International Journal of Mathematics And its Applications
Volume 4, Issue 4 (2016), 67–80.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Minkowski Inverse for the Range Symmetric Block Matrix with Two Identical Sub-blocks Over Skew Fields in Minkowski Space $\mathscr{M}$

Research Article

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Abstract: Let K be a skew field and $K^{n\times n}$ be the set of all $n \times n$ matrices over K. The purpose of this paper is to give some necessary and sufficient conditions for the existence and the representations of the minkowski inverse of the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under some conditions.

MSC: 15A09, 65F20.

Keywords: Skew fields, Block matrix, Minkowski inverse.

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1. Introduction

Let K be a skew field, C be the complex number field, $K^{m\times n}$ be the set of all $m \times n$ matrices over K, and I_n be the $n \times n$ identity matrix over K. For a matrix $A \in K^{n\times n}$, the matrix $X \in K^{n\times n}$ satisfying $A^k XA = A^K$, $XAX = X$, $(AX)^{\sim} = AX$ and $(XA)^{\sim} = XA$ is called the Minkowski inverse of A and is denoted by $X = A^m$. If A^m exists if and only if $rank(A) = rank(A^2)$ and $R(AA^m) = R(A^m) = R(A)$. we denote $I - AA^m$ by A^{π}. A matrix $A \in C^{n\times n}$ is said to be EP, if $AA^{\dagger} = A^{\dagger}A$. A matrix $A \in C^{n\times n}$ is G-unitary, if $AA^{\sim} = A^{\sim}A = I$. The Minkowski Inverse of block matrices has numerous applications in Game Theory, matrix theory such as singular differential and difference equation. In 1979, S. campbell and C. Meyer proposed an open problem to find an explicit representation for the Drazin inverse of a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the blocks A and D are supposed to be square matrices but their sizes need not be the same. A simplified problem to find an explicit representation of the Drazin (group) inverse for block matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ (A is square, 0 is null matrix) was proposed by S. Campbell in 1983. This open problem was motivated in hoping to find general expressions for the solutions of the second order system of the differential equations

$$Ex^{''}(t) + Fx^{'}(t) + Gx(t) = 0 (t \geq 0),$$

where E is a singular matrix. Detailed discussions of the importance of the problem can be found in [11].

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In this paper, we give the sufficient conditions or the necessary and sufficient conditions for the existance and the representations of the minkowski inverse for block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $(A,\ Bn \in K^{n\times n})$ when A and B satisfy one of the following conditions:

(1) B^m and $(B^\pi A)^m$ exist;

(2) B^m and $(AB^\pi)^m$ exist;

(3) B^m exists and $BAB^\pi = 0$;

(4) B^m exists and $B^\pi AB = 0$.

2. Preliminaries

Lemma 2.1. *Let $A \in K^{n\times n}$ Then A has a Minkowski Inverse if and only if there exist G-unitary matrices $P \in K^{n\times n}$ and $A_1 \in K^{n\times n}$ such that $A = P \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim$ and $A^m = P \begin{pmatrix} A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim$, where $rank(A) = r$.*

Proof. Since $rank(A) = r$, there exists G-unitary matrices $P \in K^{n\times n}$ and $A \in K^{n\times n}$, such that $A = P \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim$,

$X = P \begin{pmatrix} A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim$

(1) $AXA = P \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim.P \begin{pmatrix} A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim.P \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim = P \begin{pmatrix} A_1A_1^mA_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim = P \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim = A$

(2) $XAX = P \begin{pmatrix} A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim.P \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P^\sim.P \begin{pmatrix} A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim = P \begin{pmatrix} A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim = X$

(3) $(AX)^\sim = (XA)^\sim = \begin{pmatrix} G_1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \left(P \begin{pmatrix} A_1A_1^m & 0 \\ 0 & 0 \end{pmatrix} P^\sim\right)^* \begin{pmatrix} G_1^m & 0 \\ 0 & -I \end{pmatrix} = (P^\sim)^* \begin{pmatrix} G_1(A_1A_1^m)G_1 & 0 \\ 0 & 0 \end{pmatrix} P^* = (P^\sim)^* \begin{pmatrix} (A_1A_1^m)^\sim & 0 \\ 0 & 0 \end{pmatrix} P^*$.

