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Partitioned q-k-EP Matrices

Research Article

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Abstract: Necessary and sufficient conditions are determined for a schur complement in a q-k-EP matrix to be q-k-EP. Further it is shown that in q-k-EP$_r$ matrix, every principal sub matrix of rank r is q-k-EP$_r$. Necessary and sufficient conditions for products of q-k-EP$_r$ partitioned matrices to be q-k-EP$_r$ is given.

MSC: 15A57.

Keywords: q-k-EP matrices, Schur complements.

1. Introduction

In this section we consider an $2n \times 2n$ matrix M partitioned in the form,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

where A and D are nxn matrices. If a partitioned matrix M of the form (1) is q-k-EP, then is general, Schur complement of A in M, i.e., $(M \mid A)$ is not q-k-EP. Here, necessary and sufficient conditions for $(M \mid A)$ to be q-k-EP are obtained for both the cases $\rho(M) = \rho(A)$ and $\rho(M) \neq \rho(A)$. As an application, a decomposition of a partitioned matrix into a sum of q-k-EP matrices is obtained. Throughout this section let $k = k_1 k_2$ as in [5].

2. Schur Complements in q-k-EP Matrices

Definition 2.1. *If $M \in H_{2n\times 2n}$ is of the partitioned form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then a schur complement of A in M denoted by $(M|A)$ is defined as, $D - CA^{-}B$ where A^{-} is a generalized inverse of A satisfying $AXA = A$.*

Theorem 2.2. *Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $N(A) \subseteq N(C)$ and $N(MA) \subseteq N(B)$ then the following are equivalent.*

(i) M is a q-k-EP matrix with $k{=}k_1\ k_2$

(ii) A is a q-k_1-EP (MA) is q-k_2-EP, $N(A^*) \subseteq N(B^*)$ and $N((MA)^*) \subseteq N(C^*)$

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(iii) Both the matrices $\begin{bmatrix} A & 0 \\ C & (M|A) \end{bmatrix}$ *and* $\begin{bmatrix} A & B \\ 0 & (M|A) \end{bmatrix}$ *are q-k-EP.*

Proof. (i) $\Rightarrow$ (ii)

(i) Since M is a q-k-EP with $k = k_1k_2$, KM is EP and $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$ where K_1 and K_2 are associated permutation matrices of k_1 and k_2. Consider, $P = \begin{bmatrix} I & 0 \\ CA^- & I \end{bmatrix}$, $Q = \begin{bmatrix} I & B(M|A)^- \\ 0 & I \end{bmatrix}$ and $L = \begin{bmatrix} A & 0 \\ 0 & (M|A) \end{bmatrix}$. It is clear that P, Q are non-singular.

$$\begin{aligned} KPQL &= \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ CA^- & I \end{bmatrix} \begin{bmatrix} I & B(M|A)^- \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & (M|A) \end{bmatrix} \\ &= \begin{bmatrix} K_1A & K_1B(M|A)(M|A)^- \\ K_2CA^-A & K_2CA^-B(M|A)^-(M|A) + K_2(M|A) \end{bmatrix} \end{aligned}$$

Since $N(A) \subseteq N(C)$, by [8], we have $C = CA^-A$. Thus $K_2C = K_2CA^-A$. Also, since $N(M|A) \subseteq N(B)$, $B = B(M|A)^-(M|A)$. So, $K_2CA^-B(M|A)^-(M|A) + K_2(M|A) = K_2D$, (since $(MA) = D - CA^-B$). Thus,

$$KPQL = \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = KM.$$

Thus KM is factorized as $KM = KPQL$. Hence $\rho(KM) = (L)$ and $N(KM) = N(L)$. But M is q-k-EP. Therefore, KM is EP. $N(KM) = N((KM)^*) \Rightarrow N(L) = N(M^*K)$ [8]. By using, $M^*K = M^*KL^-L$ holds for all L^-. Choose, $L^- = \begin{bmatrix} A^- & 0 \\ 0 & (M|A) \end{bmatrix}^-$

$$M^*K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = \begin{bmatrix} A^*K_1 & C^*K_2 \\ B^*K_1 & D^*K_2 \end{bmatrix}$$

Since $M^*K = M^*KL^-L$,

$$\begin{aligned} \begin{bmatrix} A^*K_1 & C^*K_2 \\ B^*K_1 & D^*K_2 \end{bmatrix} &= \begin{bmatrix} A^*K_1 & C^*K_2 \\ B^*K_1 & D^*K_2 \end{bmatrix} \begin{bmatrix} A^- & 0 \\ 0 & (M|A) \end{bmatrix}^- \begin{bmatrix} A & 0 \\ 0 & (M|A) \end{bmatrix} \\ &= \begin{bmatrix} A^*K_1A^-A & C^*K_2(M|A)^-(M|A) \\ B^*K_1A^-A & D^*K_2(M|A)^-(M|A) \end{bmatrix} \end{aligned}$$

From the above, $A^*K_1 = A^*K_1A^-A$

$$\Rightarrow (K_1A)^* = (K_1A)^*A^-A$$

$$\Rightarrow N(A) \subseteq N(K_1A)^* = N(A^*K_1)$$

Since, $\rho(K_1A)^* = \rho(K_1A) \Rightarrow \rho(A^*K_1) = \rho(A)$. Thus, $N(A) = N(A^*K_1)$. Hence A is a q-k_1-EP. Similarly, we can prove $(M|A)$ is q-k_2-EP. Further, $C^*K_2 = C^*K_2(M \mid A)^-(M \mid A) \Rightarrow N(M \mid A) \subseteq N(C^*K_2) \Rightarrow N(K_2(M \mid A)^* N(C^*K_2) \Rightarrow N(M \mid A)^* N(C^*)$. Thus (ii) holds.

$(ii) \Rightarrow (i)$

Since $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*), N(M|A) \subseteq N(B), N((M \mid A)^*) \subseteq N(C^*)$ holds. By [2],

$$(KM)^\dagger = \begin{bmatrix} (K_1A)^\dagger + (K_1A)^\dagger(K_1B)(M|A)^\dagger K_2(K_1A)^\dagger & -(K_1A)^\dagger(K_1B)K_2(M|A)^\dagger \\ -K_2(M|A)^\dagger K_2C(K_1A)^\dagger & K_2(M|A)^\dagger \end{bmatrix}$$

From [8], $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*) \Rightarrow (M \mid A)$ is invariant for every choice of A^-. Hence $K_2D = K_2(M|A) + K_2C(K_1A)^\dagger(K_1B)$. Further using $K_2C = K_2(M \mid A)K_2(M \mid A)^\dagger K_2C$ and $K_1B = K_1A(K_1A)^\dagger K_1B$. Now,

$$(KM)(KM)^\dagger = \begin{bmatrix} K_1A(K_1A)^\dagger & 0 \\ 0 & K_2(M \mid A)K_2(M \mid A)^\dagger \end{bmatrix}$$

Again using, $K_2C = (K_2C)(K_1A)(K_1A)^\dagger$ and $K_1B = (K_1B)K_2(M \mid A)K_2(M \mid A)^\dagger$

$$(KM)^\dagger KM) = \begin{bmatrix} (K_1A)^\dagger K_1A & 0 \\ 0 & K_2(M \mid A)^\dagger K_2(M \mid A) \end{bmatrix}$$

Since A is q-k_1-EP, (M|A) is q-k_2-EP [5]. We have $(KM)(KM)^\dagger = (KM)^\dagger KM \Rightarrow M^\dagger MK = KMM^\dagger \Rightarrow M$is q-$k$-EP [5]. Thus (i) holds.

