International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(2)**(2023), 59–70
Available Online: http://ijmaa.in

Non-archimedean Pseudo-differential Operators $\mathscr{B}_{\psi_1,\psi_2}$ Connected with Fractional Fourier Transform

Abhisekh Shekhar[1,*], Nawin Kumar Agrawal[2]

[1] *Department of Mathematics, C.M.Science College (A constituent unit of L.N.M.U., Darbhanga), Darbhanga, Bihar, India*
[2] *University Department of Mathematics, L. N. Mithila University, Darbhanga, Bihar, India*

Abstract

The aim of this paper is to introduce non-archimedean pseudo-differential operators $\mathscr{B}_{\psi_1,\psi_2}$ connected with fractional Fourier transform whose symbol $\min\{|\psi_1(.)|, |\psi_2(.)|\}$ is found from the character of two mappings defined on the p-adic numbers. In this manuscript, we also study some properties of fractional heat Kernel $\mathscr{X}(\zeta,\tau)$ on $\mathbb{Q}_p$, self-adjoint of $\mathscr{B}_{\psi_1,\psi_2}$ and dissipative of $\mathscr{B}_{\psi_1,\psi_2}$ in $L^2(\mathbb{Q}_p)$.

Keywords: Non-archimedean analysis; Pseudo-differential operators; Fractional Fourier transform; dissipative operators; p-Adic analysis.

2020 Mathematics Subject Classification: 26E30, 35S05, 46E35, 47G30, 46F12.

1. Introduction

The connections of non-archimedean pseudo-differential operators with certain p-adic pseudo-differential equations that describe certain physical models [1–5]. Therefore, non-archimedean pseudo-differential operators have received a lot of attention in two decades.Non-archimedean pseudo-differential operators have gained popularity in recent years due to their utility in studying certain equations associated with new physical models/Models in physical form [6–12]. The interest in pseudo-differential operators in the p-adic context has grown significantly in recent years as a result of their utility in modelling various types of physical phenomena. For example, modelling geological processes (such as the formation of petroleum micro-scale reservoirs and fluid flows in porous media such as rock); the dynamics of complex systems such as macromolecules, glasses, and proteins; the study of Coulomb gases, etc. [13–17]. Nonlocal diffusion problems arise in a wide range of applications in the archimedean setting, including biology, image processing, particle systems, and coagulation models. This research work is motivated/ inspired by

*Corresponding author (abhi08.iitkgp@gmail.com)

the works of Ismael Gutiérrez García and Anselmo Torresblanca-Badillo [10, 16, 18, 19]. In this manuscript, we discuss the p-adic non-archimedean pseudo-differential operators of the form

$$\begin{aligned}(\mathscr{B}_{\psi_1,\psi_2}\phi)(\zeta) &= -\mathcal{F}_{\vartheta}^{-1}\big([\min\{|\psi_1(||\zeta||_p)|,|\psi_2(||\zeta||_p)|\}]\widehat{\phi}_{\vartheta}(\zeta)\big),\\ &= -\int_{\mathbb{Q}_p}\chi_p^{-\vartheta}(\zeta,\eta)[\min\{|\psi_1(||\zeta||_p)|,|\psi_2(||\zeta||_p)|\}]\widehat{\phi}_{\vartheta}(\zeta)d\zeta,\end{aligned}$$

where $\mathcal{F}_{\vartheta}^{-1}$ denotes the inverse fractional Fourier transform, $\widehat{\phi}_{\vartheta}$ is the fractional Fourier transform of ϕ, the mapping $\min\{|\psi_1(.)|,|\psi_2(.)|\}:\mathbb{Q}_p\to\mathbb{C}$ is the symbol of the p-adic non-archimedean pseudo-differential operator $\mathscr{B}_{\psi_1,\psi_2}$.

The implementation of these symbols aims to offer new classes of the p-adic non-archimedean pseudo-differential equations that are naturally associated with our p-adic non-archime. The p-adic heat equation (or Cauchy problem) relating to $\mathscr{B}_{\psi_1,\psi_2}$ has the following mathematical form:

$$\begin{cases}\frac{\partial\phi}{\partial\tau}(\zeta,\ \tau)=-\mathscr{B}_{\psi_1,\psi_2}(\phi(\zeta,\tau)),\ \ 0<\tau<\infty,\ \ \zeta\in\mathbb{Q}_p\\ \phi(\zeta,0)=\phi_0(\zeta)\in\mathscr{D}(\mathbb{Q}_p).\end{cases}\tag{1}$$

$$\phi(\zeta,\tau)=\int_{\mathbb{Q}_p}\chi_p^{-\vartheta}(\zeta,\eta)e^{-\tau[\min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\}]}(\mathcal{F}_{\vartheta}\phi_0)(\eta)d\eta,\ \ \zeta\in\mathbb{Q}_p,\ \tau\geq 0$$

is a basic solution of (1). In this article, we study some properties of fractional heat Kernel $\mathscr{Z}(\zeta,\tau)$ on $\mathbb{Q}_p$, self-adjoint of $\mathscr{B}_{\psi_1,\psi_2}$ and dissipative of $\mathscr{B}_{\psi_1,\psi_2}$ in $L^2(\mathbb{Q}_p)$. p-adic numbers, on the other hand, have become a natural structure in the last three years, thanks to their topology, to research novel mathematical models connected to the transmission of infectious illnesses (say, COVID-19) over certain sorts of population groupings, see for example, [14,20,21].

