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Impact Response of Non-Homogeneous Layer Bonded to an Elastic Homogeneous Half-Space

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Abstract

Impact response of a non-homogeneous layer bonded to an elastic homogeneous half-space is considered in this paper. Sudden torsion is applied to the non-homogeneous layer over a bonded rigid circular disc. Using Laplace and Hankel transforms the mixed boundary value problem is reduced into Fredholm integral equations of second kind. Solving the integral equations, the analytical expression of tortional impact is obtained in the Laplace transform domain. Using Laplace inversion technique numerical values of torque at the surface of the rigid disk are obtained and graphically plotted.

Keywords: Impact Response; Laplace and Hankel transforms; Fredholm integral equation; Laplace Inversion.

2020 Mathematics Subject Classification: 74B99.

1. Introduction

The study of contact problem in solid mechanics is of great interest as its wide application in geomechanics. Many researchers have studied this kind problems. One such type of problems were studied by Eason [3], Shail [8]. They have considered the sudden torsional impact problem in half-space. The problem of torsional impact of a thick elastic plate was investigated by Toshikazu Shabuya [9]. The torsional oscillation of a rigid circular disc attached to an elastic layer bonded to an elastic half-space has been considered by Keer et al. [5]. Basu and Mandal [1] have studied the impact of tortional load on a penny shaped crack in an elastic layer sandwiched between two elastic half space. Rahimian et al. [7] considered the Reissner-Sagoci problem for a transversely isotropic half-space. The Reissner-Sagoci type problem for a non-homogeneous elastic cylinder embedded in an elastic non-homogeneous half-space was studied by Singh, Dhaliwal and Vrbik [10] In this paper the problem of torsional impact of a non-homogeneous layer, with rigidity modulus μ_1 and density ρ_1 given by the law $(\mu_1, \rho_1) = (\mu_0, \rho_0)\, e^{-\beta' z}$ bonded to an elastic homogeneous half-space with rigidity

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modulus μ_2 and density ρ_2 is considered. The torsional impact is applied over the circular region by a bonded rigid disc. The geometry and coordinate system are shown in Figure 1. Torsional impact for the case of homogeneous layer bonded to homogeneous half space was discussed by R. S. Dhaliwal, B. M. Singh, Jan Vrbik [2]. Making use of Laplace and Hankel transforms the solution of the problem is reduced into Fredholm integral equation of second kind. Laplace Inversion is done using numerical Laplace inversion method to find the applied torque and presented by means of graph for different parameters.

2. Basic Equations

In the problem cylindrical co-ordinate system (r, θ, z) is used. In the dynamic problem of anti-plane shear, there exists a single non-vanishing component of displacement in the θ direction and independent of θ, i.e.

$$u_r = 0 = u_z, \quad u_\theta = u_\theta(r, z, t) \tag{1}$$

where u_r, u_θ, u_z are the displacement components in the r, θ, z directions respectively. Hence the corresponding stress components that are different from zero can be obtained from u_θ as

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{2}$$

$$\sigma_{\theta z} = \mu\left(\frac{\partial u_\theta}{\partial z}\right) \tag{3}$$

where μ denotes the shear modulus of the material. We denote all the physical quantities for the layer by superscript 1 and for the half-space by superscript 2. Two of the equations of motions are identically satisfied and the remaining one gives

$$\frac{\partial \sigma_{r\theta}}{\partial r} + 2\frac{\sigma_{r\theta}}{r} + \frac{\partial \sigma_{\theta z}}{\partial z} = \rho \ddot{u}_\theta \tag{4}$$

For the layer this equation becomes

$$\frac{\partial^2 u^1{}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u^1{}_\theta}{\partial r} - \frac{u^1{}_\theta}{r^2} + \frac{\partial^2 u^1{}_\theta}{\partial z^2} - \beta'\frac{\partial u^1{}_\theta}{\partial z} = \frac{\rho_0}{\mu_0}\frac{\partial^2 u^1{}_\theta}{\partial t^2} = \frac{1}{c_1{}^2}\frac{\partial^2 u^1{}_\theta}{\partial t^2} \tag{5}$$

where $(\mu_1, \rho_1) = (\mu_0, \rho_0)\, e^{-\beta' z}$, $c_1 = \sqrt{\frac{\mu_0}{\rho_0}}$ and for the half-space equation (4) becomes

$$\frac{\partial^2 u^2{}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u^2{}_\theta}{\partial r} - \frac{u^2{}_\theta}{r^2} + \frac{\partial^2 u^2{}_\theta}{\partial z^2} = \frac{1}{c_2{}^2}\frac{\partial^2 u^2{}_\theta}{\partial t^2}, \text{where } c_2 = \sqrt{\frac{\mu_2}{\rho_2}} \tag{6}$$