Similarly, $(XA)^\sim = (P^\sim)^* \begin{pmatrix} (A_1A_1^m)^\sim & 0 \\ 0 & 0 \end{pmatrix} P^*$. Therefore $(AX)^\sim = (XA)^\sim$. Hence $X = A^m$. □

Lemma 2.2 ([**?**]). *Let $A, G \in K^{n\times n}$, $ind(A) = K$. Then $G = A^D$ if and only if $A^KGA = A^K$, $AG = GA$, $rank(G) \leq rank(A^K)$.*

Lemma 2.3. *Let $\begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$ $S = B^\pi AB^\pi$, $A, B \in X^{n\times n}$.*

(i) B^m and $(B^\pi A)^m$ exist in $\mathscr{M}$ then S^m and M^m exist in $\mathscr{M}$.

(ii) If B^m exist in $\mathscr{M}$ and $BAB^\pi = 0$, then M^m exist if and only if $(AB^\pi)^m$ exist in $\mathscr{M}$.

Proof. Suppose $rank(B) = r$. Applying Lemma 2.1, there exist unitary matrix $P \in C^{n\times n}$ and invertible matrix $B_1 \in k^{r\times r}$, such that $B = p\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} p^{\sim}$ and $B^m = p\begin{pmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} p^{\sim}$. First we can find B^π,

$$\begin{aligned}
B^\pi &= I - BB^m \\
&= \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} - p\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} p^{\sim}.p\begin{pmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} p^{\sim} \\
&= \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} - p\begin{pmatrix} B_1B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} p^{\sim} \\
&= \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} - p\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} p^{\sim} \\
&= \begin{pmatrix} pp^r & 0 \\ 0 & (pp^{\sim})_{n-r} \end{pmatrix} - p\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} p^{\sim} \\
&= p\begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} - p\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} p^{\sim} \\
B^\pi &= p\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim}
\end{aligned}$$

(i) Because $(B^\pi A)^m$ exist in $\mathscr{M}$. We have $rank(B^\pi A) = rank(B^\pi A)^2$. That is

$$\begin{aligned}
B^\pi A &= p\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} p\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} p^{\sim} \\
&= p\begin{pmatrix} 0+0 & 0+0 \\ 0+A_3 & 0+A_4 \end{pmatrix} p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix} p^{\sim} \\
B^\pi A &= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix} p^{\sim}
\end{aligned}$$

Therefore

$$rank(B^\pi A) = rank(A_3A_4) \tag{1}$$

$$\begin{aligned}
(B^\pi A)^2 &= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix} p^{\sim} p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix} p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ A_3A_4 & (A_4)^2 \end{pmatrix} p^{\sim}. \text{ Therefore}
\end{aligned}$$

$$rank(B^\pi A)^2 = rank(A_3A_4\ A_4) \tag{2}$$

Equating (1) and (2), we get

$$\begin{aligned} rank(A_3\ A_4) &= rank(A_3A_4\ A_4^2) \\ &\leq rank(A_4(A_3\ A_4)) \\ &\leq rank(A_4) \text{ and} \\ rank(A_3\ A_4) &\geq rank(A_4) \end{aligned}$$

We have $rank(A_3\ A_4) = rank(A_4)$, so there exist a matrix $x \in k^{(n-r)\times r}$ such that $A_3 = A_4X$. Because order$(A_3) = (n-r)\times r$ and order$(A_4X) = (n-r)\times r$ corresponding entries we also equal. We get rank$(A_4) = rank(A_4^2)$. Therefore A^m exist in $\mathscr{M}$. Nothing that

$$\begin{aligned} S &= B^{\pi}AB^{\pi} \\ &= p\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} p \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} p^{\sim} p \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} \\ &= p\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} \\ &= p\begin{pmatrix} 0+0 & 0+0 \\ 0+A_3 & 0+A_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} \\ &= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} p^{\sim} \\ &= p\begin{pmatrix} 0+0 & 0+0 \\ 0+0 & A_4 \end{pmatrix} p^{\sim} = p\begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} p^{\sim} \\ S &= p\begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} p^{\sim} \end{aligned}$$

Therefore $rank(S) = rank(A_4)$. Similarly, $S^2 = p\begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} p^{\sim} \Rightarrow rank(s^2) = rank(A_4^2)$. Hence $rank(A_4) = rank(A_4^2)$. Which implies $rank(s) = rank(s^2)$. Thus $rank(s^m)$ exist in $\mathscr{M}$. Since $rank(m) = rank\begin{pmatrix} A_1 & A_2 & B_1 & 0 \\ A_3 & A_4 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = rank\begin{pmatrix} 0 & 0 & B_1 & 0 \\ 0 & A_4 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ By using $rank(B_1) = r$. Therefore $rank(B_1) = r$. Also $rank(M) = 2r + rank(A_4)$ and we can find $rank(M^2)$

