag{42}$$

By using above equation (32) reduces to

$$\overline{P}(0,z,s)=\frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]-[-\lambda z+\lambda+\alpha]\overline{Q}(s)}{z-\bar{b}(s+\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)-\sum\limits_{j=1}^{n}\bar{b}(s+\lambda-\lambda z+\alpha)\bar{b}_j(s+\lambda-\lambda z+\alpha)}. \tag{43}$$

Substituting the value of $\overline{P}(0,z,s)$ into equation (4.20), we get

$$\overline{P}(z,s)=\frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]-[-\lambda z+\lambda+\alpha]\overline{Q}(s)}{z-\bar{b}(s+\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)-\sum\limits_{j=1}^{n}\bar{b}(s+\lambda-\lambda z+\alpha)\bar{b}_j(s+\lambda-\lambda z+\alpha)}\left[\frac{1-\bar{b}(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right]. \tag{44}$$

Now using equations (41) and (43), equation (38) become

$$\begin{aligned}\overline{P}^{(j)}(z,s) &= r_j\frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]-[-\lambda z+\lambda+\alpha]\overline{Q}(s)}{z-\bar{b}(s+\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)-\sum\limits_{j=1}^{n}\bar{b}(s+\lambda-\lambda z+\alpha)\bar{b}_j(s+\lambda-\lambda z+\alpha)}\\ &\quad \bar{b}(s+\lambda-\lambda z+\alpha)\left[\frac{1-\bar{b}_j(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right],j=1,2,\ldots,n.\end{aligned} \tag{45}$$

From equation (29)

$$\overline{V}(z,s)=\frac{\alpha\overline{Q}(s)+\alpha z\{\overline{P}(z,s)+\overline{P}^{(j)}(z,s)\}}{(s+1)}\;,j=1,2,\ldots,n. \tag{46}$$

Let $\overline{P}(z,s)=\overline{P}(z,s)+\overline{P}^{(j)}(z,s), j=1,2,\ldots,n$ denote the probability generating function of the number in the queue irrespective of the type of service being provided. Then adding equations (44) and (45) we have

$$\begin{aligned}\overline{W}(z,s) &= \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]-[-\lambda z+\lambda+\alpha]\overline{Q}(s)}{z-\bar{b}(s+\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)-\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)\bar{b}_j(s+\lambda-\lambda z+\alpha)}\\ &\quad\left[\frac{1-\bar{b}(s+\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)-\sum\limits_{j=1}^{n}r_j\bar{b}(s+\lambda-\lambda z+\alpha)\bar{b}_j(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right].\end{aligned} \tag{47}$$

Thus substituting the value of $\overline{V}(z,s)$ from equation (46) into equation (47) we get

$$\overline{W}(z,s) \;=\; \frac{\overline{N}(z,s)}{\overline{D}(z,s)} \tag{48}$$

$$\begin{aligned}\overline{N}(z,s) \;=\; & \left\{\alpha\overline{Q}(s)e^{-\lambda d[1-z]} - [-\lambda z+\lambda+\alpha]\,(s+1) + \overline{Q}(s)[1-s\overline{Q}(s)](s+1)\right\} \\ & \left\{1-\overline{b}(s+\lambda-\lambda z+\alpha) + \sum_{j=1}^{n} r_j\overline{b}(s+\lambda-\lambda z+\alpha) - \sum_{j=1}^{n} r_j\overline{b}(s+\lambda-\lambda z+\alpha)\overline{b}_j(s+\lambda-\lambda z+\alpha)\right\},\end{aligned} \tag{49}$$

$$\begin{aligned}&\overline{D}(z,s) = (s+\lambda-\lambda z+\alpha)(s+1)\left[z-\overline{b}(s+\lambda-\lambda z+\alpha) + \sum_{j=1}^{n} r_j\overline{b}(s+\lambda-\lambda z+\alpha)\right.\\ &\left.-\sum_{j=1}^{n} r_j\overline{b}(s+\lambda-\lambda z+\alpha)\overline{b}_j(s+\lambda-\lambda z+\alpha)\right] - \alpha z e^{-\lambda d[1-z]}\left\{1-\overline{b}(s+\lambda-\lambda z+\alpha)\right.\\ &\qquad\left\{+\sum_{j=1}^{n} r_j\overline{b}(s+\lambda-\lambda z+\alpha) - \sum_{j=1}^{n} r_j\overline{b}(s+\lambda-\lambda z+\alpha)\overline{b}_j(s+\lambda-\lambda z+\alpha)\right\}.\end{aligned} \tag{50}$$

