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Common Fixed Point Theorem in Rational Inequality and Their Application

Research Article

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Abstract: In this paper we prove some sufficient condition for the existence and uniqueness of fixed point and common fixed point in rational inequality on complete metric spaces. As application, Some existence and uniqueness results of solution and common solution for some functional equations and system of functional equations in Dynamic programming are given by using the fixed point and common fixed point theorems.

Keywords: Common fixed point, complete metric space, common solution, functional equation, system of functional equations, dynamic programming.

1. Introduction

First Bellman [6] studied the existence of solutions for some classes of functional equation arising in Dynamic programming. Bellman and Lee [7] gives the basic form of functional equations in dynamic programming as follows:

$$f(x) = opt_{y\in D}\ H\{x, y, f(T(x,y))\}, \quad \forall\ x\in S \tag{1}$$

Where *opt* represents *sup* or inf, x and y denotes the state and decision vectors respectively, T stands for the transformation of the process and $f(x)$ represents the optimal return function with the initial state x. After Bellman, many researcher Baskaran and Subrahmanyam [8], Bhakta and Choudhury [4], Bhakta and Mitra [5], Chang and Ma [9], Liu [11–14], Liu, Agarwal, and Kang [15], Liu and Ume [19], Pathak and Fisher [3], Zhang [10] investigate the existence and uniqueness of solution and common solution for some kinds of functional equations and systems of functional equations. Ray [1] established two common fixed point theorems for the following self mapping f, g, and h in a complete metric space (X, d):

$$d(fx, gy) \le d(hx, hy) - w(d(hx, hy)), \quad \forall\ x, y\in X$$

Liu [11], introduced and studied a class contractive type mapping as:

$$d(fx, gy) \le \max\{d(hx, hy), d(hx, fx), d(hy, gy)\} - w(\max\{d(hx, hy), d(hx, fx), d(hy, gy)\}), \quad \forall\ x, y\in X$$

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Z. Liu, L. Wang, H .K. Kim and S. M. Kang [22], proved some common fixed point theorems for contractive type mappings for three self mapping as follows

$$d(fx,gy) \leq \max\left\{d\left(hx,hy\right),d\left(hx,fx\right),d\left(hy,gy\right),\frac{1}{2}\left[d\left(hx,hy\right)+d\left(fx,gy\right)\right],\frac{d\left(hx,fx\right)d\left(hy,gy\right)}{1+d\left(fx,gy\right)},\frac{d\left(hx,fx\right)d\left(hy,gy\right)}{1+d\left(hx,hy\right)}\right\}$$
$$-\,\omega\left(\max\left\{d\left(hx,hy\right),d\left(hx,fx\right),d\left(hy,gy\right),\frac{1}{2}\left[d\left(hx,hy\right)+d\left(fx,gy\right)\right],\frac{d\left(hx,fx\right)d\left(hy,gy\right)}{1+d\left(fx,gy\right)},\frac{d\left(hx,fx\right)d\left(hy,gy\right)}{1+d\left(hx,hy\right)}\right\}\right)$$

$\forall\ x,y\in X$. In this paper we give some sufficient conditions for existence and uniqueness of common fixed point in rational inequality on complete metric space (X,d):

$$d\left(fx,gy\right) \leq \max\left\{\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,fx\right)+d\left(hx,hy\right)},\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d\left(hy,gy\right)}{d\left(fx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d(hy,gy)}{d\left(fx,hy\right)+d(hx,fx)},\frac{d\left(fx,hx\right).d(fx,hy)}{d\left(fx,hy\right)+d(hx,hy)}\right\}$$
$$-\,\omega\left(\max\left\{\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,fx\right)+d\left(hx,hy\right)},\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d\left(hy,gy\right)}{d\left(fx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d(hy,gy)}{d\left(fx,hy\right)+d(hx,fx)},\frac{d\left(fx,hx\right).d(fx,hy)}{d\left(fx,hy\right)+d(hx,hy)}\right\}\right),\ \forall\ x,y\in X$$

As applications we use the fixed point theorems for the following functional equation and system of functional equations, in the dynamic programming:

$$f(x)=opt_{y\in D}\{u\left(x,y\right)+H\left(x,y,f\left(T\left(x,y\right)\right)\right\},\ \ \forall\ x{\in}S \tag{2}$$

And

$$f_i\left(x\right)=opt_{y\in D}\{u\left(x,y\right)+H_i\left(x,y,f_i\left(T\left(x,y\right)\right)\right\},\quad \forall\ x{\in}S, i=\{1,2,3\} \tag{3}$$

Here we assume $W=\{\ w|\ w:R^{+}{\to}R^{+}$ is a continuous mapping with $0<w\left(t\right)<t$ for all $t>0\}$. $R^{+}=[0,\infty)$, ω denotes the sets of all positive integer, N= set of all nonnegative integer. And define $C_f\left(X\right)=\{\ g|\ \ g:X{\to}X$ is a continuous and $fg=gf\}$, where f is a self mapping in (X,d). Let I denotes the identity mapping in X.