$(ii) \Rightarrow (iii)$

$\begin{bmatrix} K_1A & 0 \\ K_2C & K_2(M \mid A) \end{bmatrix}$ is EP $\Leftrightarrow K_1A$ and $K_2(M \mid A)$ are EP. $\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}\begin{bmatrix} A & 0 \\ C & (M \mid A) \end{bmatrix}$ is EP$\Leftrightarrow K_1A$ and $K_2(M \mid A)$ are EP. $\begin{bmatrix} A & 0 \\ C & (M \mid A) \end{bmatrix}$ is q-k-EP$\Leftrightarrow A$ is q-k_1-EP and $(M \mid A)$ is q-k_2-EP. Further $N(A) \subseteq N(C)$, $N((M|A)^*) \subseteq N(C^*)$. Also $\begin{bmatrix} K_1A & K_1B \\ 0 & K_2(M \mid A) \end{bmatrix}$ is EP $\Leftrightarrow K_1A$ and $K_2(M \mid A)$ are EP. $\begin{bmatrix} A & B \\ 0 & (M \mid A) \end{bmatrix}$ is q-k-EP$\Leftrightarrow A$ is q-k_1-EP and $(M \mid A)$ is q-k_2-EP. Further, $N(A^*) \subseteq N(B^*)$, $N(M|A) \subseteq N(B)$. Hence the equivalence of (ii) and (iii). □

Theorem 2.3. *Let M be a matrix, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $N(A^*) \subseteq N(B^*)$, $N((M|A)^*) \subseteq N(C^*)$ then the following are equivalent.*

(i). M is q-k-EP with $k = k_1k_2$.

(ii). A is q-k_1-EP and $(M \mid A)$ is q-k_2-EP. Further, $N(A) \subseteq N(C)$, $N(M|A) \subseteq N(B)$.

(iii). Both the matrices $\begin{bmatrix} A & 0 \\ C & (M \mid A) \end{bmatrix}$ and $\begin{bmatrix} A & B \\ 0 & (M \mid A) \end{bmatrix}$ are q-k-EP.

Proof. Applying the fact M is q-k-EP$\Leftrightarrow M^*$ is q-k-EP from Theorem 2.2, the proof is obvious. □

Corollary 2.4. *Let $M = \begin{bmatrix} A & C^* \\ C & D \end{bmatrix}$ with $N(A) \subseteq N(C), N(M|A) \subseteq N(C^*)$ then the following are equivalent.*

(i). M is q-k-EP with $k = k_1k_2$

(ii). A is q-k_1-EP and $(M \mid A)$ is q-k_2-EP. Further, $N(A) \subseteq N(C)$, $N(M|A) \subseteq N(B)$.

(iii). The matrix $\begin{bmatrix} A & 0 \\ C & (M \mid A) \end{bmatrix}$ *is q-k-EP.*

Remark 2.5. *The conditions on M in Theorem 2.2 and Theorem 2.4 are essential.*

For example,

Let $M = \begin{bmatrix} 1 & i & i & i \\ -i & 1 & j & i \\ -i & -j & 1 & k \\ -i & -i & -k & 1 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $KM^*K = \begin{bmatrix} 1 & i & i & i \\ -i & 1 & j & i \\ -i & -j & 1 & k \\ -i & -i & -k & 1 \end{bmatrix} = M \Rightarrow M$ is q-k-EP and rank 2 $\Rightarrow M$ is q-k-EP$_2$. More over, $A = B = C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $(M \mid A) = D - CA^{\dagger}B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$; $K_2 (M \mid A) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ is EP $\Rightarrow (M \mid A)$ is q-k_2-EP. $K_1 A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is EP $\Rightarrow A$ is q-k_1-EP. $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*)$, but $N(M|A) \not\subseteq N(B), N((M|A)^*) \subseteq N(C^*)$. Further, $K \begin{bmatrix} A & 0 \\ C & (M \mid A) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$ is not EP. $\begin{bmatrix} A & 0 \\ C & (M \mid A) \end{bmatrix}$ is not q-k-EP. Similarly, $K \begin{bmatrix} A & B \\ 0 & (M \mid A) \end{bmatrix}$ is not EP. $\begin{bmatrix} A & B \\ 0 & (M \mid A) \end{bmatrix}$ is not q-k-EP. Thus, Theorem 2.2 and Theorem 2.3 as well as Corollary 2.4 fails.

Remark 2.6. *For a q-k-EP matrix* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with $k = k_1 k_2$, the following are equivalent.*

$$N(A) \subseteq N(C), N(M|A) \subseteq N(B) \tag{2}$$

$$N(A^*) \subseteq N(B^*), N((M|A)^*) \subseteq N(C^*) \tag{3}$$

If we omit the condition, M is q-k-EP then the above fails.

For example, let

$$M = \begin{bmatrix} i & 1 & 1 & 0 \\ 1 & j & 1 & 0 \\ 1 & 1 & k & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$KM = \begin{bmatrix} i & 1 & 1 & 0 \\ 1 & j & 1 & 0 \\ 1 & 1 & k & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not EP. Therefore, M is not q-k-EP. Here $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is k_1-EP. $B = K_1 C^* K_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Thus

$N(A) \subseteq N(C), N(A^*) \subseteq\ N(B^*)$. Hence $(M|A)$ is independent of the choice of A^-.

$$K_2(M|A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$K_2(M|A)$ is not EP. $(M|A)$ is not q- k_2-EP. Thus $N((M \mid A)^*) \subseteq N(C^*)$ but $N(M|A) \subseteq N(B)$. Thus (3) holds while (2) fails.