$$M^2 = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \begin{pmatrix} A^2+B^2 & AB \\ BA & B^2 \end{pmatrix} = rank \begin{pmatrix} A^2+B^2 & AB \\ BA & B^2 \end{pmatrix}$$

$$= rank \begin{pmatrix} A^2 - ABB^mB + B^2 & 0 \\ 0 & B \end{pmatrix}$$

$$= rank \begin{pmatrix} B_1^2 + A_2A_3 & A_2A_4 & 0 & 0 \\ A_4A_3 & A_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By $A_3 = A_4X$, we get,

$$rank(M^2) = \begin{pmatrix} B_1^2 & 0 & 0 & 0 \\ 0 & A_1^2 & 0 & 0 \\ 0 & 0 & B_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank$(M^2) = 2r + rank(A_4^2)$ and $rank(M) = rank(M^2)$. That is M^m exist.

(ii) If $BAB^\pi = 0$, then $A_2 = 0$.Thus $AB^\pi = p \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} p^\sim$

$$rank(M) = rank \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ A_3 & A_4 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = rank \begin{pmatrix} 0 & 0 & B_1 & 0 \\ 0 & A_4 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2r + rank(A_4).$$

Similarly, $rank(M^2) = r \begin{pmatrix} A^2+B^2 & AB \\ BA & B^2 \end{pmatrix} = rank \begin{pmatrix} A^2 - ABB^mBA & 0 \\ 0 & 0 \end{pmatrix} = rank \begin{pmatrix} B_1^2 & 0 & 0 & 0 \\ A_4A_3 & A_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. By $A_3 = A_4X$, we get,

$$rank(M^2) = \begin{pmatrix} B_1^2 & 0 & 0 & 0 \\ 0 & A_1^2 & 0 & 0 \\ 0 & 0 & B_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$rank(M^2) = \begin{pmatrix} B_1^2 & 0 & 0 & 0 \\ 0 & A_1^2 & 0 & 0 \\ 0 & 0 & B_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$rank(M^2) = 2r + rank(A_4^2)$$

As AB^πexist in $\mathscr{M}$, we can get rank(A_4)=rank(A_4^2). Thus rank(M)=rank(M^2). That is M^m exists.

Conversely assume that M^m exist if and only if rank$(M^2) = rank(M)$. $rank(A_4) = rank(A_4^2)$. That is $(AB^\pi)^m$ exist in $\mathscr{M}$. Hence Proved. □

Lemma 2.4. *Let $A, B \in K^{n\times n}$, $S = B^{\pi}AB^{\pi}$ suppose B^m and $(B^{\pi}A)^m$ exists in $\mathscr{M}$ then s^m exist in $\mathscr{M}$ and the following conclusion holds:*

(i) $B^{\pi}As^mA = B^{\pi}A$

(ii) $B^{\pi}AS^m = S^mAB^{\pi}$

(iii) $BS^m = S^mB = B^ms^m = S^mB^m = 0.$

Proof. Suppose rank$(B) = r$, Applying Lemma 2.1 there exist G-unitary matrix $p \in K^{n\times n}$ and $B_1 \in K^{r\times r}$ such that $B = p\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}p^{\sim}$ and $B^m = p\begin{pmatrix} B_1^m & 0 \\ 0 & 0 \end{pmatrix}p^{\sim}$. Let $A = p\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}p^{\sim}$, where $A_1 \in K^{r\times r}, A_2 \in K^{r\times(n-r)}, A_3 \in K^{(n-r)\times r}, A_4 \in K^{(n-r)\times(n-r)}$. From Lemma 2.3 (i), we get S^m exist in $\mathscr{M}$ and $S^m = p\begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}p^{\sim}$

(i)

$$\begin{aligned}
B^{\pi}AS^{\pi}A &= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}p^{\sim}p\begin{pmatrix} 0 & 0 \\ 0 & A_4^m \end{pmatrix}p^{\sim}p\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & A_4^m \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0+0 & 0+0 \\ 0 & A_4A_4^m \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0+0 & 0+0 \\ 0 & A_4A_4^m \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ 0 & A_4A_4^m \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0+0 & 0+0 \\ 0+A_4A_4^mA_3 & A_4A_4^mA_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ 0+A_4A_4^mA_3 & A_4A_4^mA_4 \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}p^{\sim}
\end{aligned}$$