To remove the time variable from equations (5) and (6) we introduce the Laplace transform by

$$\overline{f(p)} = \int_0^\infty f(t) e^{-pt} dt \tag{7}$$

$$f(t) = \frac{1}{2\pi i}\int_{B_r} \overline{f(p)} e^{pt} dp \tag{8}$$

The path of integration in equation (8) is Bromwich path which is a line on the r.h.s. and parallel to the imaginary axis of the p-plane. The Laplace transform of equation (5) and (6) give

$$\frac{\partial^2 \overline{u^1}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u^1}_\theta}{\partial r} - \frac{\overline{u^1}_\theta}{r^2} + \frac{\partial^2 \overline{u^1}_\theta}{\partial z^2} - \beta' \frac{\partial \overline{u^1}_\theta}{\partial z} = \frac{p^2}{{c_1}^2}\overline{u^1}_\theta \tag{9}$$

and

$$\frac{\partial^2 \overline{u^2}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u^2}_\theta}{\partial r} - \frac{\overline{u^2}_\theta}{r^2} + \frac{\partial^2 \overline{u^2}_\theta}{\partial z^2} = \frac{p^2}{{c_2}^2}\overline{u^2}_\theta \tag{10}$$

3. Formulation and Solution of the Problem

Consider the layer $(0 < z < h)$ of non-homogeneous isotropic elastic material of thickness h bonded to an elastic homogeneous isotropic half-space $(z > h)$. It is assumed that the rigid disc of radius a is attached to the open face of the layer. At time $t = 0$, the disc is impulsively twisted through an angle α. The boundary conditions of the problem can be written in the following form for $t > 0$

$$u^1{}_\theta(r, 0, t) = \alpha r H(t) \quad 0 \le r < a \tag{11}$$

$$\sigma^1{}_{\theta z}(r, 0, t) = 0 \qquad r > a \tag{12}$$

where $H(t)$ denotes Heavy-side unit function.

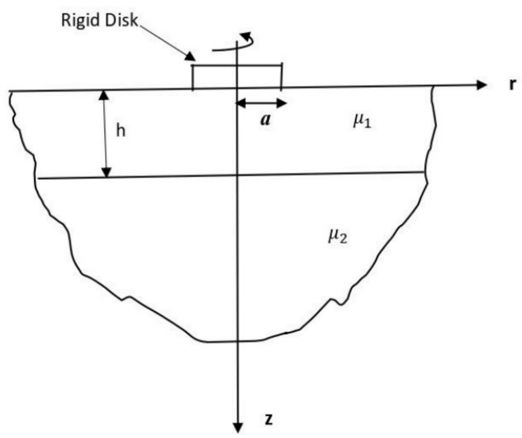

Figure 1: Geometry of the problem

In addition to the above boundary conditions, we have the following continuity conditions at $z = h$:

$$u^1{}_\theta(r, h, t) = u^2{}_\theta(r, h, t) \tag{13}$$

$$\sigma^1{}_{\theta z}(r, h, t) = \sigma^2{}_{\theta z}(r, h, t) \tag{14}$$

Now for simplicity we take $\beta' = 2\beta$. Solving equation (9) and (10) we find that Laplace transforms of the displacement component for the regions 1 $(0 < z < h)$ and 2 $(z > h)$ in the following form:

$$\overline{u^1}_\theta (r,z,p) = \int_0^\infty \left[A_1 (s,p) e^{-(\beta+\gamma_1)z} + A_2 (s,p) e^{(-\beta+\gamma_1)z}\right] J_1 (rs) \, ds \tag{15}$$

$$\overline{u^2}_\theta (r,z,p) = \int_0^\infty A_3 (s,p) e^{-\gamma_2 (z-h)} J_1 (rs) \, ds \tag{16}$$

where $J_n()$ is a Bessel function of first kind for $n \geq 0$ and

$$\begin{aligned} \gamma_1 &= \sqrt{(\beta^2 + s^2 + p^2/c_1{}^2)}, \qquad & \gamma_2 &= \sqrt{(s^2 + p^2/c_2{}^2)} \\ c_1 &= (\mu_0/\rho_0)^{1/2}, & c_2 &= (\mu_2/\rho_2)^{1/2} \end{aligned} \tag{17}$$