If we let $z=1$ in equation (48), we can easily verify that

$$\overline{Q}(s)+\overline{V}(z,s)+\overline{W}(z,s) = \frac{1}{s}, \tag{51}$$

as it should be. Thus $\overline{V}(z,s)$, $\overline{P}(z,s)$ and $\overline{P}^{(j)}(z,s), j=1,2,\ldots,n$ are completely determined from equations (46), (44) and (45) respectively.

5. Steady State Solution

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0}\overline{f}(s) = \lim_{t\to\infty} f(t). \tag{52}$$

In order to determine $\overline{P}(z,s)$, $\overline{P}^{(j)}(z,s)$, $j=1,2,\ldots,n$ and $\overline{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (44), (45) and (46) by using initial conditions (43) and (41). For that purpose, we shall use the normalizing condition

$$P(1)+\sum_{j=1}^{n}P^{(j)}(1)+V(1)+Q = 1. \tag{53}$$

Thus multiplying both sides of equations (44), (45) and (46) by s, taking limit as $s\to 0$, applying property (52) and simplifying we have

$$P(z) \;=\; \left[\frac{V(z)e^{-\lambda d[1-z]} - [-\lambda z+\lambda+\alpha]Q}{z-b(\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha) - \sum\limits_{j=1}^{n}\overline{b}(\lambda-\lambda z+\alpha)\overline{b}_j(\lambda-\lambda z+\alpha)}\right]\left[\frac{1-b(\lambda-\lambda z+\alpha)}{\lambda-\lambda z+\alpha}\right], \tag{54}$$

$$\begin{aligned}P^{(j)}(z) \;=\; & r_j\frac{V(z)e^{-\lambda d[1-z]} - [-\lambda z+\lambda+\alpha]Q}{z-b(\lambda-\lambda z+\alpha)+\sum\limits_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha) - \sum\limits_{j=1}^{n}\overline{b}(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)} \\ & b(\lambda-\lambda z+\alpha)\left[\frac{1-b_j(\lambda-\lambda z+\alpha)}{\lambda-\lambda z+\alpha}\right], j=1,2,\ldots,n.\end{aligned} \tag{55}$$

and

$$V(z) = \alpha Q + \alpha z W(z), \tag{56}$$

where

$$\begin{aligned} W(z) &= P(z) + \sum_{j=1}^{n} P^{(j)}(z),\ j = 1, 2, \ldots, n, \\ &= \frac{N(z)}{D(z)}\ , \end{aligned} \tag{57}$$

where

$$\begin{aligned} N(z) &= \left\{1 - b(\lambda - \lambda z + \alpha) + \sum_{j=1}^{n} r_j b(\lambda - \lambda z + \alpha) - \sum_{j=1}^{n} r_j b(\lambda - \lambda z + \alpha) b_j(\lambda - \lambda z + \alpha)\right\} \\ &\quad \left\{\alpha Q e^{-\lambda d[1-z]} - [-\lambda z + \lambda + \alpha]\right\} Q, \end{aligned} \tag{58}$$

$$\begin{aligned} D(z) &= (\lambda - \lambda z + \alpha)\left[z - b(\lambda - \lambda z + \alpha) + \sum_{j=1}^{n} r_j b(\lambda - \lambda z + \alpha)\right. \\ &\quad \left. - \sum_{j=1}^{n} r_j \bar{b}(\lambda - \lambda z + \alpha) b_j(\lambda - \lambda z + \alpha)\right] - \alpha z e^{-\lambda d[1-z]} \{1 - b(\lambda - \lambda z + \alpha) \\ &\quad \left. + \sum_{j=1}^{n} r_j b(\lambda - \lambda z + \alpha) - \sum_{j=1}^{n} r_j b(\lambda - \lambda z + \alpha) b_j(\lambda - \lambda z + \alpha)\right\}. \end{aligned} \tag{59}$$