2. Mathematical Background of Fractional Fourier Analysis on $\mathbb{Q}_p$

Definition 2.1 (The field of p-adic numbers). *Let p be a prime number. Through out this manuscript p will denote a prime number. Firstly we define p-adic norm $|.|_p$ on $\mathbb{Q}$ as follows*

$$|\eta|_p=\begin{cases}0, & if\ \eta=0,\\ p^{-\tau}, & if\ \ \eta=p^{\tau}\frac{\rho}{\sigma},\end{cases}$$

where ρ and σ are integers coprime with p. The integer $\tau:=ord(\eta)$, with ord(0):=$+\infty$, is called the p-adic order of η. The unique expansion of any p-adic number $\eta\neq 0$ is of the form

$$\eta=p^{ord(\eta)}\sum_{i=0}^{\infty}\eta_i p^i,\tag{2}$$

where $\eta_i\in\{0,1,2,\dots,p-1\}$ and $\eta_0\neq 0$. Using (2), we define the fractional part of $\eta\in\mathbb{Q}_p$, denoted by $\{\eta\}_p$, as the rational number

$$\{\eta\}_p = \begin{cases} 0, & if\ \eta = 0\ \ or\ ord(\eta) \geq 0, \\ p^{ord(\eta)} \sum\limits_{i=0}^{-ord_p(\eta)-1} \eta_i p^i, & if\ \ ord(\eta) < 0. \end{cases}$$

Extension of the p-adic norm on $\mathbb{Q}_p$ is given by

$$||\eta||_p = |\eta|, \quad \forall \eta \in \mathbb{Q}_p.$$

Let $r_0 \in \mathbb{Z}$ and $a_0 \in \mathbb{Q}_p$. We consider $I_{r_0}(\eta_0) = \{\eta \in \mathbb{Q}_p : ||\eta - \eta_0||_p \leq p^{r_0}\}$. The empty set and the points are the only connected subsets of $\mathbb{Q}_p$. Therefore, the topological space $(\mathbb{Q}_p,\ ||.||_p)$ is totally disconnected. The necesssary and sufficient condition for the compactness of a subset of $\mathbb{Q}_p$ is that bounded and closed subdet of $\mathbb{Q}_p$.

3. Few Functional Spaces

A function $f : \mathbb{Q}_p \to \mathbb{C}$ is called locally constant if for any $\eta \in \mathbb{Q}_p$ there exists an integer $r(\eta) \in \mathbb{Z}$ such that $f(\eta + \eta') = f(\eta)$ for all $\eta' \in I_{r(\eta)}$. A function $f : \mathbb{Q}_p \to \mathbb{C}$ is called a test function (or a Bruhat-Schwartz function) if it is a compact support with locally constant. The set of all complex valued test functions on $\mathbb{Q}_p$ is denoted by $\mathcal{D}(\mathbb{Q}_p)$ or simply $\mathcal{D}$. The set of all distributions (all continous functionals) on $\mathcal{D}$ is denoted by $\mathcal{D}'(\mathbb{Q}_p)$ or simply $\mathcal{D}'$. The mapping $\langle U, \psi \rangle : \mathcal{D}'(\mathbb{Q}_p) \times \mathcal{D}(\mathbb{Q}_p) \to \mathbb{C}$ for $U \in \mathcal{D}'(\mathbb{Q}_p)$ and $\psi \in \mathcal{D}(\mathbb{Q}_p)$ is defined as follows:

$$\langle U, \psi \rangle = \int_{\mathbb{Q}_p} U(\zeta)\psi(\zeta)d\zeta.$$

Definition 3.1 (Regular Distribution)**.** *Let M be an arbitrary compact subset of $\mathbb{Q}_p$. i.e. $M \subset \mathbb{Q}_p$. Then $L^1_{loc}(\mathbb{Q}_p) = \{\phi | \phi : \mathbb{Q}_p \to \mathbb{C}$ such that $\phi \in L^1(M)\}$. A distribution $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is defined by every function $\phi \in L^1_{loc}(\mathbb{Q}_p)$ according to the formula*

$$\langle \phi, \psi \rangle = \int_{\mathbb{Q}_p} \phi(\zeta)\psi(\zeta)d\zeta.$$

This type of distributions is known as regular distributions.
Let $\sigma \in [0,\ \infty)$. Then the set $L^\sigma(\mathbb{Q}_p,\ dx) = \{h : \mathbb{Q}_p \to \mathbb{C}$ such that $\int_{\mathbb{Q}_p} |h(x)|^\sigma dx < \infty\}$, the set $L^\infty(\mathbb{Q}_p,\ dx) = \{h : \mathbb{Q}_p \to \mathbb{C}$ such that essential supremum of $|h| < \infty\}$, the set $C(\mathbb{Q}_p,\mathbb{C}) = \{h : \mathbb{Q}_p \to \mathbb{C}$ and h is a continous function$\}$, and the set

$$C_0(\mathbb{Q}_p,\mathbb{C}) = \{h : \mathbb{Q}_p \to \mathbb{C}\ and\ h\ is\ a\ continous\ function\ and\ \lim_{||\zeta||_p \to \infty} h(\zeta) = 0\}$$

are complex vector space under the binary operation vector addition (+) and scalar multiplication (.). It also implies that $(C_0(\mathbb{Q}_p,\mathbb{C}),\ ||.||_{L^\infty})$ is a Banach space.

