International Journal of Mathematics And its Applications
Volume 4, Issue 4 (2016), 39–50.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Common Fixed Point Theorems in Multiplicative Metric Spaces Satisfying E.A. Property and (CLR) Property for Rational Contractive Map

Research Article

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Abstract: In this paper we prove common fixed point theorems for weakly compatible mappings along with E.A property and common limit range properties using Rational contraction satisfying implicit functions in multiplicative metric space.

MSC: 46S40, 47H10, 54H25.

Keywords: E.A. property, common limit range properties, implicit function, common fixed point, multiplicative metric space.

1. Introduction and Preliminaries

It is well known that the set of all positive real numbers is not complete with respect to usual metric but it is observed that set of all real numbers is a complete multiplicative metric space with respect to the multiplicative absolute value function. This problem was overcome in 2008, by Bashirov [1] by introducing the new metric space named Multiplicative Metric Space. The notion of convergence in multiplicative metric space and related fixed point theorems in multiplicative metric space was introduced by Özavsar and Cevikel initiated [7]. We start with definition and topological definitions of multiplicative metric space. Also we use the notations $\mathbb{R}$ to represent set of real numbers and $\mathbb{R}_+$ is used to represent set of all positive real numbers.

Definition 1.1 ([1]). *Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:*

(1.1) $d(x, y) \geq 1 \; \forall \; x,y \in X$ *and* $d(x,y)=1$ *if and only if* $x=y$;

(1.2) $d(x, y)=d(y,x) \; \forall \; x, y \in X$;

(1.3) $d(x, y) \leq d(x, z).d(z, y) \; x,y \in X$ *(multiplicative triangle inequality).*

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Definition 1.2. *Let f and g be two mappings of a multiplicative metric space $(X,\ d)$ into itself, then f and g are said to be*

(1.4) commutative mapping if $fgx = gfx\ \forall\ x \in X$.

(1.5) Weak commutative mapping if $d(fgx, gfx)\ \leq d(fx, gx)\ \forall x \in X$.

(1.6) Weakly compatible if f and g commute at coincidence points, that is, $ft = gt$ for some $t \in X$. Implies that $fgt = gft$.

(1.7) E.A property if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

(1.8) CLR_g property (common limit range of g property) if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$ for some $t \in X$.

(1.9) CLR_f property (common limit range of f property) if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = ft$ for some $t \in X$.

2. Main Results

The concept of implicit functions is used by Popa [11], which is an effective contractive condition in multiplicative metric space. Implicit relations on metric spaces have been used by many authors [6, 9, 12]. In this section to prove the main result we define a suitable class of the implicit function involving four real non-negative arguments as follows:
Let Ψ denote the family of functions such that $\phi : \mathbb{R}_+^4 \to \mathbb{R}_+$ is continuous and increasing in each coordinate variable and

(1) $\phi(t, t.t_1, 1, t) \leq t.t_1$

(2) $\phi(t, 1, t.t_1, t_1) \leq t.t_1$

(3) $\phi(1, t, 1, 1) \leq t$

(4) $\phi(t, 1, t, 1) \leq t$

(5) $\phi(t, t, t, 1) \leq t$

(6) $\phi(t, t, 1, 1) \leq t$

for every $t, t_1 \in \mathbb{R}_+$ $(t,\ t_1 \geq 1)$. It is obvious that $\phi(1, 1, 1, 1) = 1$. There exist many functions $\phi \in \Psi$. Now we prove the following theorems for weakly compatible mappings satisfying implicit function in a multiplicative metric space.

Theorem 2.1. *Let A, B, S, T be mappings of a multiplicative metric space $(X,\ d)$ into itself satisfying*

(E1) $SX \subset BX$ and $TX \subset AX$

(E2) $d(Sx, Ty) \leq \left\{ \phi \left\{ \begin{matrix} \frac{d(Ax,By)[d(Ax,Sx)+d(Ty,Sx)]}{d(By,Ty)+d(By,Ax)}, \frac{d(Ax,By)[d(Ty,Sx)+d(Ax,Ty)]}{d(By,Ax)+d(Sx,By)}, \\ \frac{d(Ty,Sx)[d(By,Ax)+d(Sx,By)]}{d(Ty,Sx)+d(Ax,Ty)}, \frac{d(Ax,Sx)[d(By,Ty)+d(By,Ax)]}{d(Ax,Sx)+d(Ty,Sx)} \end{matrix} \right\} \right\}^{\lambda}$ *For all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$ and $\phi \in \Psi$;*

(E3) let us suppose that the pairs $(A,\ S)$ and $(B,\ T)$ are weakly compatible;

(E4) One of the subspaces AX or BX or SX or TX is complete

Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any arbitrary point of metric space X. It is given that $SX \subset BX$, hence there exist $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$. Now for this x_1 where exists $x_2 \in X$ in such a way that $Ax_2 = Tx_1 = y_1$. In a similar way, we can define an inductive sequence $\{y_n\}$ in such a way that,

$$Sx_{2n} = Bx_{2n+1} = y_{2n}, Ax_{2n+2} = Tx_{2n+1} = y_{2n+1}.$$

Next, we prove that $\{y_n\}$ is a multiplicative cauchy sequence in X. in fact, $\forall\ n \in \mathbb{N}$, we have, From (E2), we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\leq d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \left\{\phi\left\{\begin{array}{c}
\frac{d(Ax_{2n},Bx_{2n+1})[d(Ax_{2n},Sx_{2n})+d(Tx_{2n+1},Sx_{2n})]}{d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n})}, \\
\frac{d(Ax_{2n},Bx_{2n+1})[d(Tx_{2n+1},Sx_{2n})+d(Ax_{2n},Tx_{2n+1})]}{d(Bx_{2n+1},Ax_{2n})+d(Sx_{2n},Bx_{2n+1})}, \\
\frac{d(Tx_{2n+1},Sx_{2n})[d(Bx_{2n+1},Ax_{2n})+d(Sx_{2n},Bx_{2n+1})]}{d(Tx_{2n+1},Sx_{2n})+d(Ax_{2n},Tx_{2n+1})}, \\
\frac{d(Ax_{2n},Sx_{2n})[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n})]}{d(Ax_{2n},Sx_{2n})+d(Tx_{2n+1},Sx_{2n})}
\end{array}\right\}\right\}^{\lambda} \\
&\leq \left\{\phi\left\{\begin{array}{c}
\frac{d(y_{2n-1},\ y_{2n})[d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})]}{d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})}, \\
\frac{d(y_{2n-1},y_{2n})[d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n+1})]}{d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})}, \\
\frac{d(y_{2n+1},y_{2n})[d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})]}{d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n+1})}, \\
\frac{d(y_{2n-1},y_{2n})[d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})]}{d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})}
\end{array}\right\}\right\}^{\lambda} \\
&\leq \left\{\phi\left\{\begin{array}{c}
\frac{d(y_{2n-1},y_{2n})[d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})]}{d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})}, \\
\frac{d(y_{2n-1},y_{2n})[d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1})]}{d(y_{2n},y_{2n-1})+1}, \\
\frac{d(y_{2n+1},y_{2n})[d(y_{2n},y_{2n-1})+1]}{d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1})}, \\
\frac{d(y_{2n-1},y_{2n})[d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})]}{d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})}
\end{array}\right\}\right\}^{\lambda} \\
&\leq \left\{\phi\left\{\begin{array}{c}
d(y_{2n-1}, y_{2n}), \\
d(y_{2n-1}, y_{2n}).d(y_{2n}, y_{2n+1}), \\
1, \\
d(y_{2n-1}, y_{2n})
\end{array}\right\}\right\}^{\lambda} \\
d(y_{2n}, y_{2n+1}) &\leq d^{\lambda}(y_{2n-1}, y_{2n}).d^{\lambda}(y_{2n}, y_{2n+1}) \quad \text{[using (i)]}
\end{aligned}$$