2. Main Result

In this paper we prove some fixed point and common fixed point theorems for some classes of contractive type mapping in a complete metric space (X,d), for self mapping f,g and h in (X,d) and $x_0{\in}X$ put $d_n{=}\ d(hx_n,hx_{n+1})$ for all $n{\in}\omega$. Our main result are as follows:

Theorem 2.1. *Let (X,d) be a complete metric space and f,g and h be three self mapping in X with $h{\in}C_f\left(X\right){\cap}C_g\left(X\right)$ and $f\left(X\right){\cup}g\left(X\right){\subseteq}h\left(X\right)$. If there exists a $\omega{\in}W$ satisfying*

$$d\left(fx,gy\right) \leq \max\left\{\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,fx\right)+d\left(hx,hy\right)},\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d\left(hy,gy\right)}{d\left(fx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d(hy,gy)}{d\left(fx,hy\right)+d(hx,fx)},\frac{d\left(fx,hx\right).d(fx,hy)}{d\left(fx,hy\right)+d(hx,hy)}\right\}$$
$$-\,\omega\left(\max\left\{\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,fx\right)+d\left(hx,hy\right)},\frac{d\left(hx,gy\right).d\left(hy,fx\right)}{d\left(hx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d\left(hy,gy\right)}{d\left(fx,hy\right)+d\left(fx,gy\right)},\frac{d\left(hx,fx\right).d(hy,gy)}{d\left(fx,hy\right)+d(hx,fx)},\frac{d\left(fx,hx\right).d(fx,hy)}{d\left(fx,hy\right)+d(hx,hy)}\right\}\right) \tag{4}$$

Then f,g and h have a unique common fixed point in X.

Proof. Let x_0 be any point in X, According to $f(X) \cup g(X) \subseteq h(X)$, we choose a sequence $\{x_n\}_{n\in\omega} \in X$ such that $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$ for any $n \in \omega$. By equation (4) we get,

$$\begin{aligned}
d(fx_{2n}, gx_{2n+1}) \leq \max\Bigg\{ & \frac{(d(hx_{2n}, gx_{2n+1}).d(hx_{2n+1}, fx_{2n}))}{(d(hx_{2n}, fx_{2n}) + d(hx_{2n}, hx_{2n+1}))}, \frac{(d(hx_{2n}, gx_{2n+1}).d(hx_{2n+1}, fx_{2n}))}{(d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1}))}, \\
& \frac{(d(hx_{2n}, fx_{2n}).d(hx_{2n+1}, gx_{2n+1}))}{(d(fx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1}))}, \frac{(d(hx_{2n}, fx_{2n}).d(hx_{2n+1}, gx_{2n+1}))}{(d(fx_{2n}, hx_{2n+1}) + d(hx_{2n}, fx_{2n}))}, \\
& \frac{(d(fx_{2n}, hx_{2n}).d(fx_{2n}, hx_{2n+1}))}{(d(fx_{2n}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))} \Bigg\} \\
- \omega \Bigg(\max\Bigg\{ & \frac{(d(hx_{2n}, gx_{2n+1}).d(hx_{2n+1}, fx_{2n}))}{(d(hx_{2n}, fx_{2n}) + d(hx_{2n}, hx_{2n+1}))}, \frac{(d(hx_{2n}, gx_{2n+1}).d(hx_{2n+1}, fx_{2n}))}{(d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1}))}, \\
& \frac{(d(hx_{2n}, fx_{2n}).d(hx_{2n+1}, gx_{2n+1}))}{(d(fx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1}))}, \frac{(d(hx_{2n}, fx_{2n}).d(hx_{2n+1}, gx_{2n+1}))}{(d(fx_{2n}, hx_{2n+1}) + d(hx_{2n}, fx_{2n}))}, \\
& \frac{(d(fx_{2n}, hx_{2n}).d(fx_{2n}, hx_{2n+1}))}{(d(fx_{2n}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))} \Bigg\} \Bigg) \\
d(hx_{2n+1}, hx_{2n+2}) \leq \max\Bigg\{ & \frac{(d(hx_{2n}, hx_{2n+2}).d(hx_{2n+1}, hx_{2n+1}))}{(d(hx_{2n}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))}, \frac{(d(hx_{2n}, hx_{2n+2}).d(hx_{2n+1}, hx_{2n+1}))}{(d(hx_{2n}, hx_{2n+1}) + d(hx_{2n+1}, hx_{2n+2}))}, \\
& \frac{(d(hx_{2n}, hx_{2n+1}).d(hx_{2n+1}, hx_{2n+2}))}{(d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n+1}, hx_{2n+2}))}, \frac{(d(hx_{2n}, hx_{2n+1}).d(hx_{2n+1}, hx_{2n+2}))}{(d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))}, \\
& \frac{(d(hx_{2n+1}, hx_{2n}).d(hx_{2n+1}, hx_{2n+1}))}{(d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))} \Bigg\} \\
- \omega \Bigg(\max\Bigg\{ & \frac{(d(hx_{2n}, hx_{2n+2}).d(hx_{2n+1}, hx_{2n+1}))}{(d(hx_{2n}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))}, \frac{(d(hx_{2n}, hx_{2n+2}).d(hx_{2n+1}, hx_{2n+1}))}{(d(hx_{2n}, hx_{2n+1}) + d(hx_{2n+1}, hx_{2n+2}))}, \\
& \frac{(d(hx_{2n}, hx_{2n+1}).d(hx_{2n+1}, hx_{2n+2}))}{(d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n+1}, hx_{2n+2}))}, \frac{(d(hx_{2n}, hx_{2n+1}).d(hx_{2n+1}, hx_{2n+2}))}{(d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))}, \\
& \frac{(d(hx_{2n+1}, hx_{2n}).d(hx_{2n+1}, hx_{2n+1}))}{(d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n}, hx_{2n+1}))} \Bigg\} \Bigg) \\
d_{2n+1} \leq \max\left\{0, 0, \frac{(d_{2n}d_{2n+1})}{d_{2n+1}}, \frac{(d_{2n}d_{2n+1})}{d_{2n}, 0}\right\} & - \omega\left(\max\left\{0, 0, \frac{(d_{2n}d_{2n+1})}{d_{2n+1}}, \frac{(d_{2n}d_{2n+1})}{d_{2n}}, 0\right\}\right)
\end{aligned}$$