(ii)

$$\begin{aligned}
B^{\pi}AS^m &= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}p^{\sim}p\begin{pmatrix} 0 & 0 \\ 0 & A_4^m \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & A_4^m \end{pmatrix}p^{\sim} \\
&= p\begin{pmatrix} 0+0 & 0+0 \\ 0+0 & A_4A_4^m \end{pmatrix}p^{\sim}.
\end{aligned}$$

$$\begin{aligned}
S^mAB^\pi &= p\begin{pmatrix}0 & 0\\ 0 & A_4^m\end{pmatrix}p^\sim p\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}p^\sim p\begin{pmatrix}0 & 0\\ 0 & I_{n-r}\end{pmatrix}p^\sim\\
&= p\begin{pmatrix}0 & 0\\ 0 & A_4^m\end{pmatrix}\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & I_{n-r}\end{pmatrix}p^\sim\\
&= p\begin{pmatrix}0+0 & 0+0\\ 0+A_4^mA_3 & 0+A_4^mA_4\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & I_{n-r}\end{pmatrix}p^\sim\\
&= p\begin{pmatrix}0 & 0\\ A_4^mA_3 & A_4^mA_4\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & I_{n-r}\end{pmatrix}p^\sim\\
&= p\begin{pmatrix}0+0 & 0+0\\ 0+0 & 0+A_4^mA_4\end{pmatrix}p^\sim\\
S^mAB^\pi &= p\begin{pmatrix}0 & 0\\ 0 & A_4^mA_4\end{pmatrix}p^\sim
\end{aligned}$$

Therefore $B^\pi AS^m = S^mAB^\pi = p\begin{pmatrix}0 & 0\\ 0 & A_4^mA_4\end{pmatrix}p^\sim$.

$$\begin{aligned}
BS^m &= p\begin{pmatrix}B_1 & 0\\ 0 & 0\end{pmatrix}p^\sim p\begin{pmatrix}0 & 0\\ 0 & A_4^m\end{pmatrix}p^\sim\\
&= p\begin{pmatrix}B_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & A_4^m\end{pmatrix}p^\sim\\
&= p\begin{pmatrix}0+0 & 0+0\\ 0+0 & 0+0\end{pmatrix}p^\sim = 0
\end{aligned}$$

Similarly, $S^mB = 0$.

$$B^mS^m = p\begin{pmatrix}B_1^m & 0\\ 0 & 0\end{pmatrix}p^\sim p\begin{pmatrix}0 & 0\\ 0 & A_4^m\end{pmatrix}p^\sim = p\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}p^\sim = 0.$$

Therefore $B^mS^m = 0$. Similarly, we can obtain $S^mB^m = 0$. Therefore $BS^m = S^mB = B^mS^m = S^mB^m$. Hence the Lemma. ☐

3. Main Results

Theorem 3.1. *Let $M = \begin{pmatrix}A & B\\ B & 0\end{pmatrix}$, where $A, B \in K^{n\times n}$. Suppose B^m and $(B^\pi A)^m$ exists in $\mathscr{M}$ then M^m exist in m and $M^m = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}$, where*

$U_{11} = S^m + (S^mA - I)BB^mAB^\pi AS^m - (S^mA - I)BB^mAB^\pi$

$U_{12} = B^m - S^mAB^m$

$U_{21} = B^m - B^mAS^m + B^mA(S^mA - I)BB^mAB^\pi - B^mA(S^mA - I)BB^mAB^\pi AS^m$

$U_{22} = B^m AS^m AB^m - B^m AB^m$

$S = B^\pi AB^\pi.$

Proof. The existence of M^m and S^m in $\mathscr{M}$ have given in Lemma 2.3(i).

Let $X = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, then $MX = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} AU_{11} + BU_{21} & AU_{12} + BU_{22} \\ 0 & 0 \end{pmatrix}$.