The Laplace transform of the boundary and the continuity conditions in equation (11)-(14) are

$$\overline{u^1}_\theta (r,0,p) = \alpha r/p \qquad 0 \leq r < a \tag{18}$$

$$\overline{\sigma^1}_{\theta z} (r,0,p) = 0 \qquad r > a \tag{19}$$

$$\overline{u^1}_\theta (r,h,p) = \overline{u^2}_\theta (r,h,p) \tag{20}$$

$$\overline{\sigma^1}_{\theta z} (r,h,p) = \overline{\sigma^2}_{\theta z} (r,h,p) \tag{21}$$

From the conditions (20) and (21) we get

$$A_3 = \frac{2\gamma_1 e^{-2\beta h}}{(-\beta+\gamma_1) e^{-2\beta h} + G\gamma_2} A_1 e^{-(\beta+\gamma_1)h} \tag{22}$$

$$A_2 e^{(-\beta+\gamma_1)h} = \frac{(\beta+\gamma_1) e^{-2\beta h} - G\gamma_2}{(-\beta+\gamma_1) e^{-2\beta h} + G\gamma_2} A_1 e^{-(\beta+\gamma_1)h} \tag{23}$$

where

$$G = \mu_2/\mu_0 \tag{24}$$

with the help of equations (3), (15), (22) and (23) we find from the conditions(18) and (19) that

$$\int_0^\infty B_1 (s,p) J_1 (rs) \, ds - 2\int_0^\infty B_1 (s,p) \left[\frac{1}{\frac{\gamma_1-\beta}{\gamma_1} + \frac{\gamma_1+\beta}{\gamma_1}\left(\frac{G\gamma_2+(\gamma_1-\beta)e^{-2\beta h}}{G\gamma_2-(\gamma_1+\beta)e^{-2\beta h}} e^{2\gamma_1 h}\right)}\right] J_1 (rs) \, ds = \frac{\alpha r}{p} \quad 0 < r < a \tag{25}$$

$$\int_0^\infty (\beta+\gamma_1) B_1 (s,p) J_1 (rs) \, ds = 0 \quad r > a \tag{26}$$

where

$$B_1 (s,p) = -A_1(s,p) \left[\frac{(\gamma_1-\beta)\left(e^{-2\gamma_1 h} - 1\right) - G\gamma_2 \left(1 + \frac{\gamma_1-\beta}{\gamma_1+\beta} e^{-2\gamma_1 h}\right)}{(\gamma_1-\beta) e^{-\beta h} + G\gamma_2}\right] \tag{27}$$

As usual the solution of dual integral equations (25) and (26) can be written in the following form:

$$(\gamma_1 + \beta)\, B_1(s, p) = \frac{2s}{\pi p} \int_0^a \varphi(\xi, p) \sin(s\xi)\, d\xi \tag{28}$$

where $\varphi(\xi, p)$ satisfies the following Fredholm integral equation of second kind:

$$\varphi(\xi, p) + \int_0^a \varphi(u, p) K_1(u, \xi, p)\, du + \int_0^a \varphi(u, p) K_2(u, \xi, p)\, du = 2\alpha\xi, 0 < \xi < a \tag{29}$$

where

$$K_1(u, \xi, p) = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{\beta + \gamma_1} - 1 \right) \sin(s\xi) \sin(us)\, ds \tag{30}$$

$$K_2(u, \xi, p) = -\frac{4}{\pi} \int_0^\infty \frac{s}{\beta + \gamma_1} \left[\frac{1}{\frac{\gamma_1 - \beta}{\gamma_1} + \frac{\gamma_1 + \beta}{\gamma_1} \left(\frac{G\gamma_2 + (\gamma_1 - \beta)e^{-2\beta h}}{G\gamma_2 - (\gamma_1 + \beta)e^{-2\beta h}} e^{2\gamma_1 h} \right)} \right] \sin(s\xi) \sin(us)\, ds \tag{31}$$