$$d_{2n+1} \leq \max\{d_{2n}, d_{2n+1}\} - \omega(max\{d_{2n}, d_{2n+1}\}) \tag{5}$$

Suppose that $d_{2n+1} > d_{2n}$ for some $n \in \omega$. Then from Equation (5), $d_{2n+1} \leq d_{2n+1} - \omega(d_{2n+1}) < d_{2n+1}$, a contradiction. Hence we have $d_{2n+1} \leq d_{2n}$. And so $d_{2n+1} \leq d_{2n} - \omega(d_{2n})$, for any $n \in \omega$. Consequently we get, $d_{2n} \leq d_{2n-1} - \omega(d_{2n-1})$ for all $n \in N$. It follows that,

$$d_n \leq d_{n-1} - \omega(d_{n-1}), \text{ for all } n \in N. \tag{6}$$

Next we prove that

$$\lim_{n\to\infty} d_n = 0 \tag{7}$$

also Equation (6) written as, $\sum_{i=0}^{n} \omega(d_i) \leq d_0 - d_{n+1} \leq d_0$ for all $n \in \omega$ and $\{d_n\}_{n\in\omega}$ is a decreasing sequence. Whereas the series $\sum_{n=0}^{\infty} \omega(d_n)$ and the sequence $\{d_n\}_{n\in\omega}$ are convergent. It is clear that $\lim_{n\to\infty} \omega(d_n) = 0$ and there exist some point $p \in R^+$ such that $\lim_{n\to\infty} d_n = p$. In terms of the continuity of ω, we derive that $\lim_{n\to\infty} \omega(d_n) = \omega(p) = 0$. This means that $p = 0$, that is Equation (7), holds.

Now next we show that $\{hx_n\}_{n\in\omega}$ is a Cauchy sequence, for this we prove that $\{hx_{2n}\}_{n\in\omega}$ is a Cauchy sequence. suppose that $\{hx_{2n}\}_{n\in\omega}$ is not a Cauchy sequence. Thus there exist some $\epsilon > 0$ such that for any even integer $2k$. There are even integer $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon$. Further, let $2m(k)$ denote the least even integer exceeding $2n(k)$ which satisfies that $2m(k) > 2n(k) > 2k$,

$$d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \epsilon \text{ and } d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon \tag{8}$$

Notice that for any $k \in N$

$$d\left(hx_{2m(k)}, hx_{2n(k)}\right) \leq d_{2m(k)-1} + d_{2m(k)-2} + d\left(hx_{2m(k)-2}, hx_{2n(k)}\right)$$

$$\left|d\left(hx_{2m(k)}, hx_{2n(k)+1}\right) - d\left(hx_{2m(k)}, hx_{2n(k)}\right)\right| \leq d_{2n(k)}$$

$$\left|d\left(hx_{2m(k)+1}, hx_{2n(k)+1}\right) - d\left(hx_{2m(k)}, hx_{2n(k)+1}\right)\right| \leq d_{2m(k)}$$

$$\left|d\left(hx_{2m(k)+1}, hx_{2n(k)+2}\right) - d\left(hx_{2m(k)+1}, hx_{2n(k)+2}\right)\right| \leq d_{2n(k)+1}$$