We prove $X = M^m$,

$$\begin{aligned}
AU_{11} + BU_{21} &= A(S^m + (S^m A - I)BB^m AB^\pi AS^m - (S^m A - I)BB^m AB^\pi) + B(B^m - B^m AS^m \\
&\quad + B^m A(S^m A - I)BB^m AB^\pi - B^m A(S^m A - I)BB^m AB^\pi AS^m) \\
&= AS^m + A(S^m A - I)BB^m AB^\pi AS^m - A(S^m A - I)BB^m AB^\pi) + BB^m - BB^m AS^m \\
&\quad + BB^m A(S^m A - I)BB^m AB^\pi - BB^m A(S^m A - I)BB^m AB^\pi AS^m) \\
&= BB^m + (I - BB^m)AS^m + (I - BB^m)A(S^m A - I)BB^m AB^\pi AS^m - (I - BB^m)(AS^m A - I)BB^m AB^\pi \\
&= BB^m + B^\pi AS^m + B^\pi A(S^m A - I)BB^m AB^\pi AS^m - B^\pi A(S^m - I)BB^m AB^\pi \\
&= BB^m + B^\pi AS^m + B^\pi ABB^m AB^\pi AS^m - B^\pi ABB^m AB^\pi AS^m - B^\pi ABB^m AB^\pi + B^\pi ABB^m AB^\pi \\
&= B^\pi AS^m + BB^m \quad \text{and} \\
U_{11}A + U_{12}B &= (S^m + (S^m A - I)BB^m AB^\pi AS^m - (S^m A - I)BB^m AB^\pi)A + (B^m - S^m AB^m)B \\
&= S^m A + (S^m A - I)BB^m AB^\pi AS^m A - (S^m A - I)BB^m AB^\pi A + B^m B - S^m AB^m B \\
&= S^m A + S^m ABB^m AB^\pi AS^m A - BB^m AB^\pi AS^m A - S^m ABB^m AB^\pi A + BB^m AB^\pi A + B^m B - S^m AB^m B \\
&= S^m A(I - BB^m) + S^m ABB^m AB^\pi A - BB^m AB^\pi A - S^m ABB^m AB^\pi B + BB^m AB^\pi A + B^m B \\
&= S^m AB^\pi + BB^m \\
&= BB^m + B^\pi AS^m
\end{aligned}$$

Therefore $AU_{11} + BU_{21} = U_{11}A + U_{12}B$

$$\begin{aligned}
AU_{12} + BU_{22} &= A(B^m - S^m AB^m) + B(B^m AS^m AB^m - B^m AB^m) \\
&= AB^m - AS^m AB^m + BB^m AS^m AB^m - BB^m AB^m \\
&= AB^m - B^\pi AS^m AB^m - BB^m AB^m \\
&= AB^m - B^\pi AB^m - BB^m AB^m \\
&= (I - BB^m)AB^m - B^m AB^m \\
&= 0 \\
U_{11}B &= (S^m + (S^m A - I)BB^m AB^\pi AS^m - (S^m A - I)BB^m AB^\pi)B \\
&= S^m B + (S^m A - I)BB^m AB^\pi AS^m B - (S^m A - I)BB^m AB^\pi B \\
&= -(S^m A - I)BB^m A(I - BB^m)B \\
&= -(S^m A - I)(BB^m AB - BB^m ABB^m B \\
&= -(S^m A - I)(BB^m AB - BB^m AB \\
&= 0.
\end{aligned}$$

Therefore $AU_{12}+BU_{22}=U_{11}B$. Similarly, we can get,

$$BU_{11}=U_{21}A+U_{22}B=BB^mAB^\pi(I-AS^m)$$
$$BU_{12}=U_{21}B=BB^m.$$

Consequently,

$$\begin{aligned}(MX)^\sim &= \begin{pmatrix}G_1 & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}BB^m+B^\pi AS^m & 0\\ BB^mAB^\pi(I-AS^m) & BB^m\end{pmatrix}^*\begin{pmatrix}G_1 & 0\\ 0 & -I\end{pmatrix}\\ &= \begin{pmatrix}G_1 & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}(BB^m+B^\pi AS^m)^* & (BB^mAB^\pi(I-AS^m))^*\\ 0 & (BB^m)^*\end{pmatrix}\begin{pmatrix}G_1 & 0\\ 0 & -I\end{pmatrix}\\ &= \begin{pmatrix}G_1(BB^m+B^\pi AS^m)^* & G_1(BB^mAB^\pi(I-AS^m))^*\\ 0 & -(BB^m)^*\end{pmatrix}\begin{pmatrix}G_1 & 0\\ 0 & -I\end{pmatrix}\\ &= \begin{pmatrix}G_1(BB^m+B^\pi AS^m)^*G_1 & -G_1(BB^mAB^\pi(I-AS^m))^*\\ 0 & (BB^m)^*\end{pmatrix}\end{aligned}$$

Therefore $(MX)^\sim = \begin{pmatrix}(BB^m+B^\pi AS^m)^\sim & -G_1(BB^mAB^\pi(I-AS^m))^*\\ 0 & (BB^m)^*\end{pmatrix}$. Similarly,