Remark 2.7. *For a k-EP matrix M, the Formula 2.3 gives $(KM)^\dagger$ if and only if either (2) or (3) holds.*

Corollary 2.8. $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with $k = k_1k_2$ for which $(KM)^\dagger$ is given by the Formula 2.3. Then M is q-k-EP if and only if A is q-k_1-EP and $(M \mid A)$ is q-k_2-EP.*

Proof. This follows from Theorem 2.2 and using Remark 2.11 □

Theorem 2.9. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with $\rho(M)=\rho(A)= r$, then M is q-k-EP$_r$ with $k = k_1k_2$ if and only if A is q-k_1-EP$_r$ and $CA^\dagger K_1 = (A^\dagger BK_2)^*$.*

Proof. Let $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, $KM = \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix}$. Since $\rho(M) = \rho(A) = r$, $\rho(KM) = \rho(K_1A) = r$. By [5], $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*)$ and $(KM \mid K_1A) = K_2(M \mid A) = 0$. From [8], these relations are equivalent to $K_2C = K_2CA^\dagger A$, $K_1B = K_1BAA^\dagger$ and $K_2D = K_2CA^\dagger B$.

Consider, $P = \begin{bmatrix} I & 0 \\ CA^\dagger & I \end{bmatrix}$, $Q = \begin{bmatrix} I & A^\dagger B \\ 0 & I \end{bmatrix}$ and $L = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$. P, Q are non-singular. By assumption, $CA^\dagger K_1 = (A^\dagger BK_2)^*$, we have $KP = (KQ)^*$,

$$KPLQ = \begin{bmatrix} K_1A & K_1AA^\dagger B \\ K_2CA^\dagger A & K_2CA^\dagger B \end{bmatrix} = \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix} = KM$$

Since, $KP = (KQ)^*$, $KP^*K = Q$, we have $KM = KPLKP^*K \Rightarrow KM = (KP)(LK)(KP)^*(KP)(KL)(KP)^*$, since $KL = LK$. Since A is q-k_1-EP$_r$, K_1A is EP$_r$. $KL = \begin{bmatrix} K_1A & 0 \\ 0 & 0 \end{bmatrix}$ is EP$_r$ $\Rightarrow$ L is q-k-EP$_r$. Therefore, $N(L) = N(L^*K)\, N(KL) = N(KL)^*$. By [1],

$$N((KP)(KL)(KP)^*) = N((KP)(KL)^*(KP)^*)\, N(KM) = N(KM)^*$$

$N(M) = N(M^*K)M$ is q-k-EP$_r$ [5]. Since $\rho(M) = r$, M is q-k-EP$_r$.

Conversely, let us assume that M is q-k-EP$_r$. Thus KM is EP$_r$and $KM = KPLQ$, $(KM)^- = Q^- \begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} P^-K$ is EP $\Rightarrow N(KM) = N(KM)^*$ [8]

$$(KM)^* = (KM)^*(KM)^-(KM)$$

$$\begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix}^* = \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix}^* Q^- \begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} P^-K \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix}$$

$$\begin{bmatrix} (K_1A)^* & (K_2C)^* \\ (K_1B)^* & (K_2D)^* \end{bmatrix} = \begin{bmatrix} (K_1A)^*A^\dagger A & (K_1A)^*A^\dagger B \\ (K_1B)^*A^\dagger A & (K_1B)^*A^\dagger B \end{bmatrix}$$

$$(K_1A)^* = (K_1A)^* A^\dagger AN\,(A) = N((K_1A)^*) \text{ and}$$
$$(K_2C)^* = (K_1A)^* A^\dagger BK_2C = B^* (A^\dagger)^* (K_1A)$$

Hence $N\,(A) = N(A^* K_1)A$ is q-k_1-EP, since $\rho(\ A) =$ r, A is q-k_1-EP$_r$

$$\begin{aligned} K_2CA^\dagger &= B^* \left(A^\dagger)^* (K_1A\right) A^\dagger = B^* \left(A^\dagger)^* (K_1AA^\dagger\right) \\ &= B^* \left(A^\dagger)^* (A^\dagger AK_1\right) \quad \text{([5], Theorem 2.4)} \\ &= B^* \left((A^\dagger)^* (A^\dagger A)^* (K_1)^*\right) \quad \text{(Since } A^\dagger A \text{ is hermitian)} \\ &= B^* \left((A^\dagger AA^\dagger)^* (K_1)^*\right) \\ K_2CA^\dagger &= B^* (A^\dagger)^* (K_1)^* = (K_1A^\dagger B)^* = (A^\dagger B)^* K_1 \end{aligned}$$

Also, $CA^\dagger K_1 = K_2(A^\dagger B)^* = (A^\dagger BK_2)^*$. The theorem is proved. ☐

Corollary 2.10. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *with A is a non-singular matrix and $\rho(A) = \rho(M)$, then M is q-k-EP with $k = k_1k_2 \Leftrightarrow CA^\dagger K_1 = K_2(A^\dagger B)^* = (A^\dagger BK_2)^*$.*

Remark 2.11. *The condition on rank of M is essential in Theorem 2.13.*

For example, Consider $M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $KM = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\rho(KM) = \rho(M) = 2$, but $\rho(K_1A) = \rho(A) = 1$. Hence $\rho(KM) \neq \rho(K_1A) \Rightarrow \rho(M) \neq \rho(A) KM$ is not EP. M is not q-k-EP. $K_1A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is EP. A is q-k_1-EP.

$$A^\dagger = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; CA^\dagger K_1 = \frac{1}{4} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \left(A^\dagger BK_2\right)^*$$

Thus Theorem 2.13 fails.

Corollary 2.12. *Let M be a $2n \times 2n$ matrix of rank r. Then M is q-k-EP$_r$ with $k = k_1k_2 \Leftrightarrow$ Every principal sub matrix of rank r is q-k_1-EP$_r$.*

Proof. Suppose M is q-k-EP$_r$, KM is EP$_r$. Let K_1A be any principal sub matrix of KM such that $\rho(KM) = \rho(K_1A) = r$ then there exists a permutation matrix P such that $(KM)' = P(KM)P^T \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix}$, with $(KM)' = (K_1A) = r$. By [1], $(KM)'$ is EP$_r$. By Theorem 2.13, K_1A is EP$_r \Rightarrow A$ is q-k_1-EP$_r$. Since A is arbitrary, every principal sub matrix of rank r is q-k_1-EP$_r$. ☐

Definition 2.13. *M_1 and M_2 are called complementary summands of M if $M = M_1 + M_2$ and $\rho(M) = \rho(M_1) + \rho(M_2)$.*

Theorem 2.14. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *with $\rho(M) = \rho(A) + \rho(M|A)$ where $(M|A) = D - CA^\dagger B$. If A is q-k_1-EP and $(M|A)$ is q-k_2-EP such that $CA^\dagger K_1 = \left(A^\dagger BK_2\right)^*$ and $B(M|A)^\dagger K_2 = ((M \mid A)^\dagger CK_1)^*$ then M can be decomposed into complementary summands of q-k-EP matrices.*