This implies that,

$$d(y_{2n}, y_{2n+1}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n})$$

On substituting, $h = \frac{\lambda}{1-\lambda} \in (0, \frac{1}{2})$

$$d(y_{2n}, y_{2n+1}) \leq d^{h}(y_{2n-1}, y_{2n}),$$

In a similar way we have,

$$\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &= d(Tx_{2n+1}, Sx_{2n+2}) = d(Sx_{2n+2}, Tx_{2n+1}) \\
&\leq \left\{\phi\left\{\begin{array}{c}
\frac{d(Ax_{2n+2},Bx_{2n+1})[d(Ax_{2n+2},Sx_{2n+2})+d(Tx_{2n+1},Sx_{2n+2})]}{d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})}, \\
\frac{d(Ax_{2n+2},Bx_{2n+1})[d(Tx_{2n+1},Sx_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})]}{d(Bx_{2n+1},Ax_{2n+2})+d(Sx_{2n+2},Bx_{2n+1})}, \\
\frac{d(Tx_{2n+1},Sx_{2n+2})[d(Bx_{2n+1},Ax_{2n+2})+d(Sx_{2n+2},Bx_{2n+1})]}{d(Tx_{2n+1},Sx_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})}, \\
\frac{d(Ax_{2n+2},Sx_{2n+2})[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})]}{d(Ax_{2n+2},Sx_{2n+2})+d(Tx_{2n+1},Sx_{2n+2})}
\end{array}\right\}\right\}^{\lambda}
\end{aligned}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d\left(y_{2n+1}, y_{2n},\right)\left[d\left(y_{2n+1}, y_{2n+2}\right)+d\left(y_{2n+1}, y_{2n+2}\right)\right]}{d\left(y_{2n}, y_{2n+1}\right)+d\left(y_{2n}, y_{2n+1}\right)}, \\ \frac{d\left(y_{2n+1}, y_{2n}\right)\left[d\left(y_{2n+1}, y_{2n+2}\right)+d\left(y_{2n+1}, y_{2n+1}\right)\right]}{d\left(y_{2n}, y_{2n+1}\right)+d\left(y_{2n+2}, y_{2n}\right)}, \\ \frac{d\left(y_{2n+1}, y_{2n+2}\right)\left[d\left(y_{2n}, y_{2n+1}\right)+d\left(y_{2n+2}, y_{2n}\right)\right]}{d\left(y_{2n+1}, y_{2n+2}\right)+d\left(y_{2n+1}, y_{2n+1}\right)}, \\ \frac{d\left(y_{2n+1}, y_{2n+2}\right)[d\left(y_{2n}, y_{2n+1}\right)+d(y_{2n}, y_{2n+1})]}{d\left(y_{2n+1}, y_{2n+2}\right)+d(y_{2n+1}, y_{2n+2})} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} d\left(y_{2n+1}, y_{2n+2}\right), \\ 1, \\ d\left(y_{2n+1}, y_{2n+2}\right).d\left(y_{2n+1}, y_{2n}\right), \\ d\left(y_{2n}, y_{2n+1}\right) \end{array} \right\} \right\}^{\lambda}$$

$$d\left(y_{2n+1}, y_{2n+2}\right) \leq d^{\lambda}\left(y_{2n}, y_{2n+1}\right).d^{\lambda}\left(y_{2n+1}, y_{2n+2}\right) \quad \text{[Using Symmetry and (ii)]}$$

This implies that,

$$d\left(y_{2n+1}, y_{2n+2}\right) \leq d^{\frac{\lambda}{1-\lambda}}\left(y_{2n}, y_{2n+1}\right)$$

On substituting, $h = \frac{\lambda}{1-\lambda} \in (0, \frac{1}{2})$

$$d\left(y_{2n+1}, y_{2n+2}\right) \leq d^{h}\left(y_{2n}, y_{2n+1}\right),$$

Hence

$$d\left(y_n, y_{n+1}\right) \leq d^{h^1}\left(y_{n-1}, y_n\right) \leq d^{h^2}\left(y_{n-2}, y_{n-1}\right) \leq \cdots \leq d^{h^n}\left(y_0, y_1\right)$$

For all $n \geq 2$, let $m, n \in \mathbb{N}$ such that $m \geq n$. Using the triangular multiplicative inequality, we obtain

$$\begin{aligned} d\left(y_m, y_n\right) &\leq d\left(y_m, y_{m-1}\right).d\left(y_{m-1}, y_{m-2}\right) \ldots d\left(y_{n+1}, y_n\right) \\ &\leq d^{h^{m-1}}\left(y_1, y_0\right).d^{h^{m-2}}\left(y_1, y_0\right) \ldots d^{h^n}\left(y_1, y_0\right) \\ &\leq d^{\frac{h^n}{1-h}}\left(y_1, y_0\right) \end{aligned}$$