Following (7), (8) and the above inequality, we infer that

$$\begin{aligned} \epsilon &= \lim_{k\to\infty} d\left(hx_{2m(k)}, hx_{2n(k)}\right) \\ &= \lim_{k\to\infty} d(hx_{2m(k)}, hx_{2n(k)+1}) \\ &= \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) \\ &= \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) \end{aligned} \tag{9}$$

Using Equation (4), we have $\forall\ \ k \in N$,

$$\begin{aligned} d\left(fx_{2m(k)}, gx_{2n(k)+1}\right) \leq \max\Bigg\{ &\frac{d\left(hx_{2m(k)}, gx_{2n(k)+1}\right).d\left(hx_{2n(k)+1}, fx_{2m(k)}\right)}{d\left(hx_{2m(k)}, fx_{2m(k)}\right) + d\left(hx_{2m(k)}, hx_{2n(k)+1}\right)}, \\ &\frac{d\left(hx_{2m(k)}, gx_{2n(k)+1}\right).d\left(hx_{2n(k)+1}, fx_{2m(k)}\right)}{d\left(hx_{2m(k)}, hx_{2n(k)+1}\right) + d\left(fx_{2m(k)}, gx_{2n(k)+1}\right)}, \frac{d\left(hx_{2m(k)}, fx_{2m(k)}\right).d\left(hx_{2n(k)+1}, gx_{2n(k)+1}\right)}{d\left(fx_{2m(k)}, hx_{2n(k)+1}\right) + d\left(fx_{2m(k)}, gx_{2n(k)+1}\right)}, \\ &\frac{d\left(hx_{2m(k)}, fx_{2m(k)}\right).d\left(hx_{2n(k)+1}, gx_{2n(k)+1}\right)}{d\left(fx_{2m(k)}, hx_{2n(k)+1}\right) + d\left(hx_{2m(k)}, fx_{2m(k)}\right)}, \frac{d\left(fx_{2m(k)}, hx_{2m(k)}\right).d(fx_{2m(k)}, hx_{2n(k)+1})}{d\left(fx_{2m(k)}, hx_{2n(k)+1}\right) + d(hx_{2m(k)}, hx_{2n(k)+1})} \Bigg\} \\ - \omega\Bigg(\max\Bigg\{ &\frac{d\left(hx_{2m(k)}, gx_{2n(k)+1}\right).d\left(hx_{2n(k)+1}, fx_{2m(k)}\right)}{d\left(hx_{2m(k)}, fx_{2m(k)}\right) + d\left(hx_{2m(k)}, hx_{2n(k)+1}\right)}, \\ &\frac{d\left(hx_{2m(k)}, gx_{2n(k)+1}\right).d\left(hx_{2n(k)+1}, fx_{2m(k)}\right)}{d\left(hx_{2m(k)}, hx_{2n(k)+1}\right) + d\left(fx_{2m(k)}, gx_{2n(k)+1}\right)}, \frac{d\left(hx_{2m(k)}, fx_{2m(k)}\right).d\left(hx_{2n(k)+1}, gx_{2n(k)+1}\right)}{d\left(fx_{2m(k)}, hx_{2n(k)+1}\right) + d\left(fx_{2m(k)}, gx_{2n(k)+1}\right)}, \\ &\frac{d\left(hx_{2m(k)}, fx_{2m(k)}\right).d\left(hx_{2n(k)+1}, gx_{2n(k)+1}\right)}{d\left(fx_{2m(k)}, hx_{2n(k)+1}\right) + d\left(hx_{2m(k)}, fx_{2m(k)}\right)}, \frac{d\left(fx_{2m(k)}, hx_{2m(k)}\right).d(fx_{2m(k)}, hx_{2n(k)+1})}{d\left(fx_{2m(k)}, hx_{2n(k)+1}\right) + d(hx_{2m(k)}, hx_{2n(k)+1})} \Bigg\}\Bigg) \end{aligned}$$

Letting $k \to \infty$, by Equation (9) we get,

$$\begin{aligned} \epsilon \leq \max&\left\{\frac{\epsilon.\epsilon}{d_{2m(k)}+\epsilon}, \frac{\epsilon.\epsilon}{\epsilon+\epsilon}, \frac{d_{2m(k)}.d_{2n(k)+1}}{\epsilon+\epsilon}, \frac{d_{2m(k)}d_{2n(k)+1}}{\epsilon+d_{2m(k)}}, \frac{d_{2m(k)}.\epsilon}{\epsilon+\epsilon}\right\} \\ - \omega&\left(\max\left\{\frac{\epsilon.\epsilon}{d_{2m(k)}+\epsilon}, \frac{\epsilon.\epsilon}{\epsilon+\epsilon}, \frac{d_{2m(k)}.d_{2n(k)+1}}{\epsilon+\epsilon}, \frac{d_{2m(k)}d_{2n(k)+1}}{\epsilon+d_{2m(k)}}, \frac{d_{2m(k)}.\epsilon}{\epsilon+\epsilon}\right\}\right) \end{aligned}$$

