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The Split (Nonsplit) Nomatic Number of a Graph

Research Article

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Abstract: For a given connected graph $G = (V, E)$, a set $S \subseteq V(G)$ is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. A neighborhood set S is a split (nonsplit) neighborhood set if $\langle V(G) - S \rangle$ is connected (disconnected). The maximum number of a partition of $V(G)$, all of whose are split (nonsplit) neighborhood sets, is the split(nonsplit) nomatic number $N_s(G)(N_{ns}(G))$. Our purpose in this paper is to initiate the study of split(nonsplit) nomatic number of a graph. We first study basic properties and bounds for $N_s(G)(N_{ns}(G))$. In addition, we determine the $N_s(G)(N_{ns}(G))$ of some classes of graphs.

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1. Introduction

Let $G = (V, E)$ be simple, undirected, and nontrivial graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Also $|V| = n$ and $|E| = m$ denote number of vertices and number of edges in G. The open neighborhood $N(v)$ of vertex v denotes number of vertices adjacent to v and its closed neighborhood $N[v] = N(v) \cup \{v\}$. The $\beta_1(G)$ is the minimum number of edges in a maximal independent set of edge of G. The complement $\overline{G}$ of a graph G defined to be graph which has V as its sets of vertices and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Further, a graph G is said to be self-complementary, if G is isomorphic with $\overline{G}$. For notation and graph theory terminology we generally follow [6].

A set $D \subseteq V$ is a dominating set if every vertex not in S is adjacent to one or more vertices in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [5]. They have defined the domatic number $d(G)$ of a graph G by means of sets. For some purposes it is more convenient to consider domatic colourings instead of domatic partitions. A coloring of vertices of a graph G is called domatic, if each vertex of G is adjacent to vertices of all colors different from its own. (Two vertices of the same color may be adjacent.) Then the domatic number of G is the maximum number of colors of a domatic coloring of G. In otherwords, a partition of V, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. For complete review on the concept of domatic number, we refer [2], [3], [4] and [12].

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In [10], E. Sampathkumar ad P. S. Neeralagi introduce the concepts of neighborhood number as follows. A set $S \subseteq V$ is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ of G is the minimum cardinality of a neighborhood set of a graph G. A neighborhood set S is a split (nonsplit) neighborhood set if $\langle V(G) - S \rangle$ is connected (disconnected). The split (nonsplit)neighborhood number $\eta_s(G)(\eta_{ns}(G))$ of a graph G is the minimum cardinality of a split (nonsplit)neighborhood set of a graph G. For more detail, we refer , see [1], [7], [9] and [11]. With the help of a neighborhood set, S. R. Jayaram [8] have defined the nomatic number of a graph. The maximum number of a partition of $V(G)$, all of whose are neighborhood sets, is the nomatic number $N(G)$. Here, we shall introduce the nomatic analogue of this concept and prove some assertions concerning it. A split (nonsplit) nomatic partition is a partition of $V(G)$ into split(nonsplit) neighborhood sets, and the split (nonsplit) nomatic number $N_s(G)(N_{ns}(G))$ is the largest number of sets in a split (nonsplit) nomatic partition. A neighborhood set S with minimum cardinality is called η - set of a graph G. Similarly, the other sets can be expected.

The split (nonsplit) nomatic number problem arises in various areas and scenarios. In particular, this problem is related to the task of distributing resources in a computer network, and also to the task of locating facilities in a communication network as follows:

Suppose, for example, that resources are to be allocated in a computer network such that expensive services are quickly accessible in the immediate neighborhood of each vertex. If every vertex has only a limited capacity, then there is a bound on the number of resources that can be supported. In particular, if every vertex can serve a single resource only, then the maximum number of resources that can be supported equals the nomatic number of the network graph.

2. Split Nomatic Number

Observation 2.1. *Every neighborhood set is a split neighborhood set of a graph G. Clearly, $\eta(G) \leq \eta_s(G)$. Like wise $N_s(G) \leq N(G)$.*

Observation 2.2. *Every domatic partition is a split domatic partition and every split domatic partition is a split nomatic partition of a graph. Clearly, $N_s(G) \leq d_s(G) \leq d(G)$.*

Theorem 2.3. *A neighborhood set S of a graph G is a split neighborhood set if and only if there exist atlest two vertices $x, y \in V - S$ such that every $x - y$ path contains a vertex of S.*

Proof. Let S be a η_s- set of a graph G, then $\langle V - S \rangle$ is disconnected and its contains atleast two components, say G_1 and G_2, let $x \in G_1$, $y \in G_2$. There is no path in $V - S$ containing x and y. Hence every path in G connecting x and y, which contains S.