We see that for $z = 1$, $W(z)$ in equation (57) is indeterminate of the $\frac{0}{0}$ form. Therefore, we apply L'Hopital's rule on equation (57) and on simplifying we get

$$\begin{aligned} W(1) &= \lim_{z \to 1} W(z) \\ &= \frac{\left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j b(\alpha) - \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right] [\lambda + \alpha\lambda d]\, Q}{\alpha\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j b(\alpha) + \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right] - [\lambda + \alpha\lambda d]\left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j b(\alpha) - \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right]}. \end{aligned} \tag{60}$$

Now using (60) in equation (56) we have

$$V(1) = \frac{\alpha^2 Q\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j b(\alpha) + \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right]}{\alpha\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j b(\alpha) + \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right] - [\lambda + \alpha\lambda d]\left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j b(\alpha) - \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right]}. \tag{61}$$

Now since we must have $Q + W(1) + V(1) = 1$, we have

$$Q = \frac{\alpha\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j b(\alpha) + \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right] - \left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j b(\alpha) - \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right] [\lambda + \lambda\alpha d]}{\alpha(\alpha + 1)\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j b(\alpha) + \sum_{j=1}^{n} r_j \bar{b}(\alpha) b_j(\alpha)\right]} \tag{62}$$

which is the steady state probability that the server is idle but operative. Then we substitute the value of Q from (62) into (57) we have

$$W(z) = \frac{N_1(z)}{D_1(z)}$$

where

$$
\begin{aligned}
N_1(z) &= \left[1-\overline{b}(\lambda-\lambda z+\alpha)+\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\overline{b}_j(\lambda-\lambda z+\alpha)\right] \\
&\frac{\alpha\left[\overline{b}(\alpha)-\sum_{j=1}^{n} r_j b(\alpha)+\sum_{j=1}^{n} r_j\overline{b}(\alpha)b_j(\alpha)\right]-\left[1-\overline{b}(\alpha)+\sum_{j=1}^{n} r_j b(\alpha)-\sum_{j=1}^{n} r_j\overline{b}(\alpha)b_j(\alpha)\right][\lambda+\lambda\alpha d]}{\alpha(\alpha+1)\left[\overline{b}(\alpha)-\sum_{j=1}^{n} r_j b(\alpha)+\sum_{j=1}^{n} r_j\overline{b}(\alpha)b_j(\alpha)\right]}\left\{\alpha e^{-\lambda d[1-z]}-[-\lambda z+\lambda+\alpha]\right\}, \quad (63) \\
D_1(z) &= (\lambda-\lambda z+\alpha)\left[z-b(\lambda-\lambda z+\alpha)+\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)\right. \\
&\left.-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)\right]-\alpha z e^{-\lambda d[1-z]}\left\{1-b(\lambda-\lambda z+\alpha)\right. \\
&\left.+\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)\right\}. \quad (64)
\end{aligned}
$$

Next, using above equation in $V(z)$ and simplifying, we have

$$
V(z)=\alpha Q\left\{\frac{(\lambda-\lambda z+\alpha)\left[\overline{b}(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)+\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\overline{b}_j(\lambda-\lambda z+\alpha)\right][z-1]}{D(z)}\right\}, \quad (65)
$$

where $D(z)$ is given by (59) and Q is given by (62). Thus $W(z)$ and $V(z)$ have been completely and explicitly determined in above equations.