Letting $k \to \infty$, by Equation (9) we get,

$$\begin{aligned} \epsilon &\leq \max\left\{\epsilon, \frac{\epsilon}{2}, 0, 0, 0\right\} - \omega\left(\max\left\{\epsilon, \frac{\epsilon}{2}, 0, 0, 0\right\}\right) \\ &= \epsilon - \omega(\epsilon) < \epsilon \end{aligned}$$

Which is a contradiction, hence $\{hx_n\}_{n\in\omega}$ is a Cauchy sequence. It follows from completeness of (X, d) that $\{hx_n\}_{n\in\omega}$ converges to a point $u \in X$. Since $h \in C_f(X) \cap C_g(X)$. We infer that

$$\begin{aligned} hu &= \lim_{n\to\infty} fhx_{2n} = \lim_{n\to\infty} hfx_{2n} = \lim_{n\to\infty} hhx_{2n+1} \\ &= \lim_{n\to\infty} ghx_{2n+1} = \lim_{n\to\infty} hgx_{2n+1} = \lim_{n\to\infty} hhx_{2n+2} \end{aligned} \tag{10}$$

From Equation (4), we get

$$\begin{aligned} d\left(fu, ghx_{2n+1}\right) \leq \max & \left\{\frac{d\left(hu, ghx_{2n+1}\right).d\left(hhx_{2n+1}, fu\right)}{d\left(hu, fu\right)+d\left(hu, hhx_{2n+1}\right)}, \frac{d\left(hu, ghx_{2n+1}\right).d\left(hhx_{2n+1}, fu\right)}{d\left(hu, hhx_{2n+1}\right)+d\left(fu, ghx_{2n+1}\right)}\right. \\ & \frac{d\left(hu, fu\right).d\left(hhx_{2n+1}, ghx_{2n+1}\right)}{d\left(fu, hhx_{2n+1}\right)+d\left(fu, ghx_{2n+1}\right)}, \frac{d\left(hu, fu\right).d\left(hhx_{2n+1}, ghx_{2n+1}\right)}{d\left(fu, hhx_{2n+1}\right)+d\left(hu, fu\right)} \\ & \left.\frac{d\left(fu, hu\right).d\left(fu, hhx_{2n+1}\right)}{d\left(fu, hhx_{2n+1}\right)+d\left(hu, hhx_{2n+1}\right)}\right\} \\ & -\omega\left(\max\left\{\frac{d\left(hu, ghx_{2n+1}\right).d\left(hhx_{2n+1}, fu\right)}{d\left(hu, fu\right)+d\left(hu, hhx_{2n+1}\right)}, \frac{d\left(hu, ghx_{2n+1}\right).d\left(hhx_{2n+1}, fu\right)}{d\left(hu, hhx_{2n+1}\right)+d\left(fu, ghx_{2n+1}\right)}\right.\right. \\ & \frac{d\left(hu, fu\right).d\left(hhx_{2n+1}, ghx_{2n+1}\right)}{d\left(fu, hhx_{2n+1}\right)+d\left(fu, ghx_{2n+1}\right)}, \frac{d\left(hu, fu\right).d\left(hhx_{2n+1}, ghx_{2n+1}\right)}{d\left(fu, hhx_{2n+1}\right)+d\left(hu, fu\right)} \\ & \left.\left.\frac{d\left(fu, hu\right).d(fu, hhx_{2n+1})}{d\left(fu, hhx_{2n+1}\right)+d(hu, hhx_{2n+1})}\right\}\right) \end{aligned}$$