Proof. Consider, $M_1 = \begin{bmatrix} A & AA^{\dagger}B \\ CA^{\dagger}A & CA^{\dagger}B \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & \left(I - AA^{\dagger}\right)B \\ C\left(I - AA^{\dagger}\right) & (M \mid A) \end{bmatrix}$ such that $N(A)N(CA^{\dagger}A)$, $N(A^*K_1)N(\left(AA^{\dagger}B\right)^*K_1)$ and

$$(M_1 \mid A) = CA^{\dagger}B - \left(CA^{\dagger}A\right)A^{-}(AA^{\dagger}B) = \; CA^{\dagger}B - CA^{\dagger}(AA^{-}A)A^{\dagger}B) = 0$$

By [3], $\rho(M_1) = \rho(A)$. Since A is q-k_1-EP and

$$\left(CA^{\dagger}A\right)A^{\dagger}K_1 = C(A^{\dagger}AA^{\dagger})K_1 = CA^{\dagger}K_1 = \left(A^{\dagger}BK_2\right)^* = \left(A^{\dagger}(AA^{\dagger}B)K_2\right)^*$$

By Theorem 2.13, M_1 is q-k_1-EP. Since, $\rho(M) = \rho(A) + \rho(M|A)$. By [3], $N(M|A) \subseteq N(C(I - A^{\dagger}A)B)$

$$N(M\,|A)^* \subseteq N(C\left(I - A^{\dagger}A\right)^*) \text{ and } \left(I - AA^{\dagger}\right)B(M \mid A)^{\dagger} \subseteq \left(I - A^{\dagger}A\right) = 0$$

Therefore, $(M_2\,|(M \mid A)) = 0$. By [3], $(M_2) = \rho(M|A)$. Hence, $(M) = (M_1) + (M_2)$. Further, $AA^{\dagger}K_1 = K_1A^{\dagger}A$

$$\begin{aligned}\left(I - AA^{\dagger}\right)B\,(M|A)^{\dagger}K_2 &= \left(I - AA^{\dagger}\right)((M|A)^{\dagger}CK_1)^* = \left((M|A)^{\dagger}\;\; CK_1\left(I - AA^{\dagger}\right)^*\right)^* \\ &= ((M|A)^{\dagger}C(I - A^{\dagger}A)K_1)^*\end{aligned}$$

By Theorem 2.13, M_2 is q-k_2-EP. Clearly, $M = M_1 + M_2$ and $\rho(M) = \rho(M_1) + \rho(M_2)$. Hence M_1 and M_2 are complementary summands of q-k-EP matrices. □

Remark 2.15. *Any matrix represented as the sum of complementary summands of q-k-EP matrices is q-k-EP. If* $M = \sum_{i=1}^{n} M_i$ *such that* M_i *is q-k-EP and* $(M) = \left(\sum_{i=1}^{n} M_i\right)$. *Then* $N\,(M) = \bigcap_{i=1}^{n} N(M_i) = \bigcap_{i=1}^{n} N({M_i}^*K)(M_i$ *is q-k-EP).* $N\,(M) = N(M^*K)$. *Thus* M *is q-k-EP.*

3. Factorization of q-k-EP matrices

Throughout this section, M is a $2n \times 2n$ matrix of the form,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ with } \rho(M) = \rho(A) = r \tag{5}$$

Where A is $n \times n$ and D is $n \times n$. If M is q-k-EP with $k = k_1k_2$ then the associated permutation matrix K is of the form,

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{6}$$

where K_1 is the associated permutation $n \times n$ matrix of k_1 and K_2 is the associated permutation $n \times n$ matrix of k_2.

$$KM = \begin{bmatrix} K_1A & K_1B \\ K_2C & K_2D \end{bmatrix} \text{ and } \rho(A) = \rho(M) = r \tag{7}$$

By [3],

$$N\,(K_1A) \subseteq N\,(K_2C)\,,\;\; N\,(A^*K_1) \subseteq \;N\,(B^*K_1)\,,\;\; D = CA^{\dagger}B \tag{8}$$

Also let

$$MK = \begin{bmatrix} AK_1 & BK_1 \\ CK_2 & DK_2 \end{bmatrix} \quad \text{and} \;\; \rho(A) = \rho(M) = r \tag{9}$$

Again by [3],

$$N\left(AK_1\right) \subseteq N\left(CK_1\right), \;\; N\left(K_1A^*\right) \subseteq \;\; N\left(K_2B^*\right), \;\; D = CA^{\dagger}B \tag{10}$$

Lemma 3.1. *If M is q-k-EP$_r$ of the form (5) with k =k_1k_2 then there exists a $(p \times 2n - p)$ matrix X such that*

$$KM = \begin{bmatrix} K_1A & K_1AX \\ X^*K_1A & X^*K_1AX \end{bmatrix} \tag{11}$$

And A is q-k_1-EP$_r$.

Proof. Since KM is of the form (7) and $\rho(A) = \rho(M)$ then (8) holds. Hence there is an $(p \times 2n - p)$ matrix X such that K_2C =YK_1A and $B = AX$. By [8], since M is q-k-EP$_r$, By Theorem 2.13, A is q-k_1-EP$_r$ and

$$CA^{\dagger}K_1 {=} \left(A^{\dagger}BK_2\right)^*$$

Also by Theorem 2.4 [5], A is q-k_1-EP$_r$. $K_1AA^{\dagger}{=}\ AA^{\dagger}K_1AA^{\dagger}K_1{=}K_1AA^{\dagger}$. Since, $CA^{\dagger}K_1{=}\left(A^{\dagger}BK_2\right)^*$

$$K_2CA^{\dagger}K_1{=}\left(A^{\dagger}B\right)^*YK_1A = X^*K_1A$$

Also, K_2D =$K_2CA^{\dagger}B = YK_1AX = X^*K_1AX$. Hence, KM is of the form (11). □

Lemma 3.2. *If M is q-k-EP$_r$ of the form (5) with k =k_1k_2 then there exists a $(p \times 2n - p)$ matrix X such that*

$$MK = \begin{bmatrix} AK_1 & AK_1X \\ X^*AK_1 & X^*AK_1X \end{bmatrix} \tag{12}$$

And A is q-k_1-EP$_r$.