This implies that $d\left(y_m, y_n\right)$ approaches to 1 as n and m approaches to infinity, we have. Therefore $\{y_n\}$ is a multiplicative Cauchy sequence in X. Now, suppose that AX is complete, there exist $u \in AX$ such that

$$y_{n+1} = Tx_{2n+1} = Ax_{2n+2} \to u \ (n \to \infty).$$

Consequently, we can find $v \in X$ such that $Av = u$. Further a multiplicative Cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have,

$$y_{2n} = Sx_{2n} = Bx_{2n+1} \to u(n \to \infty).$$

We claim that $Sv = u$ if possible $Sv \neq u$, substituting $x = v$ and $y =x_{2n+1}$ in (E2), we have

$$d(Sv, Tx_{2n+1}) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d\left(Av, Bx_{2n+1}\right)\left[d(Av, Sv)+d\left(Tx_{2n+1}, Sv\right)\right]}{d\left(Bx_{2n+1}, Tx_{2n+1}\right)+d\left(Bx_{2n+1}, Av\right)}, \\ \frac{d\left(Av, Bx_{2n+1}\right)\left[d\left(Tx_{2n+1}, Sv\right)+d\left(Av, Tx_{2n+1}\right)\right]}{d\left(Bx_{2n+1}, Av\right)+d\left(Sv, Bx_{2n+1}\right)}, \\ \frac{d\left(Tx_{2n+1}, Sv\right)\left[d\left(Bx_{2n+1}, Av\right)+d\left(Sv, Bx_{2n+1}\right)\right]}{d\left(Tx_{2n+1}, Sv\right)+d\left(Av, Tx_{2n+1}\right)}, \\ \frac{d(Av, Sv)[d\left(Bx_{2n+1}, Tx_{2n+1}\right)+d(Bx_{2n+1}, Av)]}{d(Av, Sv)+d(Tx_{2n+1}, Sv)} \end{array} \right\} \right\}^{\lambda}$$

Taking $n \to \infty$, On the two sides of the above inequality,

$$d(Sv,u) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(u,u)[d(u,Sv)+d(u,Sv)]}{d(u,u)+d(u,u)}, \\ \frac{d(u,u)[d(u,Sv)+d(u,u)]}{d(u,u)+d(Sv,u)}, \\ \frac{d(u,Sv)[d(u,u)+d(Sv,u)]}{d(u,Sv)+d(u,u)}, \\ \frac{d(u,Sv)[d(u,u)+d(u,u)]}{d(u,Sv)+d(u,Su)} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} d(u,Sv), \\ 1, \\ d(u,Sv), \\ 1 \end{array} \right\} \right\}$$

$$d(u,Sv) \leq d^{\lambda}(u,Sv), \quad \text{[using (iv)]}$$

a contradiction, since $\lambda \in (0, \frac{1}{2})$ hence, implies $Sv = u$. Since $u = Sv \in SX \subset BX$, there exist $w \in X$ such that $u = Bw$. Claim that $Tw = u$, if possible $Tw \neq u$. substituting $x = v$ and $y = w$ in (E2), we have

$$\begin{aligned} d(u,Tw) &= d(Sv,Tw) \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,Bw)[d(Av,Sv)+d(Tw,Sv)]}{d(Bw,Tw)+d(Bw,Av)}, \frac{d(Av,Bw)[d(Tw,Sv)+d(Av,Tw)]}{d(Bw,Av)+d(Sv,Bw)}, \\ \frac{d(Tw,Sv)[d(Bw,Av)+d(Sv,Bw)]}{d(Tw,Sv)+d(Av,Tw)}, \frac{d(Av,Sv)[d(Bw,Tw)+d(Bw,Av)]}{d(Av,Sv)+d(Tw,Sv)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \left\{ ? \left\{ \begin{array}{c} \frac{d(u,u)[d(u,u)+d(Tw,u)]}{d(u,Tw)+d(u,u)}, \frac{d(u,u)[d(Tw,u)+d(u,Tw)]}{d(u,u)+d(u,u)} \\ \frac{d(Tw,u)[d(u,u)+d(u,u)]}{d(Tw,u)+d(u,Tw)}, \frac{d(u,u)[d(u,Tw)+d(u,u)]}{d(u,u)+d(Tw,u)} \end{array} \right\} \right\}^{\lambda} \quad \left[\begin{array}{c} \text{using } Av = u \\ = Sv = Bw \end{array} \right] \\ &\leq \{\phi\{1, d(Tw,u), 1, 1\}\}^{\lambda} \\ d(u,Tw) &\leq d^{\lambda}(u,Tw) \quad \text{[using (iii)]} \end{aligned}$$

A contradiction, since $\lambda \in (0, \frac{1}{2})$ implies $u = Tw$. Hence we get $u = Av = Sv$, that is, v is a coincidence point of A, S. also $u = Bw = Tw$, that is w is coincidence point of B and T. Therefore $Av = Sv = Bw = Tw = u$. Since the pairs (A, S) and (B, T) are weakly compatible, we have

$$Su = S(Av) = A(Sv) = Au = w_1 \quad \text{(say)}$$

And

$$Tu = T(Bw) = B(Tw) = Bu = w_2 \quad \text{(say)}$$

From (E2), we have

$$d(w_1,w_2) = d(Su,Tu) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Au,Bu)[d(Au,Su)+d(Tu,Su)]}{d(Bu,Tu)+d(Bu,Au)}, \frac{d(Au,Bu)[d(Tu,Su)+d(Au,Tu)]}{d(Bu,Au)+d(Su,Bu)}, \\ \frac{d(Tu,Su)[d(Bu,Au)+d(Su,Bu)]}{d(Tu,Su)+d(Au,Tu)}, \frac{d(Au,Su)[d(Bu,Tu)+d(Bu,Au)]}{d(Au,Su)+d(Tu,Su)} \end{array} \right\} \right\}^{\lambda}$$

Using symmetry and above conditions of w_1 and w_2, we have

$$\begin{aligned} d(w_1,w_2) &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(w_1,w_2)[d(w_1,w_1)+d(w_2,w_1)]}{d(w_2,w_2)+d(w_2,w_1)}, \frac{d(w_1,w_2)[d(w_2,w_1)+d(w_1,w_2)]}{d(w_2,w_1)+d(w_1,w_2)}, \\ \frac{d(w_2,w_1)[d(w_2,w_1)+d(w_1,w_2)]}{d(w_2,w_1)+d(w_1,w_2)}, \frac{d(w_1,w_1)[d(w_2,w_2)+d(w_2,w_1)]}{d(w_1,w_1)+d(w_2,w_1)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \{\phi\{d(w_1,w_2), d(w_1,w_2), d(w_1,w_2), 1\}\}^{\lambda} \\ d(w_1,w_2) &\leq d^{\lambda}(w_1,w_2) \quad \text{[using (v)]} \end{aligned}$$

on the other hand, since $\lambda \in (0, \frac{1}{2})$ implies, $d(w_1, w_2) = 1$, which implies that $w_1 = w_2$ and hence we have $Su = Au = Tu = Bu$. Again using (E2) and symmetry of multiplicative metric space we have,