As $n \to \infty$, in above inequality

$$\begin{aligned} d\left(fu, hu\right) \leq \max & \left\{\frac{d\left(hu, hu\right).d\left(hu, fu\right)}{d\left(hu, fu\right)+d\left(hu, hu\right)}, \frac{d\left(hu, hu\right).d\left(hu, fu\right)}{d\left(hu, hu\right)+d\left(fu, hu\right)}, \frac{d\left(hu, fu\right).d\left(hu, hu\right)}{d\left(fu, hu\right)+d\left(fu, hu\right)},\right. \\ & \left.\frac{d\left(hu, fu\right).d\left(hu, hu\right)}{d\left(fu, hu\right)+d\left(hu, fu\right)}, \frac{d\left(fu, hu\right).d\left(fu, hu\right)}{d\left(fu, hu\right)+d\left(hu, hu\right)}\right\} \\ & -\omega\left(\max\left\{\frac{d\left(hu, hu\right).d\left(hu, fu\right)}{d\left(hu, fu\right)+d\left(hu, hu\right)}, \frac{d\left(hu, hu\right).d\left(hu, fu\right)}{d\left(hu, hu\right)+d\left(fu, hu\right)}, \frac{d\left(hu, fu\right).d\left(hu, hu\right)}{d\left(fu, hu\right)+d\left(fu, hu\right)}\right.\right. \\ & \left.\left.\frac{d\left(hu, fu\right).d\left(hu, hu\right)}{d\left(fu, hu\right)+d\left(hu, fu\right)}, \frac{d\left(fu, hu\right).d\left(fu, hu\right)}{d\left(fu, hu\right)+d\left(hu, hu\right)}\right\}\right) \\ & = \max\left\{0, 0, 0, 0, d\left(fu, hu\right)\right\} - \omega\left(\max\left\{0, 0, 0, 0, d\left(fu, hu\right)\right\}\right) \end{aligned}$$

$$d\left(fu, hu\right) \leq d\left(fu, hu\right) - \omega\left(d\left(fu, hu\right)\right)$$

This gives that $fu = hu$. Similarly we get $hu = gu$. Again from Equation (4), we get,

$$\begin{aligned} d\left(fhx_{2n}, gx_{2n+1}\right) \leq \max & \left\{\frac{d\left(hhx_{2n}, gx_{2n+1}\right).d\left(hx_{2n+1}, fhx_{2n}\right)}{d\left(hhx_{2n}, fhx_{2n}\right)+d\left(hhx_{2n}, hx_{2n+1}\right)}, \frac{d\left(hhx_{2n}, gx_{2n+1}\right).d\left(hx_{2n+1}, fhx_{2n}\right)}{d\left(hhx_{2n}, hx_{2n+1}\right)+d\left(fhx_{2n}, gx_{2n+1}\right)}\right. \\ & \frac{d\left(hhx_{2n}, fhx_{2n}\right).d\left(hx_{2n+1}, gx_{2n+1}\right)}{d\left(fhx_{2n}, hx_{2n+1}\right)+d\left(fhx_{2n}, gx_{2n+1}\right)}, \frac{d\left(hhx_{2n}, fhx_{2n}\right).d\left(hx_{2n+1}, gx_{2n+1}\right)}{d\left(fhx_{2n}, hx_{2n+1}\right)+d\left(hhx_{2n}, fhx_{2n}\right)} \\ & \left.\frac{d\left(fhx_{2n}, hhx_{2n}\right).d(fhx_{2n}, hx_{2n+1})}{d\left(fhx_{2n}, hx_{2n+1}\right)+d(hhx_{2n}, hx_{2n+1})}\right\} \\ & -\omega\left(\max\left\{\frac{d\left(hhx_{2n}, gx_{2n+1}\right).d\left(hx_{2n+1}, fhx_{2n}\right)}{d\left(hhx_{2n}, fhx_{2n}\right)+d\left(hhx_{2n}, hx_{2n+1}\right)}, \frac{d\left(hhx_{2n}, gx_{2n+1}\right).d\left(hx_{2n+1}, fhx_{2n}\right)}{d\left(hhx_{2n}, hx_{2n+1}\right)+d\left(fhx_{2n}, gx_{2n+1}\right)}\right.\right. \\ & \frac{d\left(hhx_{2n}, fhx_{2n}\right).d\left(hx_{2n+1}, gx_{2n+1}\right)}{d\left(fhx_{2n}, hx_{2n+1}\right)+d\left(fhx_{2n}, gx_{2n+1}\right)}, \frac{d\left(hhx_{2n}, fhx_{2n}\right).d\left(hx_{2n+1}, gx_{2n+1}\right)}{d\left(fhx_{2n}, hx_{2n+1}\right)+d\left(hhx_{2n}, fhx_{2n}\right)}, \\ & \left.\left.\frac{d\left(fhx_{2n}, hhx_{2n}\right).d(fhx_{2n}, hx_{2n+1})}{d\left(fhx_{2n}, hx_{2n+1}\right)+d(hhx_{2n}, hx_{2n+1})}\right\}\right) \end{aligned}$$