6. The Mean Number in the System

Let $P_q(z)=W(z)+V(z)$ denote the probability generating function of the queue length irrespective of whether the server is operative or in failed state. Then adding $V(z)$ and $W(z)$ and simplifying, we have

$$
P_q(z)=\frac{N_2(z)}{D_2(z)} \quad (66)
$$

where where

$$
\begin{aligned}
N_2(z) &= \alpha Q\left\{(\lambda-\lambda z+\alpha)\left[\overline{b}(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\right.\right. \\
&\left.\left.+\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\overline{b}_j(\lambda-\lambda z+\alpha)\right][z-1]\right\} \\
&+\left[1-\overline{b}(\lambda-\lambda z+\alpha)+\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\overline{b}_j(\lambda-\lambda z+\alpha)\right] \\
&\left\{\alpha e^{-\lambda d[1-z]}-[-\lambda z+\lambda+\alpha]\right\}Q, \quad (67) \\
D_2(z) &= (\lambda-\lambda z+\alpha)\left[z-b(\lambda-\lambda z+\alpha)+\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)\right. \\
&\left.-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)\right]-\alpha z e^{-\lambda d[1-z]}\left\{1-b(\lambda-\lambda z+\alpha)\right. \\
&\left.+\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)\right\}. \quad (68)
\end{aligned}
$$

and Q is given by equation (62). Let L_q denote the mean number of customers in the queue under the steady state. Then

$$
L_q=\frac{\mathrm{d}}{\mathrm{d}z}P_q(z)\Big|_{z=1}
$$

Since this formula gives 0/0 form, we write $P_q(z)$ given in (6.1) as $P_q(z) = \dfrac{N_2(z)}{D_2(z)}$ where $N_2(z)$ and $D_2(z)$ are the numerator and denominator of the right hand side of (66) respectively. Then we use following well known result in queueing theory (see Kashyap and Chaudhry [9])

$$\begin{aligned} L_q &= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}P_q(z),\\ &= P_q^{'}(1),\\ &= \lim_{z\to 1}\frac{D^{'}(z)N^{''}(z) - N^{'}(z)D^{''}(z)}{2(D^{'}(z))^2},\\ &= \lim_{z\to 1}\frac{D^{'}(1)N^{''}(1) - N^{'}(1)D^{''}(1)}{2(D^{'}(1))^2}. \end{aligned} \tag{69}$$

where primes and double primes in equation (69) denote the first and second derivative at $z = 1$. Carrying out the derivatives at $z = 1$, we have

$$\begin{aligned} N^{'}(1) &= [\alpha\lambda d + \lambda]\left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right]\\ &+\alpha^2\left[\bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right], \end{aligned} \tag{70}$$

$$\begin{aligned} N^{''}(1) &= \alpha(\lambda d)^2\left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right]\\ &+2\lambda^2[1+\alpha d]\left[\bar{b}^{'}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}^{'}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}^{'}(\alpha)\bar{b}_j(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j^{'}(\alpha)\right]\\ &-2\alpha^2\lambda\left[\bar{b}^{'}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}^{'}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}^{'}(\alpha)\bar{b}_j(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j^{'}(\alpha)\right]\\ &-2\alpha\lambda\left[\bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right], \end{aligned} \tag{71}$$

$$\begin{aligned} D^{'}(1) &= \left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right][-\lambda - \alpha\lambda d]\\ &+\alpha\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right], \end{aligned} \tag{72}$$

$$\begin{aligned} D^{''}(1) &= [1+\alpha d]\left[-2\lambda - 2\lambda^2\bar{b}^{'}(\alpha) + 2\lambda^2\sum_{j=1}^{n} r_j\bar{b}^{'}(\alpha) - 2\lambda^2\sum_{j=1}^{n} r_j\bar{b}^{'}(\alpha)\bar{b}_j(\alpha)\right.\\ &\left.-2\lambda^2\sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j^{'}(\alpha)\right] - \alpha\lambda^2 d^2\left[1 - \bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right]\\ &-2\alpha\lambda\bar{b}_1^{'}(\alpha) + 2\alpha\lambda d\left[\bar{b}(\alpha) - \sum_{j=1}^{n} r_j\bar{b}(\alpha) + \sum_{j=1}^{n} r_j\bar{b}(\alpha)\bar{b}_j(\alpha)\right] \end{aligned} \tag{73}$$