$$\begin{aligned} d(Sv,Tu) &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,Bu)[d(Av,Sv)+d(Tu,Sv)]}{d(Bu,Tu)+d(Bu,Av)}, \frac{d(Av,Bu)[d(Tu,Sv)+d(Av,Tu)]}{d(Bu,Av)+d(Sv,Bu)}, \\ \frac{d(Tu,Sv)[d(Bu,Av)+d(Sv,Bu)]}{d(Tu,Sv)+d(Av,Tu)}, \frac{d(Av,Sv)[d(Bu,Tu)+d(Bu,Av)]}{d(Av,Sv)+d(Tu,Sv)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Sv,Tu)[d(Sv,Sv)+d(Tu,Sv)]}{d(Tu,Tu)+d(Tu,Sv)}, \frac{d(Sv,Tu)[d(Tu,Sv)+d(Sv,Tu)]}{d(Tu,Sv)+d(Sv,Tu)}, \\ \frac{d(Tu,Sv)[d(Tu,Sv)+d(Sv,Tu)]}{d(Tu,Sv)+d(Sv,Tu)}, \frac{d(Sv,Sv)[d(Tu,Tu)+d(Tu,Sv)]}{d(Sv,Sv)+d(Tu,Sv)} \end{array} \right\} \right\}^{\lambda} \quad [\textit{using } Av{=}Sv \textit{ and } Bu{=}Tu] \\ &\leq \{\phi\{d(Sv,Tu), d(Sv,Tu), d(Sv,Tu), 1\}\}^{\lambda} \\ d(Sv,Tu) &\leq d^{\lambda}(Sv,Tu), \quad [\text{using v}] \end{aligned}$$

on the other hand, since $\lambda \in (0, \frac{1}{2})$ implies $d(Sv, Tu) = 1$ *i.e.* $Sv = Tu$. But $Sv = u$ which implies that $Tu = u$ and hence we have $u{=}Su{=}Au{=}Tu{=}Bu$. Therefore u is a common fixed point of A, B, S and T. Similarly, we can complete the proof for the different case in which BX or TX or SX is complete.

Uniqueness: Let p and q are two different common fixed points of A, B, S, T then using symmetry of multiplicative metric space and using equation (E2)

$$\begin{aligned} d(p,q) &= d(Sp,Tq) \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ap,Bq)[d(Ap,Sp)+d(Tq,Sp)]}{d(Bq,Tq)+d(Bq,Ap)}, \frac{d(Ap,Bq)[d(Tq,Sp)+d(Ap,Tq)]}{d(Bq,Ap)+d(Sp,Bq)}, \\ \frac{d(Tq,Sp)[d(Bq,Ap)+d(Sp,Bq)]}{d(Tq,Sp)+d(Ap,Tq)}, \frac{d(Ap,Sp)[d(Bq,Tq)+d(Bq,Ap)]}{d(Ap,Sp)+d(Tq,Sp)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(p,q)[d(p,p)+d(q,p)]}{d(q,q)+d(q,p)}, \frac{d(p,q)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)}, \\ \frac{d(q,p)[d(q,p)+d(p,?\bar{a})]}{d(q,p)+d(p,q)}, \frac{d(p,p)[d(q,q)+d(q,p)]}{d(p,p)+d(q,p)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \left\{ \phi \left\{ \begin{array}{c} d(p,q), d(p,q), \\ d(p,q), 1 \end{array} \right\} \right\}^{\lambda} \\ d(p,q) &\leq d^{\lambda}(p,q), \quad [\text{using (v)}] \end{aligned}$$

on the other hand, since $\lambda \in (0, \frac{1}{2})$ implies $d(p, q) = 1$ *i.e.* $p = q$, which proves the uniqueness. □

Corollary 2.2. *Let A, B, S be mappings of a multiplicative metric space (X, d) into itself satisfying*

(E5) $SX \subset BX$ *and* $SX \subset AX$

(E6) $d(Sx, Sy) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax,By)[d(Ax,Sx)+d(Sy,Sx)]}{d(By,Sy)+d(B?\bar{a},Ax)}, \frac{d(Ax,By)[d(Sy,Sx)+d(Ax,Sy)]}{d(By,Ax)+d(Sx,By)}, \\ \frac{d(Sy,Sx)[d(By,Ax)+d(Sx,By)]}{d(Sy,Sx)+d(Ax,Sy)}, \frac{d(Ax,Sx)[d(By,Sy)+d(By,Ax)]}{d(Ax,Sx)+d(Sy,Sx)} \end{array} \right\} \right\}^{\lambda}$. *For all* $x, y \in X$, *where* $\lambda \in (0, \frac{1}{2})$ *and* $\phi \in \Psi$;

(E3) let us suppose that the pairs (A, S) and (B, S) are weakly compatible;

(E4) One of the subspaces AX or BX or SX is complete

Then A, B and S have a unique common fixed point.

In Theorem 2.1, if we put T=S, then we obtain the Corollary 2.2.

Corollary 2.3. *Let S and T be mappings of a multiplicative metric space (X, d) into itself satisfying*

(E7) $d(Sx,Ty) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(x,y)[d(x,Sx)+d(Ty,Sx)]}{d(y,Ty)+d(y,x)}, \frac{d(x,y)[d(Ty,Sx)+d(x,Ty)]}{d(y,x)+d(Sx,y)}, \\ \frac{d(Ty,Sx)[d(y,x)+d(Sx,y)]}{d(Ty,Sx)+d(x,Ty)}, \frac{d(x,Sx)[d(y,Ty)+d(y,x)]}{d(x,Sx)+d(Ty,Sx)} \end{array} \right\} \right\}^{\lambda}$ *for all* $x,y \in X$*, where* $\lambda \in (0,\frac{1}{2})$ *and* $\phi \in \Psi$*,*

(E8) One of the subspaces SX or TX is complete.

Then S and T have a unique common fixed point.

In Theorem 2.1, if we put $A = B = 1$, then we obtain the Corollary 2.3.

Theorem 2.4. *Let A, B, S, T be mappings of a multiplicative metric space (X, d) into itself satisfying the conditions (E1), (E2), (E3) and the following conditions:*

(E9) one of the subspaces AX or BX or SX or TX is closed subset of X

(E10) the pairs (A, S) and (B, T) satisfy the E.A. property.

Then A, B, S, T has a unique common fixed point.