4. Fractional Fourier Transform (FFT) on $\mathbb{Q}_p$

In this chapter, we introduce the definition of fractional Fourier transform on the field of p-adic numbers $\mathbb{Q}_p$. Firstly, the map $\chi_p^{\vartheta}(.\ ,\ .)$ is defined on $\mathbb{Q}_p$ as follows:

$$\chi_p^{\vartheta}(\zeta,\eta) = \begin{cases} C^{\vartheta} e^{\frac{i(\zeta^2+\eta^2)\cot\vartheta}{2} - i\zeta\eta\csc\vartheta}, & \vartheta \neq n\pi, n \in \mathbb{Z} \\ \frac{1}{\sqrt{2\pi}} e^{-i\zeta\eta}, & \vartheta = \frac{\pi}{2}, \end{cases} \quad \forall\ \zeta,\ \eta \in \mathbb{Q}_p$$
$$C^{\vartheta} = \sqrt{\frac{1-i\cot\vartheta}{2\pi}}.$$

If $\psi \in L^1(\mathbb{Q}_p)$, its fractional Fourier transform of one dimension is defined as follows:

$$(\mathcal{F}_{\vartheta}\psi)(\eta) = \widehat{\psi}_{\vartheta}(\eta) = \int_{\mathbb{Q}_p} \chi_p^{\vartheta}(\zeta,\eta)\psi(\zeta)d\zeta, \quad for\ \eta \in \mathbb{Q}_p. \tag{3}$$

The inverse fractional Fourier transform of a mapping $\phi \in L^1(\mathbb{Q}_p)$ is

$$(\mathcal{F}_{\vartheta}^{-1}\phi)(\zeta) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta,\eta)\phi(\eta)d\eta, \quad for\ \zeta \in \mathbb{Q}_p. \tag{4}$$

The fractional Fourier transform is an isomorphism, continuous and linear map of $\mathcal{D}(\mathbb{Q}_p)$ onto itself holding

$$(\mathcal{F}_{\vartheta}(\mathcal{F}_{\vartheta}^{-1}\phi))(\zeta) = (\mathcal{F}_{\vartheta}^{-1}(\mathcal{F}_{\vartheta}\phi))(\zeta) = \phi(\zeta),\ \forall\ \phi \in \mathcal{D}(\mathbb{Q}_p) \tag{5}$$

5. Symbol of Two Functions with FFT Connected to the Non-archimedean Pseudo-differential Operators (NPDO) $\mathscr{B}_{\psi_1,\psi_2}$

In this chapter we introduce different types of the p-adic non-archimedean pseudo-differential operators which is denoted by $\mathscr{B}_{\psi_1,\psi_2}$. We define fractional heat Kernel related to NPDO involving FFT. Some properties of fractional heat Kernel will be proved. Throughout this manuscript, we consider $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{N} = \{1,2,3,\dots\}$.

We assume the **Hypothesis B**, which is defined as follows: We call that the continuous functions $\psi_1,\ \psi_2 : \mathbb{Q}_p \to \mathbb{C}$ holds the Hypothesis B if the following conditions satisfy.

(a) ψ_1, ψ_2 are radial functions with $\psi_1(0) = 0, \psi_2(0) = 0$. We apply the notation $\psi_1(\eta) = \psi_1(|\eta|_p)$ and $\psi_2(\eta) = \psi_2(|\eta|_p);\ \eta \in \mathbb{Q}_p$.

(b) $|\psi_1|$ and ψ_2 are increasing mappings with respect to the p-adic absolute value $|.|_p$.

(c) There is an integer $\rho = \rho(\psi_1,\ \psi_2) \in \mathbb{Z}$ such that $|\psi_2(|\eta|_p)| \geq |\psi_1(|\eta|_p)| \iff \eta \in B_\rho$.

(d) There exist constants $\mathscr{C}_1 = \mathscr{C}_1(\psi_1) > 0,\ \gamma_1 = \gamma_1(\psi_1) > 0$ such that $|\psi_1(|\eta|_p)| \geq \mathscr{C}_1|\eta|_p^{\gamma_1}, \quad \forall\eta \in \mathbb{Q}_p - B_\rho$.

Example 5.1.

(a) We consider $\psi_1(||\eta||_p) = \sqrt{||\eta||_p}$ *and* $\psi_2(||\eta||_p) = \sqrt[3]{||\eta||_p}$, $\eta \in \mathbb{Q}$. *We get that* ψ_1 *and* ψ_2 *holds the Hypothesis B.*

(b) Let $\delta_1 > 0$ *and* $\delta_2 > 0$ *constants such that* $\delta_1 \geq \delta_2 > 0$. *Then the mappings* $\psi_1(||\eta||_p) = e^{||\eta||_p^{\delta_1}} - 1$ *and* $\psi_2(||\eta||_p) = e^{||\eta||_p^{\delta_2}} - 1$, $\eta \in \mathbb{Q}_p$, *holds the Hypothesis B.*

Remark 5.2. *We obtain that* $e^{-\tau[min\{|\psi_1|,\ |\psi_2|\}]} \in L^{\sigma}(\mathbb{Q}_p)$, $\quad \sigma \in [1,\ \infty)$, $\quad \forall$ *fixed* $\tau > 0$. *In fact,* $\int_{\mathbb{Q}_p} e^{-\tau[min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\}]} d\eta$ *is exactly*

$$\int_{B_\rho} e^{-\tau\sigma|\psi_2(||\eta||_p)|} d\eta + \int_{\mathbb{Q}_p - B_\rho} e^{-\tau\sigma|\psi_1(||\eta||_p)|} d\eta = \mathscr{I}_1 + \mathscr{I}_2.$$

It implies that $\mathscr{I}_1 < \infty$, *because* $e^{-\tau\sigma|\psi_2(.)|}$ *is a continuous over* B_ρ *and* B_ρ *is a compact set. Further,*

$$\begin{aligned} \mathscr{I}_2 &= \sum_{i=\rho+1}^{\infty} e^{-\tau\sigma|\psi_1(p^i)|} \int_{||\eta||_p = p^i} d\eta \\ &\leq (1 - p^{-1}) \sum_{i=\rho+1}^{\infty} e^{-\tau\sigma\mathscr{C}_1 p^{i\gamma_1}} p^i \\ &< \infty. \end{aligned}$$

Definition 5.3. *We introduce the non-archimedean pseudo-differential operator*

$$\begin{aligned} (\mathscr{B}_{\psi_1,\psi_2}\phi)(\zeta) &= -\mathcal{F}_{\alpha}^{-1}([min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\}]\widehat{\phi}_{\alpha}(\eta)) \\ &= -\int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta,\eta)[min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\}]\widehat{\phi}_{\vartheta}(\eta) d\eta, \end{aligned}$$

where $\zeta \in \mathbb{Q}_p$ *and* $\phi \in \mathscr{D}(\mathbb{Q}_p)$, *the mapping* $min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\} : \mathbb{Q}_p \to \mathbb{R}_+$, *is the symbol of the operator* $\mathscr{B}_{\psi_1,\psi_2}$.