Then if we substitute the values of $N^{'}(1), N^{''}(1), D^{'}(1)$ and $D^{''}(1)$ from equations (70) to (73) into equation (69) we obtain L_q in closed form. Further let $P(z)$ denote the probability generating function of the number in the system. Then from above equations and from (62) and simplifying we have

$$\begin{aligned} P(z) &= Q + zP_q(z)\\ &= \frac{N_3(z)}{D_3(z)}, \end{aligned} \tag{74}$$

where

$$N_3(z) = (\lambda-\lambda z+\alpha)\left[\overline{b}(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\right.$$
$$\left.+\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)\overline{b}_j(\lambda-\lambda z+\alpha)\right]Q(1+\alpha z), \tag{75}$$

$$D_3(z) = (\lambda-\lambda z+\alpha)\left[z-b(\lambda-\lambda z+\alpha)+\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)\right.$$
$$\left.-\sum_{j=1}^{n} r_j\overline{b}(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)\right]-\alpha z e^{-\lambda d[1-z]}\left\{1-b(\lambda-\lambda z+\alpha)\right.$$
$$\left.+\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)-\sum_{j=1}^{n} r_j b(\lambda-\lambda z+\alpha)b_j(\lambda-\lambda z+\alpha)\right\}. \tag{76}$$

and Q is given by equation (62). Moreover, we find the average system size L_s using Little's formula. Thus we have

$$L_s = L_q + \rho \tag{77}$$

where L_q has been found in equation (69) and ρ is obtained from equation (62) as

$$\rho = 1 - Q. \tag{78}$$

7. The Mean Waiting Time

Let W_q and W_s denote the mean waiting time in the queue and the system respectively. Then using Little's formula, we obtain,

$$W_q = \frac{L_q}{\lambda} \tag{79}$$

$$W_s = \frac{L_s}{\lambda} \tag{80}$$

where L_q and L_s have been found in equations (69) and (77) respectively.

8. Special Cases

8.1. Case I. No Optional Services, Random Breakdowns and Deterministic Repair Times

In this case, we assume that are no optional services and the essential service is provided to all the arriving customers. Therefore the results (62) to (65), (66) to (68) and (74) to (76) reduce to

$$Q = \frac{\alpha\overline{b}(\alpha)-[1-\overline{b}(\alpha)][\lambda+\lambda\alpha d]}{\alpha(\alpha+1)\overline{b}(\alpha)} \tag{81}$$

$$W(z) = \frac{[1-\overline{b}(\lambda-\lambda z+\alpha)]\left\{\alpha e^{-\lambda d[1-z]}-[-\lambda z+\lambda+\alpha]\right\}Q}{[-\lambda z+\lambda+\alpha][z-\overline{b}(\lambda-\lambda z+\alpha)]-[1-\overline{b}(\lambda-\lambda z+\alpha)]\alpha z e^{-\lambda d[1-z]}}, \tag{82}$$

$$V(z) = \frac{\alpha(-\lambda z+\lambda+\alpha)\left\{z-\overline{b}(\lambda-\lambda z+\alpha)\right\}Q}{[-\lambda z+\lambda+\alpha][z-\overline{b}(\lambda-\lambda z+\alpha)]-[1-\overline{b}(\lambda-\lambda z+\alpha)]\alpha z e^{-\lambda d[1-z]}}, \tag{83}$$

$$P(z) = \frac{(\lambda-\lambda z+\alpha)\overline{b}(\lambda-\lambda z+\alpha)[z-1](1+\alpha z)Q}{[-\lambda z+\lambda+\alpha]\left\{z-\overline{b}(\lambda-\lambda z+\alpha)\right\}-[1-\overline{b}(\lambda-\lambda z+\alpha)]\alpha z e^{-\lambda d[1-z]}},$$