Proof. Suppose that the pairs (A, S) satisfies the E.A property. Then ∃ a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Since $SX \subset BX$, ∃ a sequence y_n in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = z$. Now suppose that BX is closed subset of X, ∃ a point $u \in X$ such that $Bu = z$. We will show that $\lim_{n\to\infty} Ty_n = z$, from inequality (E2), we have

$$d(Sx_n,Ty_n) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax_n,By_n)[d(Ax_n,Sx_n)+d(Ty_n,Sx_n)]}{d(By_n,Ty_n)+d(By_n,Ax_n)}, \\ \frac{d(Ax_n,By_n)[d(Ty_n,Sx_n)+d(Ax_n,Ty_n)]}{d(By_n,Ax_n)+d(Sx_n,By_n)}, \\ \frac{d(Ty_n,Sx_n)[d(By_n,Ax_n)+d(Sx_n,By_n)]}{d(Ty_n,Sx_n)+d(Ax_n,Ty_n)}, \\ \frac{d(Ax_n,Sx_n)[d(By_n,Ty_n)+d(By_n,Ax_n)]}{d(Ax_n,Sx_n)+d(Ty_n,Sx_n)} \end{array} \right\} \right\}^{\lambda}$$

Taking n approaches to infinity and using the symmetry property of multiplicative metric space, we have

$$d\left(z, \lim_{n\to\infty} Ty_n\right) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(z,z)[d(z,z)+d(\lim_{n\to\infty} Ty_n,z)]}{d(z,\lim_{n\to\infty} Ty_n)+d(z,z)}, \\ \frac{d(z,z)[d(\lim_{n\to\infty} Ty_n,z)+d(z,\lim_{n\to\infty} Ty_n)]}{d(z,z)+d(z,z)}, \\ \frac{d(\lim_{n\to\infty} Ty_n,z)[d(z,z)+d(z,z)]}{d(\lim_{n\to\infty} Ty_n,z)+d(z,\lim_{n\to\infty} Ty_n)}, \\ \frac{d(z,z)[d(z,\lim_{n\to\infty} Ty_n)+d(z,z)]}{d(z,z)+d(\lim_{n\to\infty} Ty_n,z)} \end{array} \right\} \right\}^{\lambda}$$

$$d\left(z, \lim_{n\to\infty} Ty_n\right) \leq \left\{ \phi \left\{ 1, d\left(z, \lim_{n\to\infty} Ty_n\right), 1, 1 \right\} \right\}^{\lambda}$$

$$d\left(z, \lim_{n\to\infty} Ty_n\right) \leq d^{\lambda}\left(z, \lim_{n\to\infty} Ty_n\right) \quad \text{[using (iii)]}$$

on the other hand, since $\lambda \in \left(0,\frac{1}{2}\right)$ implies $d\left(z, \lim_{n\to\infty} Ty_n\right) = 1$ i.e. $z = \lim_{n\to\infty} Ty_n$. Thus we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z = Bu \text{ (say)}$$

for some u in X. On substituting $x = x_n$ and $y = u$ in (E2), we have

$$d(Sx_n,Tu) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax_n,Bu)[d(Ax_n,Sx_n)+d(Tu,Sx_n)]}{d(Bu,Tu)+d(Bu,Ax_n)}, \\ \frac{d(Ax_n,Bu)[d(Tu,Sx_n)+d(Ax_n,Tu)]}{d(Bu,Ax_n)+d(Sx_n,Bu)}, \\ \frac{d(Tu,Sx_n)[d(Bu,Ax_n)+d(Sx_n,Bu)]}{d(Tu,Sx_n)+d(Ax_n,Tu)}, \\ \frac{d(Ax_n,Sx_n)[d(Bu,Tu)+d(Bu,Ax_n)]}{d(Ax_n,Sx_n)+d(Tu,Sx_n)} \end{array} \right\} \right\}^{\lambda}$$

Taking $n \to \infty$, we have

$$d(Bu,Tu) \le \left\{ \phi \left\{ \begin{array}{l} \frac{d(Bu,Bu)[d(Bu,Bu)+d(Tu,Bu)]}{d(Bu,Tu)+d(Bu,Bu)}, \\ \frac{d(Bu,Bu)[d(Tu,Bu)+d(Bu,Tu)]}{d(Bu,Bu)+d(Bu,Bu)}, \\ \frac{d(Tu,Bu)[d(Bu,Bu)+d(Bu,Bu)]}{d(Tu,Bu)+d(Bu,Tu)}, \\ \frac{d(Bu,Bu)[d(Bu,Tu)+d(Bu,Bu)]}{d(Bu,Bu)+d(Tu,Bu)} \end{array} \right\} \right\}^{\lambda}$$

$$d(Bu,Tu) \le \{\phi\{1, d(Bu,Tu), 1, 1\}\}^{\lambda}$$

$$d(Bu,Tu) \le d^{\lambda}(Bu,Tu) \quad \text{[using (iii)]}$$

On the other hand, since $\lambda \in (0, \frac{1}{2})$, it implies that $d(Bu,Tu) = 1$ i.e. $Bu = Tu$. Since the pair B and T is weakly compatible, we have $BTu = TBu$ and then $BBu = TTu = BTu = TBu$. There is also a condition that is $TX \subset AX$, $\exists\, v \in X$ such that $Tu = Av$. Next we claim that $Av = Sv$, if possible let $Av \neq Sv$ putting $x = v$ and $y = u$, we have

$$d(Sv,Tu) \le \left\{ \phi \left\{ \begin{array}{l} \frac{d(Av,Bu)[d(Av,Sv)+d(Tu,Sv)]}{d(Bu,Tu)+d(Bu,Av)}, \\ \frac{d(Av,Bu)[d(Tu,Sv)+d(Av,Tu)]}{d(Bu,Av)+d(Sv,Bu)}, \\ \frac{d(Tu,Sv)[d(Bu,Av)+d(Sv,Bu)]}{d(Tu,Sv)+d(Av,Tu)}, \\ \frac{d(Av,Sv)[d(Bu,Tu)+d(Bu,Av)]}{d(Av,Sv)+d(Tu,Sv)} \end{array} \right\} \right\}^{\lambda}$$

$$d(Sv,Av) \le \left\{ \phi \left\{ \begin{array}{l} \frac{d(Av,Av)[d(Av,Sv)+d(Av,Sv)]}{d(Av,Av)+d(Av,Av)}, \\ \frac{d(Av,Av)[d(Av,Sv)+d(Av,Av)]}{d(Av,Av)+d(Sv,Av)}, \\ \frac{d(Av,Sv)[d(Av,Av)+d(Sv,Av)]}{d(Av,Sv)+d(Av,Av)}, \\ \frac{d(Av,Sv)[d(Av,Av)+d(Av,Av)]}{d(Av,Sv)+d(Av,Sv)} \end{array} \right\} \right\}^{\lambda} \quad \text{[since, Tu=Av and Bu=Av]}$$

$$d(Sv,Av) \le \{\phi\{d(Av,Sv), 1, d(Av,Sv), 1\}\}^{\lambda}$$

$$d(Sv,Av) \le d^{\lambda}(Sv,Av) \quad \text{[using (iv)]}$$

Which is a contradiction, since $\lambda \in (0, \frac{1}{2})$, this implies that $Sv = Av$. Hence we have, $Bu = Tu = Av = Sv$. Since the pair (A, S) are weakly compatible, we have $ASv = SAv$ and then $SSv = SAv = ASv = AAv$. Next we claim that $SAv = Av$, if possible $SAv \neq Av$ on substituting $x = Av$ and $y = u$, we have