Remark 5.4. *From [22] we get* $\mathscr{D}(\mathbb{Q}_p)$ *is dense in* $L^{\sigma}(\mathbb{Q}_p)$, $\quad \sigma \in [1,\ \infty)$ *and in* $C_0(\mathbb{Q}_p)$. *Since* $\widehat{\phi}_{\vartheta} \in \mathscr{D}(\mathbb{Q}_p)$ *and* $min\{|\psi_1(.)|,\ |\psi_2(.)|\}$ *is a continuous mapping on* $\mathbb{Q}_p$, *we obtain that* $min\{|\psi_1(.)|,\ |\psi_2(.)|\widehat{\phi}_{\vartheta} \in L^1(\mathbb{Q}_p).\}$ *From Riemann-Lebesgue Theorem [22], we can prove that the NPDO* $\mathscr{B}_{\psi_1,\psi_2} : \mathscr{D}(\mathbb{Q}_p) \to C_0(\mathbb{Q}_p)L^{\sigma}(\mathbb{Q}_p)$, $\quad \sigma \in [1,\ \infty)$, *is a well-defined NPDO.*

Definition 5.5. *We introduce fractional heat Kernel related to NPDO* $\mathscr{B}_{\psi_1,\psi_2}$ *involving FFT as follows:*

$$\mathscr{Z}_\tau(\zeta) = \mathscr{Z}(\zeta,\ \tau) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta,\eta) e^{-\tau[min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\}]} d\eta, \ \ \forall\ \zeta \in \mathbb{Q}_p,\ \tau > 0. \tag{6}$$

$\mathscr{Z}(\zeta,\ \tau)$ is a mapping of ζ with $\zeta \in \mathbb{Q}_p$ for fixed $\tau > 0$.

Remark 5.6. *From [23] we obtain for* $\zeta \in \mathbb{Q}_p$ *that* $\lim_{\tau \to 0^+} \mathscr{Z}_\tau(\zeta) = \delta(\zeta)$, *where* δ *is the Dirac delta function. From [23],* $e^{-\tau[min\{|\psi_1(.)|,\ |\psi_2(.)|\}]}$ *can be proved that it is a regular distribution as well as a radial function.*

Theorem 5.7. *The fractional heat Kernel $\mathscr{Z}_\tau(.)$ holds that $\mathscr{Z}(\zeta,\ \tau) \geq 0, \quad \forall \zeta \in \mathbb{Q}_p$ and $\tau > 0$.*

Proof.

Case I: When $\zeta = 0$. It is trivial that $\mathscr{Z}(\zeta,\ \tau) \geq 0$ holds for all $\tau > 0$.

Case II: When $\zeta \neq 0 \in \mathbb{Q}_p$. We obtain that

$$\begin{aligned}
\int_{B_\rho} \chi_p^{-\vartheta}(\zeta,\eta) e^{-\tau|\psi_2(||\eta||_p)|} d\eta + \int_{\mathbb{Q}_p - B_\rho} \chi_p^{-\vartheta}(\zeta,\eta) e^{-\tau\sigma|\psi_1(||\eta||_p)|} d\eta \\
= \sum_{i=-\infty}^{\rho} e^{-\tau|\psi_2(p^i)|} \int_{||p^i\eta||_p=1} \chi_p^{-\vartheta}(\zeta,\eta) d\eta \\
+ \sum_{i=\rho+1}^{\infty} e^{-\tau|\psi_1(p^i)|} \int_{||p^i\eta||_p=1} \chi_p^{-\vartheta}(\zeta,\eta) d\eta \\
= \sum_{i=-\infty}^{\rho} e^{-\tau|\psi_2(p^i)|} p^i \int_{||t||_p=1} \chi_p^{-\vartheta}(\zeta,p^{-i}t) dt \\
+ \sum_{i=\rho+1}^{\infty} e^{-\tau|\psi_1(p^i)|} p^i \int_{||t||_p=1} \chi_p^{-\vartheta}(\zeta,p^{-i}t) dt.
\end{aligned}$$

It implies that for $\zeta \neq 0$ and $\tau > 0$, $\mathscr{Z}(\zeta,\tau)$ is equal to

$$\sum_{i=-\infty}^{\rho} e^{-\tau|\psi_2(p^i)|} p^i \int_{||t||_p=1} \chi_p^{-\vartheta}(\zeta,p^{-i}t) dt + \sum_{i=\rho+1}^{\infty} e^{-\tau|\psi_1(p^i)|} p^i \int_{||t||_p=1} \chi_p^{-\vartheta}(\zeta,p^{-i}t) dt. \tag{7}$$

For $\zeta \neq 0 \in \mathbb{Q}_p$ with $||\zeta||_p = p^{-l},\ \ l \in \mathbb{Z}$, and applying the formula

$$\int_{||t||_p=1} \chi_p^{-\vartheta}(\zeta,p^{-i}t) dt = \begin{cases} 1-p^{-1}, & \text{if } \ i \leq l, \\ -p^{-1}, & \text{if } \ i = l+1, \\ 0, & \text{if } \ i \geq l+2, \end{cases} \tag{8}$$