Taking the limit as $n \to \infty$, from Equation (10), we get

$$\begin{aligned} d\left(hu, u\right) \leq \max & \left\{\frac{d\left(hu, u\right).d\left(u, hu\right)}{d\left(hu, hu\right)+d\left(hu, u\right)}, \frac{d\left(hu, u\right).d\left(u, hu\right)}{d\left(hu, u\right)+d\left(hu, u\right)}, \frac{d\left(hu, hu\right).d\left(u, u\right)}{d\left(hu, u\right)+d\left(hu, u\right)}\right. \\ & \left.\frac{d\left(hu, hu\right).d\left(u, u\right)}{d\left(hu, u\right)+d\left(hu, hu\right)}, \frac{d\left(hu, hu\right).d(hu, u)}{d\left(hu, u\right)+d(hu, u)}\right\} \\ & -\omega\left(\max\left\{\frac{d\left(hu, u\right).d\left(u, hu\right)}{d\left(hu, hu\right)+d\left(hu, u\right)}, \frac{d\left(hu, u\right).d\left(u, hu\right)}{d\left(hu, u\right)+d\left(hu, u\right)}, \frac{d\left(hu, hu\right).d\left(u, u\right)}{d\left(hu, u\right)+d\left(hu, u\right)}\right.\right. \\ & \left.\left.\frac{d\left(hu, hu\right).d\left(u, u\right)}{d\left(hu, u\right)+d\left(hu, hu\right)}, \frac{d\left(hu, hu\right).d\left(hu, u\right)}{d\left(hu, u\right)+d\left(hu, u\right)}\right\}\right) \end{aligned}$$

$$d(hu, u) \leq \max\left\{d(hu, u), \frac{d(hu, u)}{2}, 0, 0, 0\right\} - \omega\left(\max\left\{d\left(hu, u\right), \frac{d\left(hu, u\right)}{2}, 0, 0, 0\right\}\right)$$

$$d\left(hu, u\right) \leq d\left(hu, u\right) - \omega\left(d\left(hu, u\right)\right)$$

Which shows that, $hu = u$. Thus u is a common fixed point of f, g and h.

Uniqueness: If $v \in X \setminus \{u\}$ is another common fixed point of f, g and h, from Equation (4) we get

$$d(u,v) = d(fu,gv) \leq d(u,v) - \omega(d(u,v))$$

Which shows that $u = v$. Hence f, g and h have a unique common fixed point $u \in X$. □

Theorem 2.2. *Let f and g be two self mapping from a complete metric space (X,d) into itself. If there exist a $\omega \in W$ satisfying*

$$d(fx,gy) \leq \max\left\{\frac{d(x,gy).d(y,fx)}{d(x,fx)+d(x,y)}, \frac{d(x,gy).d(y,fx)}{d(x,y)+d(fx,gy)}, \frac{d(x,fx).d(y,gy)}{d(fx,y)+d(fx,gy)}, \frac{d(x,fx).d(y,gy)}{d(fx,y)+d(x,fx)}, \frac{d(fx,x).d(fx,y)}{d(fx,y)+d(x,y)}\right\}$$
$$-\omega\left(\max\left\{\frac{d(x,gy).d(y,fx)}{d(x,fx)+d(x,y)}, \frac{d(x,gy).d(y,fx)}{d(x,y)+d(fx,gy)}, \frac{d(x,fx).d(y,gy)}{d(fx,y)+d(fx,gy)}, \frac{d(x,fx).d(y,gy)}{d(fx,y)+d(x,fx)}, \frac{d(fx,x).d(fx,y)}{d(fx,y)+d(x,y)}\right\}\right)$$

Then f and g have a unique common fixed point in X.

Proof. Taking $h = I$ in the Theorem 2.1 and remaining proof as above. □

3. Application

Let X and Y be a Banach space, $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space. $B(S)$ denotes the set of all real-valued bounded functions on S. Put

$$d(a,b) = \sup_{x \in S} |a(x) - b(x)|, \ \omega a, b \in B(S),$$

It is obvious that $(B(S), d)$ is a complete metric space. Define $u : S \times D \to R$, $T : S \times D \to S$ and $H_i : S \times D \times R \to R$ for $i \in \{1,2,3\}$. Now we study those conditions, which is guarantee the existence and uniqueness of solution and common solution for the functional Equation (2) and the system of functional Equations (3).

Theorem 3.1. *If the following conditions are satisfied*

(a). u and H_i are bounded for $i \in \{1,2,3\}$

(b). There exist a $\omega \in W$ satisfying

$$|H_1(x,y,a(t)) - H_2(x,y,b(t))| \leq \max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3bhy)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\}$$
$$-\omega\left(\max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3bhy)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\}\right)$$

for all $(x,y) \in S \times D, a, b \in B(S)$ and $t \in S$, where the mapping f_1, f_2 and f_3 are defined as follows, $\forall\, x \in S, a_i \in B(S), i \in \{1,2,3\}$,

$$f_i a_i(x) = opt_{y \in D}\{u(x,y) + H_i(x,y,a_i(T(x,y)))\} \tag{11}$$

(c). $f_1(B(S)) \cup f_2(B(S)) \subseteq f_3(B(X))$ and $f_3 \in C_{f1}(B(S)) \cap C_{f2}(B(S))$,

then the system of functional Equation (3) possess a unique common solution in $B(S)$.