$$d(SAv,Av) = d(SAv,Tu)$$

$$\le \left\{ \phi \left\{ \begin{array}{l} \frac{d(AAv,Bu)[d(AAv,SAv)+d(Tu,SAv)]}{d(Bu,SAu)+d(Bu,AAv)}, \\ \frac{d(AAv,Bu)[d(Tu,SAv)+d(AAv,Tu)]}{d(Bu,AAv)+d(SAv,Bu)}, \\ \frac{d(Tu,SAv)[d(Bu,AAv)+d(SAv,Bu)]}{d(Tu,SAv)+d(AAv,Tu)}, \\ \frac{d(AAv,SAv)[d(Bu,Tu)+d(Bu,AAv)]}{d(AAv,SAv)+d(Tu,SAv)} \end{array} \right\} \right\}^{\lambda}$$

$$\le \left\{ \phi \left\{ \begin{array}{l} \frac{d(SAv,Av)[d(SAv,SAv)+d(Av,SAv)]}{d(Av,Av)+d(Av,SAv)}, \\ \frac{d(SAv,Av)[d(Av,SAv)+d(SAv,Av)]}{d(Av,SAv)+d(SAv,Av)}, \\ \frac{d(Av,SAv)[d(Av,SAv)+d(SAv,Av)]}{d(Av,SAv)+d(SAv,Av)}, \\ \frac{d(SAv,SAv)[d(Av,Av)+d(Av,SAv)]}{d(SAv,SAv)+d(Av,SAv)} \end{array} \right\} \right\}^{\lambda} \quad \text{[since AAv=SAv, Tu=Av, Bu=Av]}$$

$$\le \{\phi\{d(SAv,Av), d(SAv,Av), d(Av,SAv), 1\}\}^{\lambda}$$

$$d(SAv,Av) \le d^{\lambda}(SAv,Av) \quad \text{[using (v)]}$$

This is a contradiction, since $\lambda \in \left(0, \frac{1}{2}\right)$, hence $Av = SAv$. Hence $SAv = Av = AAv$. Hence Av is a common fixed point of A and S. Also, it can be easily prove that $BBu = Bu = TBu$, that is, Bu is a common fixed point of B and T as $Av = Bu, Av$ is a common fixed point of A, B, S and T. Similarly we can complete the proof for cases in which AX or TX or SX is closed subset of X.

Uniqueness: Let Av and Pu are two distinct common fixed points of A, B, S and T. Using symmetry of multiplicative metric space and (E2), we have

$$\begin{aligned} d(Av, Pu) &= d(SAv, TPu) \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(AAv,BPu)[d(AAv,SAv)+d(TPu,SAv)]}{d(BPu,TPu)+d(BPu,AAv)}, \\ \frac{d(AAv,BPu)[d(TPu,SAv)+d(AAv,TPu)]}{d(BPu,AAv)+d(SAv,BPu)}, \\ \frac{d(TPu,SAv)[d(BPu,AAv)+d(SAv,BPu)]}{d(TPu,SAv)+d(AAv,TPu)}, \\ \frac{d(AAv,SAv)[d(BPu,TPu)+d(BPu,AAv)]}{d(AAv,SAv)+d(TPu,SAv)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \left\{ ? \left\{ \begin{array}{c} \frac{d(Av,Pu)[d(Av,Av)+d(Pu,Av)]}{d(Pu,Pu)+d(Pu,Av)}, \\ \frac{d(Av,Pu)[d(Pu,Av)+d(Av,Pu)]}{d(Pu,Av)+d(Av,Pu)}, \\ \frac{d(Pu,Av)[d(Pu,Av)+d(Av,Pu)]}{d(Pu,Av)+d(Av,Pu)}, \\ \frac{d(Av,Av)[d(Pu,Pu)+d(Pu,Av)]}{d(Av,Av)+d(Pu,Av)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \{\phi \{d(Av, Pu), d(Av, Pu), d(Av, Pu), 1\}\}^{\lambda} \\ d(Av, Pu) &\leq d^{\lambda}(Av, Pu) \quad \text{[using (v)]} \end{aligned}$$

A contradiction, since $\lambda \in (0, \frac{1}{2})$, hence $Av = Pu$. Which completes the proof. □

Finally, we prove the following theorems for weakly compatible mappings with common limit range property satisfying the implicit function in a multiplicative metric space.

Lemma 2.5. *Let A, B, S, T be mappings of a multiplicative metric space (X, d) satisfying the conditions (E1) and (E2) and the following condition:*

(E11) the pairs (A, S) satisfies CLR_A property or the pair (B, T) satisfies CLR_B property,

Then the pairs (A, S) and (B, T) share the common limit in the range of A property or B property.

Proof. Let us first assume that the pair (A, S) satisfies the common limit range of A property. Then ∃ a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Az$ for some $z \in X$. Since $SX \subset BX$, so for each x_n there exists y_n in X such thst $Sx_n = By_n$. Then $\lim_{n\to\infty} By_n = Az$. hence, we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = Az$. Now we claim that $\lim_{n\to\infty} Ty_n = Az$. On substituting $x = x_n$ and $y = y_n$ in (E2), we have

$$d(Sx_n, Ty_n) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax_n,By_n)[d(Ax_n,Sx_n)+d(Ty_n,Sx_n)]}{d(By_n,Ty_n)+d(By_n,Ax_n)}, \\ \frac{d(Ax_n,By_n)[d(Ty_n,Sx_n)+d(Ax_n,Ty_n)]}{d(By_n,Ax_n)+d(Sx_n,By_n)}, \\ \frac{d(Ty_n,Sx_n)[d(By_n,Ax_n)+d(Sx_n,By_n)]}{d(Ty_n,Sx_n)+d(Ax_n,Ty_n)}, \\ \frac{d(Ax_n,Sx_n)[d(By_n,Ty_n)+d(By_n,Ax_n)]}{d(Ax_n,Sx_n)+d(Ty_n,Sx_n)} \end{array} \right\} \right\}^{\lambda}$$

Taking $n \to \infty$, we have

$$d\left(Az, \lim_{n\to\infty} Ty_n\right) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Az,Az)[d(Az,Az)+d(\lim_{n\to\infty} Ty_n,Az)]}{d(Az,\lim_{n\to\infty} Ty_n)+d(Az,Az)}, \\ \frac{d(Az,Az)[d(\lim_{n\to\infty} Ty_n,Az)+d(Az,\lim_{n\to\infty} Ty_n)]}{d(Az,Az)+d(Az,Az)}, \\ \frac{d(\lim_{n\to\infty} Ty_n,Az)[d(Az,Az)+d(Az,Az)]}{d(\lim_{n\to\infty} Ty_n,Az)+d(Az,\lim_{n\to\infty} Ty_n)}, \\ \frac{d(Az,Az)[d(Az,\lim_{n\to\infty} Ty_n)+d(Az,Az)]}{d(Az,Az)+d(\lim_{n\to\infty} Ty_n,Az)} \end{array} \right\} \right\}^{\lambda}$$

$$d\left(Az, \lim_{n\to\infty} Ty_n\right) \leq \left\{\phi\left\{1, d\left(Az, \lim_{n\to\infty} Ty_n\right), 1, 1\right\}\right\}^{\lambda}$$

$$d\left(Az, \lim_{n\to\infty} Ty_n\right) \leq d^{\lambda}\left(Az, \lim_{n\to\infty} Ty_n\right) \quad \text{[using (iii)]}$$

On the other hand, since $\lambda \in \left(0, \frac{1}{2}\right)$, hence $d\left(Az, \lim_{n\to\infty} Ty_n\right) = 1$. Therefore, $Az = \lim_{n\to\infty} Ty_n$ i.e. $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = Az$. Then the pairs (A, S) and (B, T) share the common limit range of A property for the other pair (B, T) which shares common limit range of B property, Since the pair (B, T) satisfies common limit range of B property.