There are different cases for l. Firstly we consider $l \leq \rho - 1$, then from (7) and (8) we get

$$\begin{aligned}
\mathscr{Z}(\zeta,\tau) &= (1-p^{-1}) \sum_{i=-l}^{\infty} e^{-\tau|\psi_2(p^{-i})|} p^{-i} - e^{-\tau|\psi_2(p^{l+1})|} p^l \\
&\geq e^{-\tau|\psi_2(p^l)|} \sum_{i=-l}^{\infty} (p^{-i} - p^{-(i+1)}) - e^{-\tau|\psi_2(p^{l+1})|} p^l \\
&= ||\zeta||_p^{-1} \{ e^{-\tau|\psi_2(||\zeta||_p^{-1})|} - e^{-\tau|\psi_2(||\zeta||_p^{-1}p)|} \} \\
&\geq 0.
\end{aligned}$$

If $l \geq \rho$, from (7) and (8) and using the mathematical induction on $l - \rho$, we get

If $l - \rho = 0$, then

$$\mathscr{Z}(\zeta,\tau) = (1-p^{-1}) \sum_{i=-l}^{\infty} e^{-\tau|\psi_2(p^{-i})|} p^{-i} - e^{-\tau|\psi_1(p^{l+1})|} p^l$$

$$
\begin{aligned}
&\geq e^{-\tau|\psi_2(p^l)|}\sum_{i=-l}^{\infty}(p^{-i}-p^{-(i+1)})-e^{-\tau|\psi_1(p^{l+1})|}p^l \\
&= ||\zeta||_p^{-1}\{e^{-\tau|\psi_2(||\zeta||_p^{-1})|}-e^{-\tau|\psi_1(||\zeta||_p^{-1}p)|}\} \\
&\geq ||\zeta||_p^{-1}\{e^{-\tau|\psi_2(||\zeta||_p^{-1})p|}-e^{-\tau|\psi_1(||\zeta||_p^{-1}p)|}\} \\
&\geq 0.
\end{aligned}
$$

Here we suppose that the assertion is true for $l-\rho=q$, for some $q\geq 1$, i.e.,

$$
\begin{aligned}
\mathscr{Z}(\zeta,\tau) &= (1-p^{-1})\sum_{i=-\infty}^{\rho}e^{-\tau|\psi_2(p^i)|}p^i+(1-p^{-1})\sum_{i=\rho+1}^{\rho+q}e^{-\tau|\psi_1(p^i)|}p^i-p^{-1}e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q+1} \\
&\geq 0.
\end{aligned}
$$

If the hypothesis for $l-\rho=q+1$ or $l=\rho+q+1$, we obtain that

$$
\begin{aligned}
\mathscr{Z}(\zeta,\tau) &= (1-p^{-1})\sum_{i=-\infty}^{\rho}e^{-\tau|\psi_2(p^i)|}p^i+(1-p^{-1})\sum_{i=\rho+1}^{\rho+q+1}e^{-\tau|\psi_1(p^i)|}p^j-p^{-1}e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q+2} \\
&= (1-p^{-1})\sum_{i=-\infty}^{\rho}e^{-\tau|\psi_2(p^i)|}p^i+(1-p^{-1})\sum_{i=\rho+1}^{\rho+q}e^{-\tau|\psi_1(p^i)|}p^j \\
&+ (1-p^{-1})e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q+1}-p^{-1}e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q+2} \\
&\geq (1-p^{-1})\sum_{i=-\infty}^{\rho}e^{-\tau|\psi_2(p^i)|}p^i+(1-p^{-1})\sum_{i=\rho+1}^{\rho+q}e^{-\tau|\psi_1(p^i)|}p^j \\
&+ (1-p^{-1})e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q+1}-e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q+1} \\
&= (1-p^{-1})\sum_{i=-\infty}^{\rho}e^{-\tau|\psi_2(p^i)|}p^i+(1-p^{-1})\sum_{i=\rho+1}^{\rho+q}e^{-\tau|\psi_1(p^i)|}p^j-e^{-\tau|\psi_1(p^{\rho+q+1})|}p^{\rho+q} \\
&\geq 0.
\end{aligned}
$$

This completes the proof of the theorem. □

Remark 5.8. *A mapping $f:\mathbb{Q}_p\to\mathbb{C}$ is said to be negative definite, if*

$$\sum_{l,m=1}^{q}\big(f(\zeta_l)+\overline{f(\zeta_m)}-f(\zeta_l-\zeta_m)\big)\gamma_l\overline{\gamma_m}\geq 0$$

$\forall\ q\neq 0\in\mathbb{N},\ \zeta_1,\ \zeta_2,\ldots,\zeta_q\in\mathbb{Q}_p,\ \ \gamma_1,\ \gamma_2,\ldots,\gamma_q\in\mathbb{C}$. *From Theorem 5.7 and [24] we obtain that the mapping min$\{|\psi_1(||.||_p)|,\ |\psi_2(||.||_p)|\}$ is negative definite on $\mathbb{Q}_p$.*

Remark 5.9. *We introduce the Cauchy problem*

$$
\begin{cases}
\frac{\partial\phi}{\partial\tau}(\zeta,\ \tau)=-\mathscr{B}_{\psi_1,\psi_2}(\phi(\zeta,\tau)),\ \ 0<\tau<\infty,\ \ \zeta\in\mathbb{Q}_p \\
\phi(\zeta,0)=\phi_0(\zeta)\in\mathscr{D}(\mathbb{Q}_p).
\end{cases}
\tag{9}
$$

Similarly from [12], we can show that

$$\phi(\zeta,\tau) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta,\eta) e^{-\tau[min\{|\psi_1(||\eta||_p)|,\ |\psi_2(||\eta||_p)|\}]} (\mathcal{F}_\vartheta \phi_0)(\eta) d\eta, \ \ \zeta \in \mathbb{Q}_p, \ \tau \geq 0$$

is a basic solution of (9). Further,

(a) $\phi(\zeta,\tau)$ *holds the principles of mass conservation and comparison i.e.,* $\int_{\mathbb{Q}_p} \phi(\zeta,\tau) d\zeta = \int_{\mathbb{Q}_p} \phi_0(\zeta) d\zeta$ *and if* $\phi_0(\zeta) \geq \varphi_0(\zeta), \ \ \forall \zeta \in \mathbb{Q}_p$, *then* $\phi(\zeta,\tau) \geq \varphi(\zeta,\tau)$.