Conversely, suppose that there are vertices x and $y(\notin S)$ such that S is an path connecting x and y. We shall prove that S is a split neighborhood set. Let, if possible S be a nonsplit neighborhood set, so $\langle V - S \rangle$ is connected, there is a path in G, which does not contain S. This contradicts to proves that S is a spilt neighborhood set of a graph G. □

Theorem 2.4. *For every connected graph G with $n \geq 3$ vertices,*

$$N_s(G) \leq Min.\{\delta(G) + 1, \frac{n}{\eta_s(G)}\}.$$

Proof. By Observation 1.1, the desired result follows. □

Theorem 2.5. *For any non trivial connected graph G,* *(i)* $N_s(G) = 1$ *if and only if* $G = P_3$ *or* $C_n; n = 2k + 1$ *for* $k \geq 1$ *or* $K_{1,n}$ *or* W_n *or* $K_{r_1,r_2,r_3,\ldots,r_t}$,

(ii) $N_s(G) = 2$ *if and only if* G *is bipartite graph with partite set* $\{|V_1|, |V_2|\} \geq 2$, *where* $V_1, V_2 \in V(G)$.

Proof. Let G be a graph of P_3 or $C_n; n = 2k + 1$ for $k \geq 1$ or $K_{1,n}$ or W_n or $K_{r_1,r_2,r_3,\ldots,r_t}$. If S is a spilt neighborhood set of G, then $V - S$ is not contain a spilt neighborhood set and $N_s(G) = 1$.

Let G be a bipartite graph. If $V(G)$ can be partitioned into two sets V_1 and V_2 so that every edge of G joins a vertex of V_1 with a vertex of V_2 with $\{|V_1|, |V_2|\} \geq 2$. Then graph G necessarily has its oddly subscripted vertices in V_1 and others in V_2, so that its length n is even that is, for every even cycle $\{v_1v_2v_3 \ldots v_nv_1\}$. Otherwise, there exist atleast four vertices of $\{v_1$ - v_2 - ... - $v_n\}$ such that $\{v_1v_2$ - v_2v_3 - ... - $v_{n-1}v_n\}$ of path length atleast 3. This implies that the set $V - S$ is contain a spilt neighborhood set with $S \subseteq \frac{V}{2}$and $N_s(G) = 2$. Thus, the converses of (i) and (ii) are obvious. □

Theorem 2.6. *Let* T *be a tree with* $n \geq 4$ *vertices. Then* $N_s(T) = \delta(T) + 1$, *provided tree* T *is not contain* $K_{1,n}$ *for* $n \geq 1$.

Proof. Let T be a tree with $n \geq 4$ vertices. If tree T is not contain $K_{1,n}$ for $n \geq 1$, then $N_s(T) = 2$, due to the fact of Theorem 1. of (ii) and $\delta(T) = 1$. Thus $N_s(T) = \delta(T) + 1$ follows. Conversely, suppose tree T is isomorphic with $K_{1,n}$ for $n \geq 1$ implies that $N_s(T) < \delta(T) + 1$, a contradiction. Hence the result follows. □

3. Nonsplit Nomatic Number

Observation 3.1. *Every neighborhood set is a nonsplit neighborhood set of a graph* G. *Clearly,* $\eta(G) \leq \eta_{ns}(G)$. *Like wise* $N_{ns}(G) \leq N(G)$.

Observation 3.2. *Every domatic partition is a nonsplit domatic partition and every nonsplit domatic partition is a nonsplit nomatic partition of a graph* G. *Clearly,* $N_{ns}(G) \leq d_{ns}(G) \leq d(G)$.

Proposition 3.3. *For any Path* P_n, *Cycle* C_n, *Tree* T *and Complete bipartite graph* $K_{r,s}$ *with* $n \geq 3$ *and* $1 \leq r \leq s$ *vertices,*

$$N_{ns}(P_n) = N_{ns}(C_n) = N_{ns}(T) = N_{ns}(K_{r,s}) = 1.$$

Proposition 3.4. *For any complete graph* K_n *with* $n \geq 1$ *vertices,*

$$N_{ns}(K_n) = n.$$

Theorem 3.5. *For any connected graph* G *with* $n \geq 3$ *vertices,*

(i) $\gamma(G) = Min.\{\gamma_s(G), \gamma_{ns}(G)\}$,

(ii) $N(G) = Max.\{N_s(G), N_{ns}(G)\}$,

(iii) $N_{ns}(G) \leq Min.\{\delta(G) + 1, \frac{n}{\eta_{ns}(G)}\}$.

Proof. By Observation 1.1 and 2.1, (i)-(iii) follows. □

Theorem 3.6. *Let* $G = K_{r_1,r_2,r_3,\ldots,r_t}$ *be a complete multipartite graph with* $1 \leq r_1 \leq r_2 \leq r_3, \ldots, \leq r_t$ *and* $t \geq 3$ *vertices. Then* $N_{ns}(G) = t$.