Normalising with the help of the following transformation of variables

$$\xi = a\xi_1, u = au_1, s = \frac{s_1}{a}, P = \frac{ap}{c_2}, k = \frac{c_2}{c_1} = \sqrt{\frac{\mu_2\rho_0}{\mu_0\rho_2}}$$

$$\varphi(a\xi_1, p) = 2a\alpha\varphi_1(\xi_1, P), \beta = \frac{\beta_1}{a}, \gamma_1{}' = \sqrt{\left({\beta_1}^2 + {s_1}^2 + k^2P^2\right)}, \gamma_2{}' = \sqrt{({s_1}^2 + P^2)} \tag{32}$$

we get

$$\varphi_1(\xi_1, P) + \int_0^1 \varphi_1(u_1, P) K_1(u_1, \xi_1, P)\, du_1 + \int_0^1 \varphi_1(u_1, P) K_2(u_1, \xi_1, P)\, du_1 = \xi_1 \quad 0 < \xi_1 < 1 \tag{33}$$

where

$$K_1(u_1, \xi_1, P) = \frac{2}{\pi} \int_0^\infty \left(\frac{s_1}{\beta_1 + \gamma_1{}'} - 1 \right) \sin(s_1\xi_1) \sin(u_1 s_1)\, ds_1 \tag{34}$$

$$K_2(u_1, \xi_1, P) = -\frac{4}{\pi} \int_0^\infty \frac{s_1}{\beta_1 + \gamma_1{}'} \left[\frac{1}{\frac{\gamma_1{}' - \beta_1}{\gamma_1{}'} + \frac{\gamma_1{}' + \beta_1}{\gamma_1{}'} \left(\frac{G\gamma_2{}' + (\gamma_1{}' - \beta_1)e^{-2\beta_1 h/a}}{G\gamma_2{}' - (\gamma_1{}' + \beta_1)e^{-2\beta_1 h/a}} e^{2\gamma_1{}' h/a} \right)} \right] \sin(s_1\xi_1) \sin(u_1 s_1)\, ds_1 \tag{35}$$

The couple $M(t)$ required to hold the disc in displaced position is given by

$$M(t) = -2\pi \int_0^a r^2 (\sigma^1{}_{\theta z})_{z=0} dr \tag{36}$$

We can easily find that

$$\left(\overline{\sigma^1}_{\theta z} \right)_{z=0} = \frac{2\mu_0}{\pi p} \frac{\partial}{\partial r} \int_r^a \frac{\varphi(\xi, p)}{\sqrt{(\xi^2 - r^2)}} d\xi \tag{37}$$

In the Laplace transformation domain eqn.(36) can be written in the following form

$$\overline{M(p)} = -(2\pi)\int_0^a r^2\left(\overline{\sigma^1}_{\theta z}\right)_{z=0} dr \tag{38}$$

Making use of (37) we find that

$$\overline{M(p)} = \frac{8\mu_0}{p}\int_0^a \xi\varphi\left(\xi, p\right) d\xi \tag{39}$$

Making use of (32) and (8) we get

$$M_1\left(T\right) = \frac{M(T)}{16\mu_0\alpha a^3} = \frac{1}{2\pi i}\int_{B_r}\frac{e^{PT}}{P}dP\int_0^1 \xi_1\varphi_1\left(\xi_1, P\right) d\xi_1 \tag{40}$$

where

$$T = \frac{c_2 t}{a} \tag{41}$$

4. Numerical Results and Discussion

Impact response of a non-homogeneous layer bonded to a homogeneous elastic half-space is considered. Solving the integral equation (33) numerically and then using Laplace inversion the numerical value of couple is obtained for different values of the parameters. In solving the problem, $\rho_2 = \rho_0$ and $G = 50$ are taken. First the integral equation (33) is solved numerically by the method of Fox and Goodwin [4]. It is seen that the Kernel given by (35) of the integral equation (33) changes very little for large values of h therefore the numerical values of $\varphi_1\left(\xi_1, P\right)$ are obtained for $\frac{a}{h} = 2.0, 10.0$, and for different values of β. For the Laplace inversion, the method due to Miller and Guy [6] is used. After solving the integral equation (33) and then equation (40) the Torsional Impact $M_1(T)$ has been calculated and plotted against the time T for different values of β in Figure 2 and Figure 3.