6. Some Properties of Fractional Heat Kernel $\mathscr{Z}(\zeta,\tau)$ on $\mathbb{Q}_p$

In this section, motivated by Taira Kazuaki [25] we introduce the existence of some properties such as Feller semigroups and Strong Markov processes related to fractional Fourier transform on $\mathbb{Q}_p$. Using the fractional heat Kernel given in (6), and for $\phi \in C_0(\mathbb{Q}_p)$, $\zeta \in \mathbb{Q}_p$, the operator Ψ_τ is defined as follows:

$$\Psi_\tau \phi(\zeta) = \begin{cases} \phi(\zeta), & \text{if } \tau = 0, \\ \int_{\mathbb{Q}_p} \mathscr{Z}(\zeta-\eta,\tau)\phi(\eta) d\eta, & \text{if } \tau > 0. \end{cases} \tag{10}$$

Theorem 6.1. $\Psi_\tau : C_0(\mathbb{Q}_p) \to C_0(\mathbb{Q}_p), \ \ \tau \geq 0$ *is well-defined contraction operator.*

Proof. When $\tau = 0$, the result is obvious/trivial.
When $\tau > 0$, $\phi \in C_0(\mathbb{Q}_p)$ and $\zeta \in \mathbb{Q}_p$.

$$\begin{aligned} |\Psi_\tau \phi(\zeta)| = \left| \int_{\mathbb{Q}_p} \mathscr{Z}(\zeta-\eta,\tau)\phi(\eta) d\eta \right| &\leq ||\phi||_{L^\infty(\mathbb{Q}_p)} \int_{\mathbb{Q}_p} \mathscr{Z}(\zeta-\eta,\tau) d\eta \\ &= ||\phi||_{L^\infty(\mathbb{Q}_p)}. \end{aligned} \tag{11}$$

From [23] similary, we can show that $\mathscr{Z}_\tau$ is continous. Since $\phi \in C_0(\mathbb{Q}_p)$ and we consider that $Supp(\phi) \subset \mathscr{B}_m, \ \ m \in \mathbb{Z}$. Applying the compactness of $\mathscr{B}_m$ and Theorem 5.7, we obtain

$$\begin{aligned} 0 \leq |\Psi_\tau \phi(\zeta)| &\leq ||\phi||_{L^\infty(\mathbb{Q}_p)} \int_{\mathscr{B}_m} \mathscr{Z}(\zeta-\eta,\tau) d\eta \\ &\leq \tau ||\phi||_{L^\infty(\mathbb{Q}_p)} \int_{\mathscr{B}_m} ||\zeta-\eta||_p^{-1} d\eta \\ &= \tau ||\phi||_{L^\infty(\mathbb{Q}_p)} ||\zeta||_p^{-1} Vol(\mathscr{B}_m) \\ &= 0. \end{aligned}$$

It implies that $|\Psi_\tau \phi(\zeta)| \to 0$ as $||\zeta||_p \to \infty$. This proves that $\Psi_\tau : C_0(\mathbb{Q}_p) \to C_0(\mathbb{Q}_p)$ is a well-defined bounded linear transform. ☐

Theorem 6.2. *The operators $\Psi_\tau,\ \tau \in [0,\infty)$, holds the semigroup axiom,*

$$\Psi_\tau(\Psi_\omega\phi)(\zeta) = \Psi_{\tau+\omega}\phi(\zeta), \quad \forall\phi \in C_0(\mathbb{Q}_p), \quad \forall\zeta \in \mathbb{Q}_p, \ \ \tau,\ \omega\ \in [0,\infty).$$

Proof. Proof is similar to Lemma 2 in [19]. □

Theorem 6.3. *The operators $\Psi_\tau,\ \ \tau \in [0,\infty)$, holds the following property for fixed $\phi \in C_0(\mathbb{Q}_p)$:*

$$||\Psi_\tau\phi - \phi||_{L^\infty(\mathbb{Q}_p)} \to 0 \ \ as \ \ \tau \to 0^+. \tag{12}$$

Proof. We consider

$$\big(\Psi_\tau\phi - \phi\big)(\zeta) = \int_{\mathbb{Q}_p} \mathscr{Z}(\zeta-\eta,\tau)[\phi(\zeta)-\phi(\eta)]d\eta, \quad \text{for fixed } \zeta \in \mathbb{Q}_p. \tag{13}$$