$$P_q(z) = \frac{P_q(Nr)}{P_q(Dr)} \tag{84}$$

where

$$P_q(Nr) = \Big\{[1-\bar{b}(\lambda-\lambda z+\alpha)][\alpha e^{-\lambda d[1-z]} - (\lambda-\lambda z+\alpha)] + \alpha(\lambda-\lambda z+\alpha)\bar{b}(\lambda-\lambda z+\alpha)[z-1]\Big\} Q \quad (85)$$

$$P_q(Dr) = [-\lambda z+\lambda+\alpha][z-\bar{b}(\lambda-\lambda z+\alpha)] - [1-\bar{b}(\lambda-\lambda z+\alpha)]\alpha z e^{-\lambda d[1-z]}, \quad (86)$$

where Q in the right hand side of equations (82) to (85) is given by (81).

8.2. Case II. Exponential Essential Service, no Optional Services, Random Breakdowns and Deterministic Repair Times.

In this case, we assume that there are no optional services and the essential service follows exponentional service time. Therefore, we have

$$\bar{b}(\alpha) = \int_0^\infty e^{-\alpha x}\mu e^{-\mu x}\mathrm{d}x = \frac{\mu}{\alpha+\mu}$$

and similarly

$$\bar{b}(\lambda-\lambda z+\alpha) = \frac{\mu}{\lambda+\mu+\alpha-\lambda z}.$$

With these substitutions into the results (62) to (65), (66) to (68) and (74) to (76), we obtain

$$Q = \frac{\mu-[\lambda+\lambda\alpha d]}{(\alpha+1)\mu}, \quad (87)$$

$$W(z) = \frac{[\frac{\lambda-\lambda z+\alpha}{\lambda-\lambda z+\alpha+\mu}][\alpha e^{-\lambda d[1-z]} - (\lambda-\lambda z+\alpha)]\left[\frac{\mu-[\lambda+\lambda\alpha d]}{(\alpha+1)\mu}\right]}{(\lambda-\lambda z+\alpha)\left[\frac{(\lambda-\lambda z+\alpha+\mu)z-\mu}{\lambda-\lambda z+\alpha+\mu}\right] - \left[\frac{\lambda-\lambda z+\alpha}{\lambda-\lambda z+\alpha+\mu}\right]\alpha z e^{-\lambda d[1-z]}}, \quad (88)$$

$$V(z) = \frac{\left[\frac{\alpha\mu(\lambda-\lambda z+\alpha)}{\lambda-\lambda z+\alpha+\mu}\right][z-1]\left[\frac{\mu-[\lambda+\lambda\alpha d]}{(\alpha+1)\mu}\right]}{(\lambda-\lambda z+\alpha)\left[\frac{(\lambda-\lambda z+\alpha+\mu)z-\mu}{\lambda-\lambda z+\alpha+\mu}\right] - \left[\frac{\lambda-\lambda z+\alpha}{\lambda-\lambda z+\alpha+\mu}\right]\alpha z e^{-\lambda d[1-z]}} \quad (89)$$

$$P(z) = \frac{\left[\frac{(\lambda-\lambda z+\alpha)\mu}{\lambda-\lambda z+\alpha+\mu}\right][z-1](1+\alpha z)\left[\frac{\mu-[\lambda+\lambda\alpha d]}{(\alpha+1)\mu}\right]}{(\lambda-\lambda z+\alpha)\left[\frac{(\lambda-\lambda z+\alpha+\mu)z-\mu}{\lambda-\lambda z+\alpha+\mu}\right] - \left[\frac{\lambda-\lambda z+\alpha}{\lambda-\lambda z+\alpha+\mu}\right]\alpha z e^{-\lambda d[1-z]}}, \quad (90)$$

$$P_q(z) = \frac{\left\{\left[\frac{\lambda-\lambda z+\alpha}{\lambda-\lambda z+\alpha+\mu}\right][\alpha e^{-\lambda d[1-z]} - (\lambda-\lambda z+\alpha)] + \frac{\alpha\mu(\lambda-\lambda z+\alpha)}{\lambda-\lambda z+\alpha+\mu}[z-1]\right\} Q}{(\lambda-\lambda z+\alpha)\left[\frac{(\lambda-\lambda z+\alpha+\mu)z-\mu}{\lambda-\lambda z+\alpha+\mu}\right] - \left[\frac{\lambda-\lambda z+\alpha}{\lambda-\lambda z+\alpha+\mu}\right]\alpha z e^{-\lambda d[1-z]}}$$