Proof. Since MK is of the form (9) and $\rho(A) = \rho(M)$ then (10) holds. Hence there is an $(2n - p \times p)$ matrix Y such that $BK_2{=}AK_1X$ and C= YA. By [8], since M is q-k-EP$_r$, by Theorem 2.13, A is q-k_1-EP$_r$ and

$$CA^{\dagger}K_1 = \left(A^{\dagger}BK_2\right)^*$$
$$YAA^{\dagger}K_1 = (A^{\dagger}AK_1X)^*YAK_1{=}X^*AK_1$$

Also,

$$\begin{aligned} DK_2 &= CA^{\dagger}BK_2 \\ &= YAK_1X \\ &= X^*AK_1X \end{aligned}$$

Hence, MK is of the form (12). □

Theorem 3.3. *If M is q-k-EP$_r$ of the form (5) and A is q-k_1-EP$_r$, then M is a product of q-k-EP$_r$ matrices.*

Proof. If M is q-k-EP$_r$ of the form (5) then it satisfies $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$, $D = CA^\dagger B$, hence there exists X and Y such that $C = YA$, $B = AX$, $D = CA^\dagger B = YAA^\dagger AX = YAX$. Consider the matrices, $SK = \begin{bmatrix} A^\dagger AK_1 & AA^\dagger Y^* K_2 \\ YAA^\dagger K_1 & YAA^\dagger Y^* K_2 \end{bmatrix}$, $KL = \begin{bmatrix} K_1 A & 0 \\ 0 & 0 \end{bmatrix}$ and $TK = \begin{bmatrix} A^\dagger AK_1 & AA^\dagger XK_2 \\ X^* A^\dagger AK_1 & X^* A^\dagger AXK_2 \end{bmatrix}$. By Theorem 2.13, S, L and T are q-k-EP$_r$. Also,

$$(SK)(KL)(TK) = \begin{bmatrix} AK_1 & AXK_2 \\ YAK_1 & YAXK_2 \end{bmatrix} = \begin{bmatrix} AK_1 & BK_2 \\ CK_1 & DK_2 \end{bmatrix} = MK$$

Thus, MK is a product of SK, KL and TK are all q-k-EP$_r$ matrices. Therefore, $M = SLT$. □

Lemma 3.4. *Let $L = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ be a $2n \times 2n$ matrix of rank r. If E is an $n \times n$ non-singular matrix, then $L = S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T$, where S, T are q-k-EP$_r$ matrices.*

Proof. $L = KP \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} KQ$, where P, Q are non-singular matrix and K is the permutation matrix $\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$. If we write $P = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$, $P = \begin{bmatrix} \widehat{A_1} & \widehat{B_1} \\ \widehat{C_1} & \widehat{D_1} \end{bmatrix}$ then $L = \begin{bmatrix} (K_1 A_1)(K_1 \widehat{A_1}) & (K_1 A_1)(K_1 \widehat{B_1}) \\ (K_2 C_1)(K_1 \widehat{A_1}) & (K_2 C_1)(K_1 \widehat{B_1}) \end{bmatrix}$ and $(K_1 A_1)(K_1 \widehat{A_1}) = E$ is non-singular. Thus, $K_1 A$, $(K_1 \widehat{A})$ are non-singular. So, $\begin{bmatrix} K_1 A_1 \\ K_2 C_1 \end{bmatrix}$ and $\begin{bmatrix} K_1 \widehat{A_1} & K_2 \widehat{B_1} \end{bmatrix}$ have rank r. Thus there is an $2n - r \times r$ matrix X and $r \times 2n - r$ matrix Y such that $XK_1A_1 = K_2C_1$ and $\widehat{A_1} Y = \widehat{B_1}$. Put $S = \begin{bmatrix} K_1 A_1 & K_1 A_1 X^* \\ XK_1 A_1 & XK_1 A_1 X^* \end{bmatrix}$, $T = \begin{bmatrix} K_1 \widehat{A_1} & K_1 \widehat{A_1} Y \\ Y^* K_1 \widehat{A_1} & Y^* K_1 \widehat{A_1} Y \end{bmatrix}$. Now,

$$S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} K_1 A_1 & K_1 A_1 X^* \\ XK_1 A_1 & XK_1 A_1 X^* \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 \widehat{A_1} & K_1 \widehat{A_1} Y \\ Y^* K_1 \widehat{A_1} & Y^* K_1 \widehat{A_1} Y \end{bmatrix} = L$$

By [1], KS and KT are EP$_r$ matrices. Hence, S, T are q-k-EP$_r$ matrices. Any matrix $AH_{2n \times 2n}$ of rank r is called a P_r matrix if it has a principal $r \times r$ non-singular matrix. □

Lemma 3.5. *Let M be a $2n \times 2n$ matrix of order r. If M is a P_r matrix then M is a product of q-k-EP$_r$ matrices.*

Proof. Let M be a $2n \times 2n$ matrix of order r having E as a principal $r \times r$ non-singular sub matrix, there is a permutation matrix P such that $PMP^T = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$. By Lemma 3.12, $\begin{bmatrix} E & F \\ G & H \end{bmatrix} = S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T$, where S,T are q-k-EP$_r$ matrices. Hence,

$$PMP^T = S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T$$

$$M = P^T S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} TP$$

$$M = (P^T SP)P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P(PTP)$$

Since S, T are q-k-EP$_r$ matrices, $P^T SP$ and $P^\dagger TP$ are q-k-EP$_r$ matrices. Thus, M is a product of q-k-EP$_r$ matrices. □

Remark 3.6. *The converse of Theorem 3.13 need not be true.*

Example 3.7. *Let* $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ -i & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & j \\ 0 & 0 & -j \\ 1 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & k & 0 \\ 0 & -k & 0 \end{bmatrix}$. *For* $K = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *where* $A,\ B,\ C$ *are q-k-EP matrices of rank 2. But* $ABC = \begin{bmatrix} 0 & 0 & 0 \\ i & j & -i \\ 0 & 1 & 0 \end{bmatrix}$ *has rank 2, does not have a* P_2 *matrices. More over, ABC is not q-k-EP.*

Lemma 3.8. *Let* $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ *be a q-k-EP*$_r$ *matrix with* $k = k_1k_2$. K_1E *is an r x r matrix and* $\begin{bmatrix} K_1E & K_1F \end{bmatrix}$ *has rank r, then* K_1E *is non-singular.*

Proof. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} K_1E & K_1F \\ 0 & 0 \end{bmatrix}$ where I_r is the $r \times r$ identity matrix. By [2], $\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} K_1E & 0 \\ K_2G & 0 \end{bmatrix}$ has rank r. By [8], K_1E has rank r. Thus K_1E is non-singular. □

Theorem 3.9. *Let A and B be* $2n \times 2n$ *q-k-EP matrices with* $k = k_1k_2$. *If AB has rank r, then AB is unitarily similar to a* P_r *matrix.*

Proof. Since A is q-k-EP$_r$, by [5], there is a unitary matrix U such that A is unitarily k-similar to a diagonal block q-k-EP$_r$ matrix $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where D is a r x r non-singular matrix.

$$A = KUK\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^{*} \Rightarrow U^{*}(KA)U = K\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Put $U^{*}(BK)U = \begin{bmatrix} E & F \\ H & G \end{bmatrix}$ where E is r x r matrix. Then

$$U^{*}(KA)(BK)U = K\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E & F \\ H & G \end{bmatrix}$$

$$(KU)^{*}AB(KU) = \begin{bmatrix} K_1DE & K_1DF \\ 0 & 0 \end{bmatrix} \text{ has rank r.}$$

Thus $K_1D\begin{bmatrix} E & F \end{bmatrix}$ has rank r, it follows $\begin{bmatrix} E & F \end{bmatrix}$ has rank r. By Lemma 3.16, K_1E is non-singular. Thus $(KU)^{*}AB(KU)$ is a P$_r$ matrix. AB is unitarily similar to a P$_r$ matrix. □