For a given arbitrary number $\epsilon > 0$, however small, $\exists$ some numbers $l(\zeta,\epsilon) \in \mathbb{Z}$ such that $||\phi(\zeta) - \phi(\eta)||_{L^\infty(\mathbb{Q}_p)} < \epsilon$, for $||\zeta-\eta||_p < p^l$. From Theorem 5.7 and (13), we obtain that

$$\begin{aligned}
\Big|\big(\Psi_\tau\phi - \phi\big)(\zeta)\Big| &\le \int_{||\zeta-\eta||_p<p^l} \mathscr{Z}(\zeta-\eta,\tau)|\phi(\zeta)-\phi(\eta)|d\eta \\
&+ \int_{||\zeta-\eta||_p\ge p^l} \mathscr{Z}(\zeta-\eta,\tau)|\phi(\zeta)-\phi(\eta)|d\eta \\
&\le \epsilon + 2||\phi||_{L^\infty(\mathbb{Q}_p)} \int_{||\xi||_p\ge p^l} \mathscr{Z}(\xi,\tau)d\xi \\
&\le \epsilon + 2\tau||\phi||_{L^\infty(\mathbb{Q}_p)} \int_{||\xi||_p\ge p^l} ||\xi||_p^{-1}d\xi.
\end{aligned}$$

Since $\int_{||\xi||_p\ge p^l} ||\xi||_p^{-1}d\xi = K < \infty$ (say), then we get

$$\Big|\big(\Psi_\tau\phi - \phi\big)(\zeta)\Big| \le \epsilon + 2K\tau||\phi||_{L^\infty(\mathbb{Q}_p)}.$$

Hence, given any $\epsilon > 0$ we obtain that

$$\lim_{\tau\to0^+} max\Big|\big(\Psi_\tau\phi - \phi\big)(\zeta)\Big| \le \epsilon, \ \ \forall\zeta \in \mathbb{Q}_p.$$

Thus,

$$||\Psi_\tau\phi - \phi||_{L^\infty(\mathbb{Q}_p)} \to 0 \ \ \text{as} \ \ \tau \to 0^+.$$

This completes the proof of the Theorem 6.3. □

Definition 6.4. *An operator Φ in a complete norm linear space $\big(\mathbb{N}(\mathbb{F}),\ ||.||\big)$ is said to be dissipative if $||\psi - \gamma\Phi\psi|| \ge ||\psi||, \quad \forall\psi \in D(\Phi)$ and for all $\gamma \in]0,\infty[$.*

Theorem 6.5. *The non-archimedean pseudo-differential operators $\mathscr{B}_{\psi_1,\psi_2}$ is dissipative in $L^2(\mathbb{Q}_p)$.*

Proof. From [23] we get for $\phi \in \mathscr{D}(\mathbb{Q}_p)$ that

$$\begin{aligned}
\left(\mathscr{B}_{\psi_1,\psi_2}\phi, \phi\right) &= -\Big(\mathcal{F}_{\vartheta}^{-1}(min\{\psi_1, \psi_2\}]\widehat{\phi}_{\vartheta}), \phi\Big) \\
&= -(min\{\psi_1, \psi_2\}\widehat{\phi}_{\vartheta}, \widehat{\phi}_{\vartheta}) \\
&= -\int_{\mathbb{Q}_p} min\{\psi_1, \psi_2\}|\widehat{\phi}_{\vartheta}(\zeta)|^2 d\zeta \\
&\leq 0.
\end{aligned}$$

Therefore, the non-archimedean pseudo-differential operators $\mathscr{B}_{\psi_1,\psi_2}$ is dissipative in $L^2(\mathbb{Q}_p)$. □

Definition 6.6. *[26] If Φ is a linear operator in the complete inner product space $\big(\mathbb{H}(K), \langle ., . \rangle\big)$ with dense domain, then*

$$G(\Phi^*) = \{(\psi, \phi) \in \mathbb{H} \times \mathbb{H} : \langle \phi, g \rangle = \langle \psi, h \rangle \ \forall \ (g, h) \in G(\Phi)\},$$

introduces a linear operator Φ^* (the adjoint of Φ). The domain of Φ^* is

$$D(\Phi^*) = \{g \in \mathbb{H} : \exists C < \infty, \ |\langle \Phi h, g \rangle| \leq C||h||, \ \ \forall h \in D(\Phi)\},$$

and Φ^* satisfies $\langle \Phi^* g, h \rangle = \langle g, \Phi h \rangle$, $\ \forall h \in D(\Phi)$. Then, Φ is said to be self-adjoint if $\Phi^* = \Phi$.

Theorem 6.7. *The non-archimedean pseudo-differential operators $\mathscr{B}_{\psi_1,\psi_2}$ is a self-adjoint operator i.e. to prove that*

$$\left\langle \mathscr{B}_{\psi_1,\psi_2}\phi, \varphi \right\rangle = \left\langle \phi, \mathscr{B}_{\psi_1,\psi_2}\varphi \right\rangle, \quad \textit{for } \phi, \varphi \in \mathscr{D}(\mathbb{Q}_p).$$

Proof. We consider for $\phi, \ \varphi \in \mathscr{D}_p(\mathbb{Q}_p)$ and using of Parseval-Steklov equality that

$$\begin{aligned}
\left\langle \mathscr{B}_{\psi_1,\psi_2}\phi, \varphi \right\rangle &= \left\langle -\mathcal{F}_{\alpha}^{-1}(\min\{|\psi_1|, \ |\psi_2|\}\widehat{\phi}_{\vartheta}), \varphi \right\rangle \\
&= -\int_{\mathbb{Q}_p} \widehat{\phi_{\vartheta}}(\zeta)\overline{\min\{|\psi_1(\zeta)|, \ |\psi_2(\zeta)|\}\widehat{\varphi}_{\vartheta}(\zeta)} d\zeta \\
&= \langle \phi, \ -\mathcal{F}_{\vartheta}^{-1}(\min\{|\psi_1|, |\psi_2|\})\widehat{\varphi}_{\vartheta} \rangle \\
&= \left\langle \phi, \mathscr{B}_{\psi_1,\psi_2}\varphi \right\rangle.
\end{aligned}$$

It implies that the non-archimedean pseudo-differential operators $\mathscr{B}_{\psi_1,\psi_2}$ is a self-adjoint operator. □

References

[1] Alexandra V Antoniouk, Andrei Yu Khrennikov and Anatoly N Kochubei, *Multidimensional nonlinear pseudo-differential evolution equation with p-adic spatial variables*, Journal of Pseudo-Differential Operators and Applications, 11(2020), 311–343.