$$\begin{aligned}(XM)^\sim &= \begin{pmatrix}(BB^m+B^\pi AS^m)^\sim & -G_1(BB^mAB^\pi(I-AS^m))^*\\ 0 & (BB^m)^*\end{pmatrix}\\ MXM &= \begin{pmatrix}BB^m+B^\pi AS^m & 0\\ BB^mAB^\pi(I-AS^m) & BB^m\end{pmatrix}\begin{pmatrix}A & B\\ B & 0\end{pmatrix}\\ &= \begin{pmatrix}BB^mA+B^\pi AS^mA & BB^mB+B^\pi AS^mB\\ BB^mAB^\pi(I-AS^m)A+BB^mB & BB^mAB^\pi(I-AS^m)B\end{pmatrix}\\ &= \begin{pmatrix}BB^mA+(I-BB^m)AS^mA & B\\ BB^mAB^\pi A-BB^mAB^\pi AS^mA+B & BB^mAB^\pi B-BB^mAB^\pi AS^m)B\end{pmatrix}\\ &= \begin{pmatrix}BB^mA+A-BB^m)A & B\\ BB^mAB^\pi A-BB^mAB^\pi AS^mA+B & BB^mAB^\pi B-BB^mAB^\pi AS^mB\end{pmatrix}\\ &= \begin{pmatrix}BB^mA+A-BB^m)A & B\\ B & BB^mAB-BB^mABB^mB\end{pmatrix}\\ &= \begin{pmatrix}A & B\\ B & 0\end{pmatrix}.\end{aligned}$$

Therefore $MXM=M$. Suppose $rank(B)=r$. By Lemma 2.1, there exist G-unitary matrices $P\in K^{n\times n}$ and invertible matrices $B=P\begin{pmatrix}B_1 & 0\\ 0 & 0\end{pmatrix}P^\sim$ and $B^m=P\begin{pmatrix}B_1^m & 0\\ 0 & 0\end{pmatrix}P^\sim$. Let $A=P\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}P^\sim$, where $A_1\in K^{r\times r},\ A_2\in K^{r\times(n-r)},\ A_3\in$

$K^{(n-r)\times r}$, $A_4 \in K^{(n-r)\times(n-r)}$. Since $X = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$

$$\begin{aligned} rank(X) &= rank \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \\ &= rank \begin{pmatrix} S^m + (S^m A - I)BB^m AB^\pi AS^m - (S^m A - I)BB^m AB^\pi & B^m - S^m AB^m \\ B^m & 0 \end{pmatrix} \\ &= rank \begin{pmatrix} S^m & B^m - S^m AB^m \\ B^m & 0 \end{pmatrix} \\ &= rank \begin{pmatrix} 0 & 0 & B_1^{-1} & 0 \\ 0 & A_4^m & 0 & 0 \\ B_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= 2r + rank(A_4^m) \\ rank(X) &= 2r + rank(A_4^m) \\ &= 2r + rank(A_4) \\ &= rank(M) \end{aligned}$$

From Lemma 2.3, we get $X = M^m$. □

Theorem 3.2. *Let* $M = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$, *where* $A, B \in K^{n\times n}$, *suppose* B^m *and* $(AB^\pi)^m$ *exists in* $\mathscr{M}$ *then* M^m *exist in* $\mathscr{M}$ *and* $M^m = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, *where*

$$\begin{aligned} U_{11} &= S^m + S^m AB^\pi ABB^m (AS^{m-I}) - B^\pi ABB^m (AS^{m-I}) \\ U_{12} &= B^m - S^m AB^m - S^m AB^\pi - S^m AB^\pi ABB^m (AS^m - I)AB^m + B^\pi ABB^m (I - AS^m)AB^m \\ U_{21} &= B^m - B^m AS^m \\ U_{22} &= B^m AS^m AB^m - B^m AB^m \\ S &= B^\pi AB^\pi. \end{aligned}$$

Theorem 3.3. *Let* $M = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$, *where* $A, B \in K^{n\times n}$, *suppose* B^m *exists in* $\mathscr{M}$ *and* $BAB^\pi = 0$ *then*

(i). M^m *exists in* $\mathscr{M}$ *iff* $(AB^\pi)^m$ *exists in* $\mathscr{M}$.

(ii). If M^m *exists in* $\mathscr{M}$, *then* $M^m = \begin{pmatrix} U & V \\ B^m & -B^m AB^m \end{pmatrix}$, *where*

$$\begin{aligned} U &= B^\pi A(B^m)^2 - (AB^\pi)^m AB^\pi A(B^m)^2 + (AB^\pi)^m \\ V &= -B^\pi A(B^m)^2 AB^m + (AB^\pi)^m AB^\pi A(B^m)^2 AB^m - (AB^\pi)^m AB^m + B^m. \end{aligned}$$

Proof.