Theorem 3.10. *Let A and B be* $n \times n$ *matrices. If A has rank r, B and AB are q-k-EP*$_r$ *matrices, then A is a product q-k-EP*$_r$ *of matrices.*

Proof. Since B is q-k-EP$_r$, BK is EP$_r$. By [5], $B = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}KU^{*}K$, D is r x r non-singular and U is a unitary matrix.

$$U^{*}BKU = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}K, U^{*}BKU = \begin{bmatrix} DK_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Put $U^*(KA)U = \begin{bmatrix} E & F \\ H & G \end{bmatrix}$ where E is r x r matrix and U is unitary. Then

$$\begin{aligned}(U^*KAU)\,((U^*BKU) &= \begin{bmatrix} E & F \\ H & G \end{bmatrix}\begin{bmatrix} DK_1 & 0 \\ 0 & 0 \end{bmatrix}\\ \Rightarrow U^*KABKU &= \begin{bmatrix} EDK_1 & 0 \\ GDK_1 & 0 \end{bmatrix}\\ \Rightarrow (KU)^*AB\,(KU) &= \begin{bmatrix} EDK_1 & 0 \\ GDK_1 & 0 \end{bmatrix}\end{aligned}$$

Since AB is q-k-EP$_r$, by [5], $GDK_1 = 0$. Hence $G = 0$. E is non-singular. Applying Lemma 3.12, A is a product of q-k-EP$_r$ matrices. □

Remark 3.11. *The condition on $\rho(A) = r$ is essential. If $\rho(A) \neq r$ then Theorem 3.18 fails.*

For example, Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and let $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Here $\rho(A) = 1$, $\rho(B) = 0$. B is q-k-EP$_0$. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is q-k-EP$_0$. Here $B = AB$. Hence the Statement of 3.18 fails.

Theorem 3.12. *Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $L = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ be q-k-EP$_r$ matrices with $k = k_1k_2$ and ML be of rank r. Then the following are equivalent.*

(1). ML is q-k-EP$_r$

(2). AP is q-k_1-EP$_r$ and $CA^\dagger K_1 = K_2RP^\dagger$

(3). AP is q-k_1-EP$_r$ and $A^\dagger BK_2 = K_1P^\dagger Q$

Proof.

$$\begin{aligned}MK &= \begin{bmatrix} AK_1 & BK_2 \\ CK_1 & DK_2 \end{bmatrix}, \quad KL = \begin{bmatrix} K_1P & K_1Q \\ K_2R & K_2S \end{bmatrix}\\ (MK)\,(KL) &= \begin{bmatrix} AK_1(1+XY^*)K_1P & AK_1\,(1+XY^*)K_1PY \\ X^*AK_1(1+XY^*)K_1P & X^*AK_1(1+XY^*)K_1PY \end{bmatrix}\\ ML &= \begin{bmatrix} AK_1ZK_1P & AK_1\,ZK_1PY \\ X^*AK_1ZK_1P & X^*AK_1ZK_1PY \end{bmatrix}, \; Z = 1+XY^*\end{aligned}$$

Clearly,

$$N\,(AK_1ZK_1P) \subseteq N(X^*AK_1ZK_1PY)$$
$$N(AK_1ZK_1P)^* \subseteq N(X^*AK_1ZK_1PY)^*$$

Schur complement of AK_1ZK_1P in ML,

$$(ML\,|AK_1ZK_1P\,) = (X^*AK_1ZK_1PY) - (X^*AK_1ZK_1P)\,(AK_1ZK_1P)^\dagger\,(AK_1\;ZK_1PY) = 0$$

By [3], $\rho(AK_1ZK_1P) = \rho(ML) = r$. Hence by Theorem 2.13, A and P are both q-k_1-EP$_r$ matrices.

$$CA^{\dagger}K_1{=}(A^{\dagger}BK_2)^*,\ RP^*K_1{=}(P^{\dagger}QK_2)^* \tag{13}$$

$$R\,(AK_1ZK_1P) \subseteq R\,(AK_1) = R(A)$$
$$R(AK_1ZK_1P)^* \subseteq R\,(P^*K_1) = R\,(P^*) = R\,(K_1P) \quad \text{(Since } P \text{ is q-}k_1\text{-EP)}$$
$$\text{and}\ \ \rho\,(AK_1ZK_1P) = \rho\,(A) = \rho(K_1P) = r$$

Hence, $R\,(AK_1ZK_1P) = R\,(A)\,;\ R(AK_1ZK_1P)^* = R\,(K_1P)$

$$(AK_1ZK_1P)(AK_1ZK_1P)^{\dagger} = (AK_1)(AK_1)^{\dagger} \tag{14}$$

By [2],

$$(AK_1ZK_1P)^{\dagger}(AK_1ZK_1P) = (K_1P)(K_1P)^{\dagger} \tag{15}$$

ML is q-k-EP$_r$ $\Leftrightarrow$ $(MK)\,(KL)$ is EP$_r$ $\Leftrightarrow$ AK_1ZK_1P is EP$_r$ (By Theorem 2.13)

$$(X^*AK_1ZK_1P)(AK_1ZK_1P)^{\dagger} = (AK_1ZK_1P)^{\dagger}\,(AK_1ZK_1PY)^*$$
$$\Leftrightarrow R\,(AK_1ZK_1P) = R\,(AK_1ZK_1P)^* \quad (By \ \ (15))$$
$$X^*(AK_1)(AK_1)^{\dagger} = Y^*\,(K_1P)\,(K_1P)^{\dagger}$$
$$R(A) = R(K_1P) \ \text{ and by } (14)$$
$$(X^*AK_1)\left(K_1A^{\dagger}\right) = (Y^*K_1P)(P^{\dagger}K_1)$$

Since A and P are both q-k_1-EP$_r$ matrices, $\Leftrightarrow$ AP is q-k_1-EP$_r$, $CK_1K_1A^{\dagger}{=}K_2RP^{\dagger}K_1$ $\Leftrightarrow$ AP is q-k_1-EP$_r$ and $CA^{\dagger}K_1{=}K_2RP^{\dagger}$ $\Leftrightarrow$ AP is q-k_1-EP$_r$ and $(A^{\dagger}BK_2)^* = K_2(P^{\dagger}QK_2)^*$ $\Leftrightarrow$ AP is q-k_1-EP$_r$ and $A^{\dagger}BK_2{=}K_1P^{\dagger}Q$. Thus, ML is q-k-EP$_r$ $\Leftrightarrow$ AP is q-k_1-EP$_r$ and $A^{\dagger}BK_2{=}K_1P^{\dagger}Q$. □

4. Pivotal Transform on q-k-EP Matrices

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ then a principal re-arrangement of square matrix M (i.e) P^TMP, where P is a permutation matrix, $P^TMP = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$, where P is a permutation matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let us consider a system of Linear equations, $Mz = t$, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying $N\,(A) \subseteq N\,(C)\,,\ N\,(A^*) \subseteq\ N\,(B^*)$. If z and t are partitioned conformably as $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $t = \begin{bmatrix} u \\ v \end{bmatrix}$. Then $Ax + By = u,\ Cx + Dy = v$. Then by [7, P.21] we can solve for x and v as

$$x = A^{\dagger}u^{-}A^{\dagger}By,\ v = CA^{\dagger}u + \left(D - CA^{\dagger}B\right)y$$

Thus a matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying $N(A)\, N(C)$, $N(A^*)\ \ N(B^*)$ can be transformed into the matrix,

$$\widehat{M} = \begin{bmatrix} A^\dagger & -A^\dagger B \\ CA^\dagger & (M|A) \end{bmatrix} \tag{16}$$

$\widehat{M}$ is called a principal pivot transform of M.