[2] Andrei Khrennikov, Klaudia Oleschko and Maria de Jesus Correa Lopez, *Modeling fluid's dynamics with master equations in ultrametric spaces representing the treelike structure of capillary networks*, Entropy, 18(7)(2016).

[3] Klaudia Oleschko and A Yu Khrennikov, *Applications of p-adics to geophysics: Linear and quasilinear diffusion of water-in-oil and oil-in-water emulsions*, Theoretical and mathematical physics, 190(1)(2017), 154-163.

[4] Ehsan Pourhadi, Andrei Khrennikov, Reza Saadati, Klaudia Oleschko, and Maria de Jesus Correa Lopez, *Solvability of the p-adic analogue of navierstokes equation via the wavelet theory*, Entropy, 21(11)(2019).

[5] Wilson A Zuniga-Galindo, *Pseudodifferential equations over non archimedean spaces*, Volume 2174, Springer, (2016).

[6] Victor A Aguilar-Arteaga and Samuel Estala-Arias, *Pseudodifferential operators and markov processes on adeles*, p-Adic Numbers, Ultrametric Analysis and Applications, 11(2019), 89-113.

[7] Victor A Aguilar-Arteaga, Manuel Cruz-López and Samuel Estala-Arias, *Non-archimedean analysis and a wave-type pseudodifferential equation on finite adeles*, Journal of Pseudo-Differential Operators and Applications, 11(3)(2020), 1139-1181.

[8] Alexandra V Antoniouk, Klaudia Oleschko, Anatoly N Kochubei and Andrei Yu Khrennikov, *A stochastic p-adic model of the capillary flow in porous random medium*, Physica A: Statistical Mechanics and its Applications, 505(2018), 763-777.

[9] Ismael Gutierrez Garcia and Anselmo Torresblanca-Badillo, *Strong markov processes and negative definite functions associated with non-archimedean elliptic pseudo-differential operators*, Journal of Pseudo-Differential Operators and Applications, 11(1)(2020), 345-362.

[10] Ismael Gutierrez Garcia and Anselmo Torresblanca-Badillo, *Some classes of non-archimedean pseudo-differential operators related to bessel potentials*, Journal of Pseudo-Differential Operators and Applications, 11(3)(2020), 1111-1137.

[11] Anselmo Torresblanca-Badillo and WA Zuniga-Galindo, *Non-archimedean pseudodifferential operators and feller semigroups*, p-Adic Numbers, Ultrametric Analysis and Applications, 10(2018), 57-73.

[12] Anselmo Torresblanca-Badillo and WA Zuniga-Galindo, *Ultrametric diffusion, exponential landscapes, and the first passage time problem*, Acta Applicandae Mathematicae, 157(1)(2018), 93-116.

[13] LF Chacon-Cortes, Ismael Gutierrez-Garcia, Anselmo Torresblanca-Badillo and Andres Vargas, *Finite time blow-up for a p-adic nonlocal semilinear ultradiffusion equation*, Journal of Mathematical Analysis and Applications, 494(2)(2021), 124599.

[14] Andrei Khrennikov and Klaudia Oleschko, *An ultrametric random walk model for disease spread taking into account social clustering of the population*, Entropy, 22(9)(2020), 931.

[15] Anselmo Torresblanca-Badillo, *Non-archimedean generalized bessel potentials and their applications*, Journal of Mathematical Analysis and Applications, 497(2)(2021), 124874.

[16] Anselmo Torresblanca-Badillo, *Non-archimedean pseudo-differential operators on sobolev spaces related to negative definite functions*, Journal of Pseudo-Differential Operators and Applications, 12(1)(2021).

[17] WA Zuniga-Galindo and Sergii M Torba, *Non-archimedean coulomb gases*, Journal of Mathematical Physics, 61(1)(2020), 013504.

[18] Anselmo Torresblanca-Badillo and Edwin A Bolano-Benitez, *New classes of p-adic evolution equations and their applications*, Journal of PseudoDifferential Operators and Applications, 14(1)(2023), 12.

[19] Anselmo Torresblanca-Badillo and Adriana A Albarracín-Mantilla, *Some further classes of pseudo-differential operators in the p-adic context and their applications*, Journal of Pseudo-Differential Operators and Applications, 14(2)(2023), 24.

[20] V Aguilar-Arteaga, García I Gutiérrez and Anselmo Torresblanca Badillo, *Energy landscapes and non-archimedean pseudo-differential operators as tools for studying the spreading of infectious diseases in a situation of extreme social isolation*, Kragujev. J. Math., 48(6)(2024), 827-844.

[21] Andrei Khrennikov, *Utrametric diffusion equation on energy landscape to model disease spread in hierarchic socially clustered population*, (2021).

[22] M. H. Taibleson, *Fourier analysis on local fields*, Princeton University Press, (1975).

[23] Sergio Albeverio, A Yu Khrennikov and Vladimir M Shelkovich, *Theory of p-adic distributions: linear and nonlinear models*, Number 370. Cambridge University Press, (2010).

[24] Christian van den Berg and Gunnar Forst, *Potential theory on locally compact abelian groups*, Volume 87, Springer Science & Business Media, (2012).

[25] Kazuaki Taira, *Boundary value problems and markov processes*, Lecture Notes in Mathematics, 1499, (1991).

[26] Thierry Cazenave and Alain Haraux, *An introduction to semilinear evolution equations*, Volume 13, Oxford University Press on Demand, (1998).