Proof. Let $G = K_{r_1,r_2,r_3,...,r_t}$ be a complete multipartite graph with $1 \leq r_1 \leq r_2 \leq r_3, \ldots, \leq r_t$ and $t \geq 3$ vertices. If $V(G)$ is a finite set of a graph G with $1 \leq t \leq n$, where r is an positive integer and $S_1, S_2, ..., S_t$ is a partition of $V(G)$, then there exists a graph G such that $N_{ns}(G) = t$, since $S_1, S_2, ..., S_t$ is a nonsplit nomatic partition of a graph G with $S_i \cap S_j = \phi$ for $i \neq j$. Suppose, if $t = 2$ vertices, which is a complete bipartite graph with partite set V_1 and V_2. This implies that $V - u$ is a unique nonsplit neighborhood set of a complete bipartite graph G and hence $N_{ns}(G) < t = 2$, which is contradiction. Thus the result follows. □

Definition 3.7. *The join $G_1 + G_2$ of two graphs G_1 and G_2 is the disjoint union of G_1 and G_2 together with all possible edges connecting a vertex of G_1 with a vertex of G_2.*

Theorem 3.8. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$1 + N_{ns}(G) \leq N_{ns}(K_1 + G).$$

Proof. If the partitions $\{S_1, S_2, ..., S_t\}$ of the vertex set V into nonsplit neighborhood sets of a graph G, then the nonsplit nomatic partitions $\{S_1, S_2, ..., S_t, S_{t+1}\}$ of a graph $K_1 + G$, where $u \in S_{t+1}$ for $u \in V(K_1)$. Thus the result follows. □

Proposition 3.9. *For any Fan $F_n = K_1 + P_{n-1}; n \geq 3$ and wheel $W_n = K_1 + C_{n-1}; n \geq 4$ vertices,*

(i) $N_{ns}(F_n) = N_{ns}(W_{2k+3}) = 3$,

(ii) $N_{ns}(W_4) = 4$,

(iii) $N_{ns}(W_{2k+4}) = 2$, *where k is an positive integer.*

Observation 3.10. *For any connected spanning subgraph H of a graph G, that is $\eta_{ns}(G) \leq \eta_{ns}(H)$. But $N_{ns}(H) \leq N_{ns}(G)$.*

By this observation, we have the following Proposition, which is straight forward.

Proposition 3.11. *For any connected graph G with $n \geq 3$ vertices,*

$$N_{ns}(G) - 1 \leq N_{ns}(G - x) \leq N_{ns}(G),$$

for any edge $x \in E(G)$.

Observation 3.12. *A nonsplit neighborhood vertex which forms a nonsplit neighborhood set, that is a vertex adjacent to all other vertices. if u is a nonsplit neighborhood vertex of a nontrivial graph G, then G is isomorphic to $(G - u) + K_1$.*

Proposition 3.13. *If u is a nonsplit neighborhood vertex of a graph G, then*

$$N_{ns}(G) = N_{ns}(G - u) + 1.$$

Proof. Since a nonsplit nomatic partition of u forms a nonsplit nomatic partition of G, $N_{ns}(G) \geq N_{ns}(G - u) + 1$. On the otherhand, suppose $S_1, S_2, ..., S_t$ is a nonsplit nomatic partition of a graph G, where $t = N_{ns}(G)$. Assume $u \in S_1$. Note that $S_1 \cup S_2 - \{u\}, S_3, ..., S_t$ is a nonsplit nomatic partition of $G - u$. So $N_{ns}(G - u) \geq t - 1 = N_{ns}(G) + 1$. Thus the result follows. □

Theorem 3.14. *For any connected graph G,*

$$N_{ns}(G) \leq \Delta(G) + 1.$$

Further, the bound is attained if and only if $G \cong K_n$ with $n \geq 2$ vertices.

Proof. By Theorem 2.1 and due to the fact of $\eta_{ns}(G) \geq \frac{n}{\Delta(G)+1}$, the desired result follow.
Now we prove the next part. Suppose $N_{ns}(G) = \Delta(G)+1$ holds. On contrary, suppose given condition is not satisfied, then there exists atleast three vertices u, v and w such that v is adjacent to both u and w, and u is not adjacent to w. Thus, $V-u$ or $V-w$ form a unique nonsplit neighborhood set and this implies that $N_{ns}(G) < \Delta(G)+1$, which is a contradiction. Converse is easy to cheek. □

By above theorem, we obtain a Nordhaus-Gaddum type results.