Lemma 4.1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $N(A) \subseteq N(C)$, $N(D) \subseteq N(B)$ *then the following are equivalent.*

(i). M is q-k-EP with $k = k_1 k_2$, $N(M|A) \subseteq N(B)$, $N(M|D) \subseteq N(C)$

(ii). A and $M|D$ are q-k_1-EP and D and $(M|A)$ are q-k_2-EP.

Further, $N(A) = N(M|D) \subseteq N(B^* K_1)$ and $N(D) = N(M|A) \subseteq N(C^* K_2)$.

Proof. $(i) \Rightarrow (ii)$: Since M is q-k-EP with $k = k_1 k_2$, $N(A) \subseteq N(C)$, $N(M|A) \subseteq N(B)$. By Theorem 2.2, A is q-k_1-EP and $(M|A)$ is q-k_2-EP; $N(A^* K_1) \subseteq N(B^* K_1)$ and $N((M|A)^* K_2) \subseteq N(C^* K_2)$. Since A is q-k_1-EP, $N(A^* K_1) = N(A)$ (By Definition of q-k-EP). Therefore, $N(A) = N(B^* K_1)$. Since M is q-k-EP, KM is EP , implies the principal rearrangement $P^T KMP = \begin{bmatrix} K_1 D & K_2 C \\ K_1 B & K_1 A \end{bmatrix}$ is also EP.
Further $N(K_2 D) \subseteq N(K_1 B)$ and $N(K_1(M|D)) \subseteq N(K_2 C)$ holds. Hence by Theorem 2.2, $K_2 D$ is EP. $K_1(M|D)$ is EP. $N((K_2 D)^*) \subseteq N((K_2 C)^*)$ and $N(K_1(M|D)) \subseteq N((K_1 B)^*)$. Thus We have, D is q-k_2-EP, $(M|D)$ is q-k_1-EP. $N(D^* K_2) \subseteq N(C^* K_2)$ and $N(K_1(M|D)) \subseteq N(B^* K_1)$. Since, D is q-k_2-EP, by Definition, $N(D^* K_2) = N(D)$. Thus, $N(D) \subseteq N(C^* K_2)$. Since the relations, $N(A) \subseteq N(C)$,$N(A^* K_1) \subseteq N(B^* K_1)$, $N(M|A) \subseteq N(B)$ and $N((M|A)^* K_2) \subseteq N(C^* K_2)$ holds for $K_1 A$. According to the assumptions and from [7],

$$(KM)^\dagger = \begin{bmatrix} (K_1A)^\dagger + (K_1A)^\dagger (K_1B)(M|A)^\dagger K_2 (K_1A)^\dagger & -(K_1A)^\dagger (K_1B) K_2 (M|A)^\dagger \\ -K_2 (M|A)^\dagger K_2 C (K_1A)^\dagger & K_2 (M|A)^\dagger \end{bmatrix} \tag{17}$$

Using $K_2 C = (K_2(M|A)(K_2(M|A)^\dagger (K_2 C)$ and $K_1 B = (K_1 A)(K_1 A))^\dagger (K_1 B)$

$$(KM)^\dagger KM) = \begin{bmatrix} (K_1A)(K_1A)^\dagger & 0 \\ 0 & (K_2\,(M \mid A))\,(K_2(M \mid A)\,)^\dagger \end{bmatrix} \tag{18}$$

Since the relations, $N(D) \subseteq N(B)$, $N(D^* K_2) \subseteq N(C^* K_2)$, $N((M|D)) \subseteq N(C)$ and $N((M|D)^* K_1) \subseteq N(B^* K_1)$ holds for $K_1 D$, according to the assumptions by Theorem 1.2,

$$(KM)^\dagger = \begin{bmatrix} (K_1(M|D))^\dagger & -(K_1A)^\dagger (K_1B) K_2 (M|A)^\dagger \\ -(K_2D)^\dagger K_2 C\ (K_1(M|D))^\dagger & K_2 (M|A)^\dagger \end{bmatrix} \tag{19}$$

Using $K_2 C = (K_2 D)(K_2(D)^\dagger (K_2 C)$, $C = DD^\dagger C$ and $K_1 B = (K_1 A)(K_1 A)^\dagger (K_1 B)$, $B = AA^\dagger B$ in (19)

$$(KM)(KM)^\dagger = \begin{bmatrix} K_1(M|D)(K_1(M|D))^\dagger & 0 \\ 0 & K_2\,(M \mid A)\,K_2(M \mid A)^\dagger \end{bmatrix} \tag{20}$$

Comparing (18) and (20),

$$(K_1A)(K_1A))^{\dagger}{=}K_1(M|D)(K_1(M|D))^{\dagger}\Rightarrow K_1AA^{\dagger}K_1{=}K_1(M|D)(K_1(M|D)^{\dagger}K_1$$

Thus, $AA^{\dagger}= (M|D)(M|D)^{\dagger}$. Since A and $(M|D)$ are q-k_1-EP, $(K_1A)^{\dagger}(K_1A) = (K_1(M \mid D)^{\dagger}(K_1(M \mid D))$; $A^{\dagger}A = (M|D)^{\dagger}(M|D)$. Thus, $N(A) = N(M|D)$ [3]. Similarly, we can obtain the expressions for $(KM)^{\dagger}(KM)$. Comparing $D^{\dagger}D = (M|A)^{\dagger}(M|A) \Rightarrow N(D) = N(M|A)$.