(i) The existence of M^m has been given in Lemma 2.3 (ii)

(ii) Let $X = \begin{pmatrix} U & V \\ B^m & -B^mAB^m \end{pmatrix}$ then

$$MX = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} \begin{pmatrix} U & V \\ B^m & -B^mAB^m \end{pmatrix} = \begin{pmatrix} AU + BB^m & AV - BB^mAB^m \\ BU & BV \end{pmatrix}$$

$$XM = \begin{pmatrix} U & V \\ B^m & -B^mAB^m \end{pmatrix} \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \begin{pmatrix} UA + VB & UB \\ B^mAB^\pi & B^mB \end{pmatrix}$$

$$\begin{aligned}
AU + BB^m &= AB^\pi A(B^m)^2 - A(AB^\pi)^m AB^\pi A(B^m)^2 + A(AB^\pi)^m + BB^m \\
&= AB^\pi A(B^m)^2 - AB^\pi A(B^m)^2 + A(AB^\pi)^m + BB^m \\
&= A(AB^\pi)^m + BB^m \\
UA + VB &= (B^\pi A(B^m)^2 - (AB^\pi)^m AB^\pi A(B^m)^2 + (AB^\pi)^m)A + (-B^\pi A(B^m)^2 AB^m \\
&\quad + (AB^\pi)^m AB^\pi A(B^m)^2 AB^m - (AB^\pi)^m AB^m + B^m)B \\
&= B^\pi A(B^m)^2 A - (AB^\pi)^m AB^\pi A(B^m)^2 A + (AB^\pi)^m A - B^\pi A(B^m)^2 AB^m B \\
&\quad - (AB^\pi)^m AB^\pi A(B^m)^2 AB^m B - (AB^\pi)^m AB^m B + B^m B \\
&= B^\pi A(B^m)^2 A(I - BB^m) - (AB^\pi)^m AB^\pi A(B^m)^2 A(I - BB^m) + (AB^\pi)^m A(I - BB^m) + BB^m \\
&= B^\pi A(B^m)^2 AB^\pi - (AB^\pi)^m AB^\pi A(B^m)^2 AB^\pi + (AB^\pi)^m AB^\pi + BB^m \\
&= (AB^\pi)^m AB^\pi + BB^m \\
&= A(AB^\pi)^m + BB^m.
\end{aligned}$$

Therefore $AU + BB^m = UA + VB$.

$$\begin{aligned}
AV - BB^mAB^m &= A(-B^\pi A(B^m)^2 AB^m + (AB^\pi)^m AB^\pi A(B^m)^2 AB^m - (AB^\pi)^m AB^m + B^m) - BB^m AB^m \\
&= -AB^\pi A(B^m)^2 AB^m + A(AB^\pi)^m AB^\pi A(B^m)^2 AB^m - A(AB^\pi)^m AB^m + AB^m - BB^m AB^m \\
&= -AB^\pi A(B^m)^2 AB^m + AB^\pi A(B^m)^2 AB^m - A(AB^\pi)^m AB^m - (I - BB^m)AB^m \\
&= -A(AB^\pi)^m AB^m + B^m AB^m \\
UB &= [B^\pi A(B^m)^2 - (AB^\pi)^m AB^\pi A(B^m)^2 + (AB^\pi)^m]B \\
&= B^\pi A(B^m)^2 B - (AB^\pi)^m AB^\pi A(B^m)^2 B + (AB^\pi)^m B \\
&= B^\pi AB^m B - (AB^\pi)^m AB^\pi AB^m B + (AB^\pi)^m B \\
&= B^\pi AB^m BB^m - (AB^\pi)^m AB^\pi AB^m BB^m + (AB^\pi)^m B \\
&= B^\pi AB^m - (AB^\pi)^m AB^\pi AB^m + (AB^\pi)^m B \\
&= B^\pi AB^m - A(AB^\pi)^m AB^m + (AB^\pi)^m B \\
&= -A(AB^\pi)^m AB^m + B^\pi AB^m.
\end{aligned}$$

Hence $AV - BB^m AB^m = UB$.

$$\begin{aligned} BU &= B[B^\pi A(B^m)^2 - (AB^\pi)^m AB^\pi A(B^m)^2 + (AB^\pi)^m] \\ &= BB^\pi A(B^m)^2 - B(AB^\pi)^m AB^\pi A(B^m)^2 + B(AB^\pi)^m \\ &= B(I - B^m)A(B^m)^2 \\ &= BA(B^m)^2 - BB^m A(B^m)^2 \\ &= BA(B^m)^2 - BB^m BA(B^m)^2 \\ &= BA(B^m)^2 - BA(B^m)^2 \\ &= 0. \\ B^m AB^\pi &= 0. \end{aligned}$$

Hence $BU = B^m AB^\pi$.