Then $\exists$ a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Bz$ for some $z \in X$. Since $TX \subset AX$, so for each $y_n \exists\ x_n$ in X such that $Ty_n = Ax_n$. Then $\lim_{n\to\infty} Ax_n = Bz$. Hence, we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Bz$.

Now we claim that $\lim_{n\to\infty} Sx_n = Bz$. If possible $\lim_{n\to\infty} Sx_n \neq Bz$, On substituting $x = x_n$ and $y = y_n$ in (E2), we have

$$d\left(Sx_n, Ty_n\right) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax_n,By_n)[d(Ax_n,Sx_n)+d(Ty_n,Sx_n)]}{d(By_n,Ty_n)+d(By_n,Ax_n)}, \\ \frac{d(Ax_n,By_n)[d(Ty_n,Sx_n)+d(Ax_n,Ty_n)]}{d(By_n,Ax_n)+d(Sx_n,By_n)}, \\ \frac{d(Ty_n,Sx_n)[d(By_n,Ax_n)+d(Sx_n,By_n)]}{d(Ty_n,Sx_n)+d(Ax_n,Ty_n)}, \\ \frac{d(Ax_n,Sx_n)[d(By_n,Ty_n)+d(By_n,Ax_n)]}{d(Ax_n,Sx_n)+d(\ Ty_n,Sx_n)} \end{array} \right\} \right\}^{\lambda}$$

taking $n \to \infty$, we have

$$d\left(\lim_{n\to\infty} Sx_n, Bz\right) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Bz,Bz)[d(Bz,\lim_{n\to\infty} Sx_n)+d(Bz,\lim_{n\to\infty} Sx_n)]}{d(Bz,Bz)+d(Bz,Bz)}, \\ \frac{d(Bz,Bz)[d(Bz,\lim_{n\to\infty} Sx_n)+d(Bz,Bz)]}{d(Bz,Bz)+d(\lim_{n\to\infty} Sx_n,Bz)}, \\ \frac{d(Bz,\lim_{n\to\infty} Sx_n)[d(Bz,Bz)+d(\lim_{n\to\infty} Sx_n,Bz)]}{d(Bz,\lim_{n\to\infty} Sx_n)+d(Bz,Bz)}, \\ \frac{d(Bz,\lim_{n\to\infty} Sx_n)[d(Bz,Bz)+d(Bz,Bz)]}{d(Bz,\lim_{n\to\infty} Sx_n)+d(Bz,\lim_{n\to\infty} Sx_n)} \end{array} \right\} \right\}^{\lambda}$$

$$d\left(\lim_{n\to\infty} Sx_n, Bz\right) \leq \left\{\phi\left\{d\left(Bz, Sx_{2n}\right), 1, d(Bz, Sx_{2n)}, 1\right\}\right\}^{\lambda}$$

$$d\left(\lim_{n\to\infty} Sx_n, Bz\right) \leq d^{\lambda}\left(\lim_{n\to\infty} Sx_n, Bz\right) \quad \text{[using (iv)]}$$

This is a contradiction, since $\lambda \in \left(0, \frac{1}{2}\right)$, hence, therefore $\lim_{n\to\infty} Sx_n = Bz$. Then the pairs (A, S) and (B, T) share the common limit range of B property. □

Theorem 2.6. *Let A, B, S, T be mappings of a multiplicative metric space (X, d) satisfying the conditions (E1) and (E2) and (E11). Then the pairs (A, S) and (B,T) have a coincidence point. Moreover, assume that the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point.*

Proof. Using the Lemma 2.5, the pairs (A, S) and (B, T) share the common limit range of A property, that is there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = Av$ for some $v \in X$. First, we claim that $Av = Sv$, substituting $x = v$ and $y = y_n$ in (E2), we have

$$d(Sv, Ty_n) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,By_n)[d(Av,Sv)+d(Ty_n,Sv)]}{d(By_n,Ty_n)+d(By_n,Av)}, \\ \frac{d(Av,By_n)[d(Ty_n,Sv)+d(Av,Ty_n)]}{d(By_n,Av)+d(Sv,By_n)}, \\ \frac{d(Ty_n,Sv)[d(By_n,Av)+d(Sv,By_n)]}{d(Ty_n,Sv)+d(Av,Ty_n)}, \\ \frac{d(Av,Sv)[d(By_n,Ty_n)+d(By_n,Av)]}{d(Av,Sv)+d(Ty_n,Sv)} \end{array} \right\} \right\}^{\lambda}$$

Taking n approaches to infinity, we have

$$d(Sv, Av) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,Av)[d(Av,Sv)+d(Av,Sv)]}{d(Av,Av)+d(Av,Av)}, \\ \frac{d(Av,Av)[d(Av,Sv)+d(Av,Av\bar{a})]}{d(Av,Av)+d(Sv,Av)}, \\ \frac{d(Av,Sv)[d(Av,Av)+d(Sv,Av)]}{d(Av,Sv)+d(Av,Av)}, \\ \frac{d(Av,Sv)[d(Av,Av)+d(Av,Av)]}{d(Av,Sv)+d(Av,Sv)} \end{array} \right\} \right\}^{\lambda}$$

$$d(Sv, Av) \leq \{\phi \{d(Av, Sv), 1, d(Av, Sv), 1\}\}^{\lambda}$$

$$d(Sv, Av) \leq d^{\lambda}(Sv, Av) \quad \text{[using (iv)]}$$

On the other hand $\lambda \in \left(0, \frac{1}{2}\right)$, implies that $d(Sv, Av) = 1$ i.e., $Sv = Av$. Since $SX \subset BX$, $\exists\, w \in X$ such that $Bw = Sv$. Now we claim that $Bw = Tw$, if possible $Bw \neq Tw$. Putting $x = v$ and $y = w$, in (E2), we have