(ii) $\Rightarrow$ *(i)* : $N(M|A) \subseteq N(B)$ follows directly from $N(M \mid A) = N(D) \subseteq N(B)$. Similarly, $N(M \mid D) \subseteq N(C)$ follows from $N(M \mid D) = N(A) \subseteq N(C)$. Now A is q-k_1-EP and $(M \mid A)$ is q-k_2-EP satisfying the relations $N(A) \subseteq N(C)$, $N(A^*K_1) \subseteq N(B^*K_1)$, $N(M|A) \subseteq N(B)$ and $N((M|A)^*K_2) \subseteq N(C^*K_2)$. Hence by Theorem 2.2, M is q-k-EP. Thus (i) holds. ☐

Theorem 4.2. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a q-k-EP*$_r$ *matrix with* $k = k_1k_2$, $N(A) \subseteq N(C)$, $N(D) \subseteq N(B)$, $N(M|A) \subseteq N(B)$ *and* $N(M|D) \subseteq N(C)$. *Then the following are hold.*

(i). Principal sub-matrix A *is q-*k_1*-EP and principal sub-matrix* D *is q-*k_2*-EP.*

(ii). The schur complement $(M|A)$ *is q-*k_2*-EP and the schur complement* $(M|D)$ *is q-*k_1*-EP.*

(iii). Each principal pivot transforms of M *is q-k-EP*$_r$*.*

Proof. (i) and (ii) are consequences of Lemma 4.1

(iii): By Lemma 4.1, KM satisfies $N(A) \subseteq N(C)$,$N(A^*K_1) \subseteq N(B^*K_1)$ hence by pivoting the block K_1A, the principal pivot transform $\widehat{KM}$ of KM is of the form $\widehat{KM}= \begin{bmatrix} (K_1A)^{\dagger} & -(K_1A)^{\dagger}(K_1B) \\ (K_2C)(K_1A)^{\dagger} & K_2(M|A) \end{bmatrix}$

$$\widehat{KM}= \begin{bmatrix} A^{\dagger}K_1 & -A^{\dagger}B \\ K_2CA^{\dagger}K_1 & K_2(M|A) \end{bmatrix} \tag{21}$$

In $\widehat{KM}$, $N(A^{\dagger}K_1)N\left(K_2CA^{\dagger}K_1\right) = N\left(CA^{\dagger}K_1\right)$, $N((A^{\dagger}K_1)^*) \subseteq N((A^{\dagger}B)^*)$. Further,

$$\begin{aligned}(\widehat{KM})|(K_1A)^{\dagger}) &= K_2\,(M \mid A) + \left(K_2CA^{\dagger}K_1\right)\left(A^{\dagger}K_1\right)^{\dagger}(A^{\dagger}B)\\ &= K_2\,(M \mid A) + K_2CA^{\dagger}K_1K_1AA^{\dagger}B\\ &= K_2(\,(M \mid A) + CA^{\dagger}B)\\ &= K_2D\\ \Rightarrow (\widehat{KM})|(K_1A)^{\dagger}) &= K_2D\end{aligned}$$

By the assumption, $N(K_2(\widehat{M}|A^{\dagger})) = N(K_2D)$ which implies

$$N((\widehat{M}|A^{\dagger})) = N(D) \subseteq N(B).$$

From Lemma 4.1, A is q-k_1-EP and$D$ is q-k_2-EP. Therefore, $A^{\dagger}$ is q-k_1-EP and $(\widehat{M}|A^{\dagger})$ is q-k_2-EP(By [5], Theorem 2.4). Hence, $D = (\widehat{M}|A^{\dagger})$. Also, $N(K_2(\widehat{M}|A^{\dagger}))^* = N(K_2D)^*)$

$$N((\widehat{M}|A^{\dagger})^*K_2) = N(D^*K_2) \subseteq N(C^*K_2)$$

Now applying Theorem 2.2, we see that $\widehat{M}$ is q-k-EP. Now,

$$\begin{aligned} r = \rho(M) &= \rho(A) + (M|A) \quad (\text{By [3]}) \\ &= \rho(A^{\dagger}) + (D) \quad (\text{By [2]}) \\ &= \rho(A^{\dagger}) + (\widehat{M}|A^{\dagger}) \\ &= (\widehat{M}) \quad (\text{By [3]}) \end{aligned}$$

Thus $\widehat{M}$ is q-k-EP$_r$. Similarly, under the conditions given on M, M can be transformed to its principal pivot transform by pivoting the block K_2D without changing the rank. Hence the Theorem. □

Remark 4.3. *For $k\,(i) = i$, (the identity transposition), Theorem 4.2 reduces to the Theorem 1 of [6].*

Remark 4.4. *In the special case when M is non-singular with A and D non-singular, then the conditions $N\,(A)\,N(C)$ and $N\,(D)\,N(B)$. Automatically hold and by [3], $(M|A)$ and $(M|D)$ are non-singular. Further, $\rho(\widehat{M}) = \rho(A) + \rho(D)$. Hence it follows that each principal pivot transform of M is non-singular. We note that the non-singularity of $\widehat{M}$ need not imply M is non-singular.*

Example 4.5. *Let* $M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}$ *with* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and the associated permutation matrix* $K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ *and* $KM = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. *Here* $K_1A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $(K_1B) = (K_2C)^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $K_2D = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. *Here K_1A and K_2D are non-singular and* $K_2\,(M \mid A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *is EP_1. Therefore, $(M \mid A)$ is q-k_2-EP_1.*

$$\rho(M) = \rho(A) + \rho(M|A) = 3$$

Since KM is symmetric, KM is EP_3 which implies M is q- k EP_3. By (21), $(\widehat{KM}) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ *is non-singular. Thus $\widehat{KM}$ is EP_4 which implies $\widehat{M}$ is q-k EP_4.*

Remark 4.6. *By considering the matrix M in Example 4.5, we note that the conditions $N\,(M|A) \subseteq N(B)$ and $N\,(M|D) \subseteq N(C)$ fail and the statement (iii) of Theorem 4.2 does not hold.*

References

[1] T.S.Baskett and I.J.Katz, *Theorems on products of EP_r matrices*, Lin. Alg. Appl., 2(1969), 87-103.

[2] A.Ben Israel and T.N.E.Greville, *Generalized inverses, Theory and applications*, Wiley and sons, Newyork, (1974).

[3] D.Carlson and E.Haynsworth, *A generalization of the schur complements by means of the moore penrose inverse*, SIAM J. Appl. Math., 26(1974), 169-175.

[4] K.Gunasekaran and K.Gnanabala, *On sums and products of q-k-EP matrices*, IJMER, 6(3)(2016), 54-60.

[5] A.R.Meenakshi and S.Krishnamoorthy, *On k-EP matrices*, Linear Algebra and its Applications, 269(13)(1998), 219232.

[6] A.R.Meenakshi, *Principal pivot transforms of an EP matrix*, C. R. Math. Rep. Acad. Sci. Canada, 8(1986), 121-126.

[7] R.Penrose, *On best approximate solutions of linear matrix equations*, Proc. Cambridge. Phil. Soc., 52(1959), 17-19.

[8] C.R.Rao and S.K.Mitra, *Generalized inverse of matrices and its applications*, Wiley and sons, Newyork, (1971).

[9] H.Schwerdtfeger, *Introduction to Linear Algebra and the theory of matrices*, P.Nooedhoff, Groningen. (1962).