$$\begin{aligned} BV &= B[-B^\pi A(B^m)^2 AB^m + (AB^\pi)^m AB^\pi A(B^m)^2 AB^m - (AB^\pi)^m AB^m + B^m] \\ &= -BB^\pi A(B^m)^2 AB^m + B(AB^\pi)^m AB^\pi A(B^m)^2 AB^m - B(AB^\pi)^m AB^m + BB^m \\ &= -B(I - BB^m)A(B^m)^2 AB^m + BB^m \\ &= -BA(B^m)^2 AB^m + BBB^m A(B^m)^2 AB^m + BB^m \\ &= -BA(B^m)^2 AB^m + BB^m BA(B^m)^2 AB^m + B^m B \\ &= -BA(B^m)^2 AB^m + BA(B^m)^2 AB^m + B^m B \\ &= B^m B. \end{aligned}$$

Hence $BV = B^m B$. Consequently,

$$\begin{aligned} (MX)^\sim &= \begin{pmatrix} G_1 & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} A(AB^\pi)^m + BB^m & -A(AB^\pi)^m AB^m + B^\pi AB^m \\ 0 & B^m B \end{pmatrix}^* \begin{pmatrix} G_1 & 0 \\ 0 & -I \end{pmatrix} \\ &= \begin{pmatrix} G_1(A(AB^\pi)^m + BB^m)^* G_1 & 0 \\ (A(AB^\pi)^m AB^m + B^\pi AB^m)^* G_1 & (B^m B)^* \end{pmatrix} \\ &= \begin{pmatrix} (A(AB^\pi)^m + BB^m)^\sim & 0 \\ (A(AB^\pi)^m AB^m + B^\pi AB^m)^* G_1 & (B^m B)^* \end{pmatrix}. \end{aligned}$$

Similarly, $(XM)^\sim = \begin{pmatrix} (A(AB^\pi)^m + BB^m)^\sim & 0 \\ (A(AB^\pi)^m AB^m + B^\pi AB^m)^* G_1 & (B^m B)^* \end{pmatrix}$.

$$\begin{aligned} MXM &= \begin{pmatrix} A(AB^\pi)^m + BB^m & -A(AB^\pi)^m AB^m + B^\pi AB^m \\ 0 & B^m B \end{pmatrix} \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} \\ &= \begin{pmatrix} A(AB^\pi)^m A + BB^m A - A(AB^\pi)^m AB^m B + B^\pi AB^m B & A(AB^\pi)^m B + BB^m B \\ BB^m B & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}. \end{aligned}$$

Therefore $MXM = M$. Suppose $rank(B) = r$. By Lemma 2.1, there exist G-unitary matrices $P \in K^{n\times n}$ and invertible matrices $B_1 \in K^{r\times r}$ such that $B = P \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} P^{\sim}$ and $B^m = P \begin{pmatrix} B_1^m & 0 \\ 0 & 0 \end{pmatrix} P$. Let $A = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P$, where $A_1 \in K^{r\times r}$, $A_2 \in K^{r\times(n-r)}$, $A_3 \in K^{(n-r)\times r}$, $A_4 \in K^{(n-r)\times(n-r)}$. Then

$$\begin{aligned}
rank(X) &= rank \begin{pmatrix} U & V \\ B^m & -B^m AB^m \end{pmatrix} \\
&= rank \begin{pmatrix} U & B^m \\ B^m & 0 \end{pmatrix} \\
&= rank \begin{pmatrix} (AB^{\pi})^m & B^m \\ B^m & 0 \end{pmatrix} \\
&= rank \begin{pmatrix} 0 & 0 & B_1^{-1} & 0 \\ 0 & A_4^m & 0 & 0 \\ B_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= 2r + rank(A_4^m).
\end{aligned}$$

Since $rank(M) = 2r + rank(A_4) = rank(r)$. Then $X = M^m$. □

Theorem 3.4. *Let* $M = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$, *where* $A, B \in K^{n\times n}$, *suppose* B^m *exists in* $\mathscr{M}$ *and* $B^{\pi}AB = 0$ *then*

(i). M^m *exists in* $\mathscr{M}$ *iff* $(B^{\pi}A)^m$ *exists in* $\mathscr{M}$.

(ii). If M^m *exists in* $\mathscr{M}$, *then* $M^m = \begin{pmatrix} U & B^m \\ V & -B^m AB^m \end{pmatrix}$, *where*

$$\begin{aligned}
U &= (B^m)^2 AB^{\pi} - (B^m)^2 AB^{\pi} A(B^{\pi}A)^m + (B^{\pi}A)^m, \\
V &= -B^m B^m B^{\pi} + B^m A(B^m)^2 AB^{\pi} A(B^{\pi}A)^m - B^m A(B^{\pi}A)^m + B^m.
\end{aligned}$$

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