$$\begin{aligned} d(Bw, Tw) &= d(Sv, Tw) \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,Bw)[d(Av,Sv)+d(Tw,Sv)]}{d(Bw,Tw)+d(Bw,Av)}, \frac{d(Av,Bw)[d(Tw,Sv)+d(Av,Tw)]}{d(Bw,Av)+d(Sv,Bw)}, \\ \frac{d(Tw,Sv)[d(Bw,Av)+d(Sv,Bw)]}{d(Tw,Sv)+d(Av,Tw)}, \frac{d(Av,Sv)[d(Bw,Tw)+d(Bw,Av)]}{d(Av,Sv)+d(Tw,Sv)} \end{array} \right\} \right\}^{\lambda} \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Bw,Bw)[d(Bw,Bw)+d(Tw,Bw)]}{d(Bw,Tw)+d(Bw,Bw)}, \frac{d(Bw,Bw)[d(Tw,Bw)+d(Bw,Tw)]}{d(Bw,Bw)+d(Bw,Bw)}, \\ \frac{d(Tw,Bw)[d(Bw,Bw)+d(Bw,Bw)]}{d(Tw,Bw)+d(Bw,Tw)}, \frac{d(Bw,Bw)[d(Bw,Tw)+d(Bw,Bw)]}{d(Bw,Bw)+d(Tw,Bw)} \end{array} \right\} \right\}^{\lambda} \quad \text{[since, Av=Bw, Sv=Bw]} \\ d(Bw, Tw) &\leq \{\phi \{1, d(Tw, Bw), 1, 1\}\}^{\lambda} \\ d(Bw, Tw) &= d^{\lambda}(Bw, Tw), \quad \text{[using (iii)]} \end{aligned}$$

Which is a contradiction, since $\lambda \in \left(0, \frac{1}{2}\right)$, hence $Bw = Tw$ and hence *Tw=Av=Sv=Bw*. Since the pairs (A, S) and (B, T) are weakly compatible and *Sv=Av* and *Tw=Bw*. Hence *ASv = SAv = AAv = SSv, TBw = BTw = BBw = TTw*. Finally we claim that *SAv=Av*. Putting *x=Av* and *y=w* in (E2) we have

$$\begin{aligned} d(SAv, Av) &= d(SAv, Tw) \\ &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(AAv,Bw)[d(AAv,SAv)+d(Tw,SAv)]}{d(Bw,Tw)+d(Bw,AAv)}, \\ \frac{d(AAv,Bw)[d(Tw,SAv)+d(AAv,Tw)]}{d(Bw,AAv)+d(SAv,Bw)}, \\ \frac{d(Tw,SAv)[d(Bw,AAv)+d(SAv,Bw)]}{d(Tw,SAv)+d(AAv,Tw)}, \\ \frac{d(AAv,SAv)[d(Bw,Tw)+d(Bw,AAv)]}{d(AAv,SAv)+d(Tw,SAv)} \end{array} \right\} \right\}^{\lambda} \quad \text{[since AAv=SAv, Tw=Av, Bw=Av]} \\ d(SAv, Av) &\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(SAv,Av)[d(SAv,SAv)+d(Av,SAv)]}{d(Av,Av)+d(Av,SAv)}, \\ \frac{d(SAv,Av)[d(Av,SAv)+d(SAv,Av)]}{d(Av,SAv)+d(SAv,Av)}, \\ \frac{d(Av,SAv)[d(Av,SAv)+d(SAv,Av)]}{d(Av,SAv)+d(SAv,Av)}, \\ \frac{d(SAv,SAv)[d(Av,Av)+d(Av,SAv)]}{d(SAv,SAv)+d(Av,SAv)} \end{array} \right\} \right\}^{\lambda} \\ d(SA, v, Av) &\leq \{\phi \{d(SAv, Av), d(SAv, Av), d(SAv, Av), 1\}\}^{\lambda} \\ d(SAv, Av) &= d^{\lambda}(SAv, Av), \quad \text{[using (v)]} \end{aligned}$$

on the other hand since $\lambda \in \left(0, \frac{1}{2}\right)$, implies that $d(SAv, Av)=1$, i.e. $SAv=Av$ and hence $SAv=Av= AAv$, which implies that Av is a common fixed point of A and S. Also, one can easily prove that $BBw=Bw=TBw$, *i.e.* Bw is common fixed point of B and T. As $Av=Bw$, Av is a common fixed point of A, B, S and T. The uniqueness follows easily from (E2). This completes the proof. □

References

[1] A.E Bashirov, E.M.kurpnara and A.Ozyapici, *Multiplicative calculus and its applications*, J. Math Anal. Appl., 337(2008), 36-48.

[2] Chahn Yong Jung, Prveen Kumar, Sanjay Kumar and Shin Ming Kang, *Common fixed point theorems for weakly compatible mapping satisfying implicit functions in multiplicative metric space*, International J. Pure and Applies Math., 102(3)(2015), 547-561.

[3] F.Gu, l.M.Cui and Y.H.Wn, *Some fixed point theorems for new contractive type mappings*, J. Qiqihar Univ., 19(2013), 85-89.

[4] G.Jungck, *Common fixed points for noncontineous nonself maps on nonmetric spaces*, Far east J. Math. Sci., 4(1996), 199-215.

[5] Kamal Kumar, Nisha Sharma, Mamta Rani and Rajeev Jha, *Generalized Contractive-Type Mapping on Multiplicative-Metric Space*, International Journal of Mathematics and Physical Sciences Research, 4(1)(2016), 67-73.

[6] M.Aamri and D.El Moutawaskil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal., 270(2002), 181-188.

[7] M.Özavsar and A.C. Çevikel, *Fixed points of multiplicative contraction mappings on multiplicative metric space*, arXiv:1205.5131v1[math.GM], (2012).

[8] Nisha Sharma, Kamal Kumar, Sheetal Sharma and Rajeev Jha, *Rational Contractive Condition in multiplicative Metric Space and Common Fixed Point Theorem*, International Journal of Innovative Research in Science, Engineering and Technology, 5(6)(2016).

[9] S.Sharma and B.Deshpande, *On compatible mapping satisfying an implicit relation in common fixed point consideration*, Tamkang J. Math., 33(2002), 245-252.

[10] W.Sintunavarat and P.Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric space*, J. Appl. Math., 2001(2011), Article ID 637958.

[11] V.Popa and M.Mocanu, *Altering distance and common fixed points under implicit relations*, Hacet. J. Math. Stat., 38(2009), 329-337.

[12] V.Popa, *A fixed point theorem for mappings in d-complete topological spaces*, Math. Moravica, 3(1999), 43-48.

[13] X.He, M.Song and D.Chen, *Common fixed point theorems for weak commutative mapping on a multiplicative metric space*, Fixed point theory Appl., 48(2014).