Proof. It follows from (a) and (b) that f_1, f_2 and f_3 are self mapping in $B(S)$. Let $a, b \in B(S)$ and $x \in S$. We now have to consider two possible cases

Case 1: Suppose that $opt_{y\in D} = \sup_{y\in D}$ for any $\epsilon > 0$, there exist $y, z \in D$ satisfying

$$\begin{aligned} f_1a(x) &< u(x,y) + H_1(x,y,a(T(x,y))) + \epsilon \\ f_2b(x) &< u(x,z) + H_2(x,z,b(T(x,z))) + \epsilon \\ f_1a(x) &\geq u(x,z) + H_1(x,z,a(T(x,z))) \\ f_2b(x) &\geq u(x,y) + H_2(x,y,b(T(x,y))) \end{aligned} \tag{12}$$

From the Equation (12) and (b), we get

$$\begin{aligned} |f_1a(x) - f_2b(x)| &< \epsilon + \max\{|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|, |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\} \\ &\leq \epsilon + \max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \right. \\ &\left. \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3b)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\} \\ &- \omega\left(\max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \right.\right. \\ &\left.\left. \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3b)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\}\right) \end{aligned}$$

Which yields that

$$\begin{aligned} d(f_1a,f_2b) &\leq \epsilon + \max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \right. \\ &\left. \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3b)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\} \\ &- \omega\left(\max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \right.\right. \\ &\left.\left. \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3b)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\}\right) \end{aligned} \tag{13}$$

Case 2: Suppose that $opt_{y\in D} = \inf_{y\in D}$. By using a method similar to the proof of Case 1, we see Equation (13) holds, letting $\epsilon \to 0$ in Equation (13), we get

$$\begin{aligned} d(f_1a,f_2b) &\leq \epsilon + \max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \right. \\ &\left. \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3b)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\} \\ &- \omega\left(\max\left\{\frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_1a)+d(f_3a,f_3b)}, \frac{d(f_3a,f_2b).d(f_3b,f_1a)}{d(f_3a,f_3b)+d(f_1a,f_2b)}, \right.\right. \\ &\left.\left. \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_1a,f_2b)}, \frac{d(f_3a,f_1a).d(f_3b,f_2b)}{d(f_1a,f_3b)+d(f_3a,f_1a)}, \frac{d(f_1a,f_3a).d(f_1a,f_3b)}{d(f_1a,f_3b)+d(f_3a,f_3b)}\right\}\right) \end{aligned}$$

Therefore, in the theorem (4), f_1, f_2 and f_3 have a unique common fixed point $v \in B(S)$.That is the system of functional Equation (3) possess a unique common solution $v \in B(S)$. □

Theorem 3.2. *Suppose that the following condition hold:*

(d). u and H_i are bounded for $i = \{1, 2\}$;

(e). There exist a $\omega \in W$ satisfying

$$|H_1(x,y,a(t))-H_2(x,y,b(t))| \leq \max\left\{\frac{d(a,f_2b).d(b,f_1a)}{d(a,f_1a)+d(a,b)},\frac{d(a,f_2b).d(b,f_1a)}{d(a,b)+d(f_1a,f_2b)},\frac{d(a,f_1a).d(b,f_2b)}{d(f_1a,b)+d(f_1a,f_2b)}\right.$$
$$\left.\frac{d(a,f_1a).d(b,f_2b)}{d(f_1a,b)+d(a,f_1a)},\frac{d(f_1a,a).d(f_1a,b)}{d(f_1a,b)+d(a,b)}\right\}$$
$$-\omega\left(\max\left\{\frac{d(a,f_2b).d(b,f_1a)}{d(a,f_1a)+d(a,b)},\frac{d(a,f_2b).d(b,f_1a)}{d(a,b)+d(f_1a,f_2b)},\frac{d(a,f_1a).d(b,f_2b)}{d(f_1a,b)+d(f_1a,f_2b)}\right.\right.$$
$$\left.\left.\frac{d(a,f_1a).d(b,f_2b)}{d(f_1a,b)+d(a,f_1a)},\frac{d(f_1a,a).d(f_1a,b)}{d(f_1a,b)+d(a,b)}\right\}\right)$$

for all $(x,y) \in S \times D$, $a,b \in B(S)$ and $t \in S$, where the mapping f_1 and f_2 are defined as follows, $\forall\ x \in S$, $a_i \in B(S)$, $i \in \{1,2\}$,

$$f_ia_i(x) = opt_{y\in D}\{u(x,y)+H_i(x,y,a_i(T(x,y))\}$$

Then the system of functional equations

$$f_i(x) = opt_{y\in D}\{u(x,y)+H_i(x,y,f_i(T(x,y))\},\ \forall\ x{\in}S,\ i{\in}\{1,2\}$$

possesses a unique common solution in $B(S)$.

Proof. In the Theorem 3.1, we take $f_3 = I$ and remaining proof as above. □

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