Theorem 3.15. *Let G be a graph such that both G and $\overline{G}$ are connected with $n \geq 4$ vertices. Then*

$$N_{ns}(G) + N_{ns}(\overline{G}) \leq n+1.$$

Theorem 3.16. *Let G be a connected graph. Then*

(i) $N_{ns}(G) = n$ if and only if the graph $G \cong K_n$ with $n \geq 1$ vertices.

(ii) $N_{ns}(G) = n-1$ if and only if the graph $G \cong K_n - e$, where e is an edge of K_n with $n \geq 3$ vertices.

(iii) $N_{ns}(G) = \frac{n}{2}$ if and only if the graph G is obtained from the complete graph K_n by deleting maximum independent edges; Provided n is an even integer with $n \geq 4$ vertices.

Proof.

(i) Let G be a connected graph with $n \geq 1$ vertices. If G is isomorphic with K_n, then each vertex of a complete graph K_n forms a one-element, which is a nonsplit neighborhood set. Thus nonsplit nomatic partition of the vertex set $V(G)$ into n-disjoint nonsplit neighborhood sets of a graph G and hence the result follows.
Conversely, suppose $N_{ns}(G) = n$ holds but that G is not isomorphic with $K_n; n \geq 1$. Then there exist at least three vertices u, v and w such that two adjacent vertices u and v. Suppose w is adjacent to v. Then u, v or v, w or u, w is one and only nonsplit neighborhood set of G, this implies that $N_{ns}(G) < n$, which is a contradiction.

(ii) If $G \cong K_n - e$, where $e = uv \in E(K_n)$, then $G - e$ is a graph consisting of an independent vertices u and v. Hence, every nonsplit nomatic partiton of G must contain u and v in one of the nonsplit neighborhood set. Hence $N_{ns}(G) \geq n-1$. The inequality $N_{ns}(G) \leq n-1$ follows from the fact that $\{V-u\}$ is a nonsplit nomatic partition of a graph G. Hence $N_{ns}(G) = n-1$.
Conversely, suppose $N_{ns}(G) = n-1$ holds but that the graph G is not isomorphic with $K_n - e$, where e is an edge of K_n with $n \geq 3$ vertices. Then there exists atleast three vertices u, v and w such that v is adjacent to both u and w, and u is not adjacent to w. This implies that the removable of any one end vertex of u or v in a graph G which contains atleast one end vertices, which form a unique nonsplit neighborhood set of a graph G. Thus $N_{ns}(G) < n-1$, which is a contradiction.

(iii) Suppose $d_{ns}(G) = \frac{n}{2}$. On contrary, if the graph G is not obtained from the complete graph K_n by deleting maximum independent edges, where n is an even integer with $n \geq 4$ vertices. Then there exist atleast four vertices of complete graph K_n by deleting set of independent edges $X(G)$ for all $X(G) \subset E(K_n)$ such that $| X(G) | < \beta_1(G)$, where $\beta_1(G)$ is a maximum number of independent set of a graph G. This implies that $d_{ns}(G) > \frac{n}{2}$, which is a contradiction. Converse is obvious. □

Theorem 3.17. *Let G be a connected graph. If $N_{ns}(G) \geq 3$, then $diam(G) \leq 2$, where $diam(G)$ denote the diameter of G.*

Proof. Let $N_{ns}(G) = r \geq 3$. Then there exists a nonsplit nomatic partition of $\{S_1, S_2, ..., S_r\}$ of a graph G. Let u and v be two vertices of a graph G. As $r \geq 3$, atleast one of the sets $S_1, S_2, ..., S_r$ contains neither u nor v. Without loss of generality, let it be D_1. We have $\{u, v\} \subseteq V(G) - D_1$ and therefore there exists a vertex $w \in D_1$ such that $\langle u, v, w \rangle$ is connected in G. If u, v are adjacent, then the distance between u and v is one. If u, v are not adjacent, then w must be adjacent to both u and v and the distance between u and v is two. As u and v were chosen arbitrarily, we have $diam(G) \leq 2$. □

Now, We have the following observations and properties of global nonsplit nomatic number $N_{gns}(G)$ of a graph G.
Note that a global nonsplit neighborhood set of G is simultaneously a nonsplit neighborhood set of G and its complement $\overline{G}$. A global nonsplit nomatic partition is a partition of $V(G)$ into global nonsplit neighborhood sets, and the global nonsplit nomatic number $N_{gns}(G)$ is the largest number of sets in a global nonsplit nomatic partition.

Theorem 3.18. *Let G be a graph such that both G and $\overline{G}$ are connected. Then*

(i) $N_{gns}(G) = N_{gns}(\overline{G})$,

(ii) $N_{gns}(G) = Min.\{N_{ns}(G), N_{ns}(\overline{G})\}$,

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