(91)

8.3. Case III. Exponential Essential Service, no Optional Services, no Random Breakdowns

If the system suffers no breakdowns, then letting $\alpha = 0$ in the main results, we obtain $V(z) = 0$,

$$Q = \frac{\mu-\lambda}{\mu} \quad (92)$$

$$P(z) = \frac{\bar{b}(\lambda-\lambda z)\left[1-\frac{\lambda}{\mu}\right](1-z)}{\bar{b}(\lambda-\lambda z)-z}, \quad (93)$$

$$P_q(z) = \frac{[\bar{b}(\lambda-\lambda z)-1]\left[1-\frac{\lambda}{\mu}\right]}{z-\bar{b}(\lambda-\lambda z)}. \quad (94)$$

Acknowledgement

The author thanks the management of SSN College of Engineering for providing the necessary requirements during the preparation of this paper.

References

[1] A.Aissani and J.R.Artajelo, *On the single server retrial queue subject to breakdowns*, Queueing Systems, 30(1998), 309-321.

[2] M.Kramer, *Stationary distributions in Queueing systems with vacation times and limited service*, Queueing Systems, 4(1)(1989), 57-68.

[3] B.T.Doshi, *Queueing systems with vacations-A survey*, Queueing System, 1(1986), 29-66.

[4] A.Federgruen and K.C.So, *Optimal maintenance policies for single server queueing systems subject to breakdowns*, Oper. Res., 38(1990), 330-343.

[5] S.W.Fuhrmann and R.B.Cooper, *Stochastic decompositions in the $M/G/1$ queue with generalized vacations*, Oper. Res., 83(1985), 1117-1129.

[6] D.P.Gaver, *Imbedded Markov chain analysis of a waiting time process in continuous time*, Ann. Math. Statist., 30(1959), 698-720.

[7] A.K.Jayawardene and O.Kella, *$M/G/\infty$ with altering renewal breakdowns*, Queueing Systems, 22(1996), 79-95.

[8] J.Keilson and L.D.Servi, *Oscillating random walk models for $GI/G/1$ vacation systems with Bernoulli schedules*, J.Appl.Prob., 25(1986), 453-471.

[9] V.G.Kulkarni and B.D.Choi, *Retrial queues with server subject to breakdowns and repairs*, Queueing Systems, 7(1990), 191-209.

[10] Y.Levy and U.Yechiali, *An $M/M/s$ queue with server vacations*, INFOR: Information Systems and Operational Research, 14(1976), 153-163.

[11] K.C.Madan, *On a single server queue with two hetrogeneous service and deterministic server vacations*, Int. J . Syt. Sci., 32(2001), 837-844.

[12] K.C.Madan, W.Abu Dayyeh and M.Gharaibeh, *On two parallel servers with random breakdowns*, Soochow. J. Math., 29(2003), 413-423.

[13] K.C.Madan, W.Abu Dayyeh and M.Gharaibeh, *Steady state analysis of two $M^x/M^{a,b}/1$ queue models with random breakdowns* , Inf. Mag. Sci., 14(2003), 37-51.

[14] J.G.Shantikumar, *On stochastic decomposition in $M/G/1$ type queues with generalised vacations*, Oper. Res., 36(1988), 566-569.

[15] H.Takagi, *Queueing Analysis : Vacation and Priority Systems*, Vol. 1, North Holland, Amsterdam, (1991).

[16] J.Wang, J.Cao and Q.Li, *Reliability analysis of the retrial queue with server breakdowns and repairs*, Queueing Systems, 38(2001), 363-380.

[17] B.Vinck and H.Bruneel, *System delay versus system content for discrete time queueing systems subject to server interruptions*, Euro. J. Oper. Res., 175(2006), 362-375.