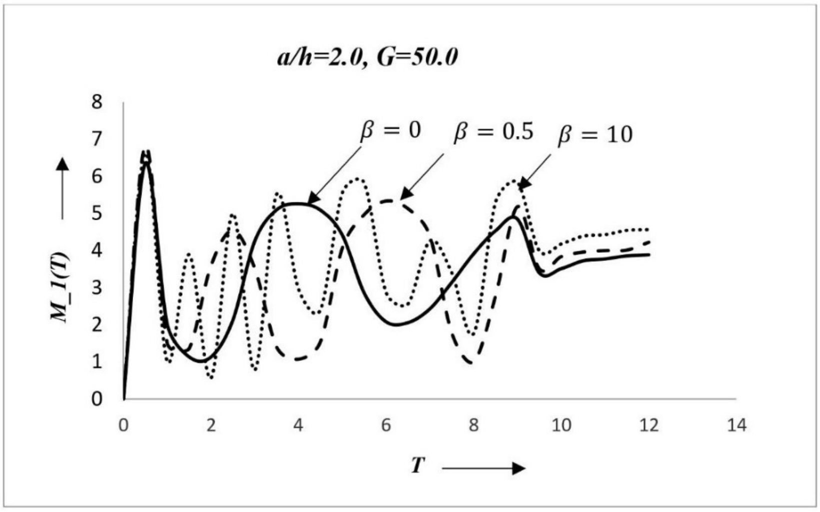

Figure 2: $M_1(T)$ versus T for $\frac{a}{h} = 2.0$, $G = 50.0$ and for $\beta = 0, 0.5, 10$

In Figure 2, $M_1(T)$ has been plotted against the time T for $\frac{a}{h} = 2.0$ and for $\beta = 0, 0.5, 10$. The value of the torque is infinity when sudden torsion is applied i.e. when $T = 0$. It is clear from the graph as the dynamic torque reaches a peak immediately when $T > 0$. Then it gradually decreases in magnitude and showing wave like nature. Also, it is shown from the graph that the frequency of oscillation increases for higher the value of β and finally decreases as T increases.

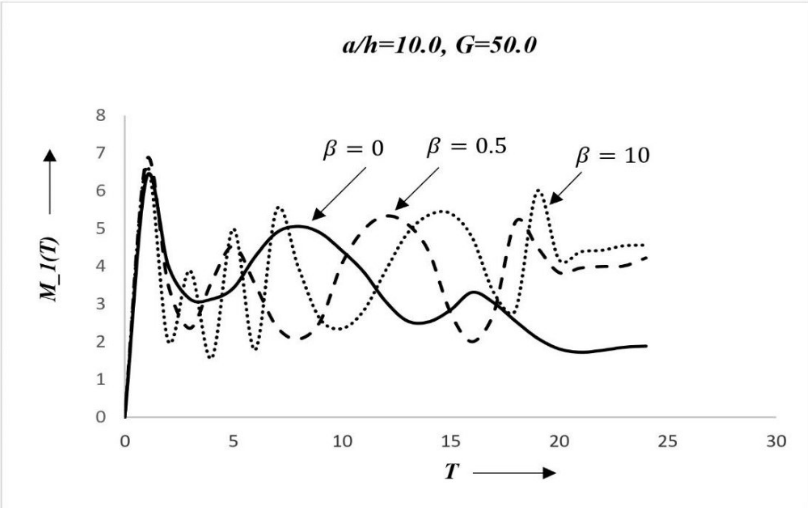

Figure 3: $M_1(T)$ versus T for $\frac{a}{h} = 10.0$, $G = 50.0$ and for $\beta = 0, 0.5, 10$

In Figure 3, $M_1(T)$ has been plotted against the time T for $\frac{a}{h} = 10.0$ and for $\beta = 0, 0.5, 10$. For higher value of $\frac{a}{h}$ the graph shows same wave like nature but takes more time to decrease.

5. Conclusion

In this problem the effect of non-homogeneity in bi-material due to sudden impact is considered. Here the exponential form of non-homogeneity is considered. The analytical expression of torsional impact and numerical values of impact load are calculated for different constants. From the obtained results the following conclusions can be drawn:

(1). Immediately after the application of sudden load the torque is infinity then gradually decreases in magnitude and oscillate.

(2). The value of impact load increases with increasing β, the exponential variation.

(3). If β is taken as zero the problem reduces to the problem considered by R. S. Dhaliwal, B. M. Singh and Jan Vrbik. In their problem they have obtained the value of Torsional impact for various values of G and they showed that the value of torque is infinity for $T = 0$ due to sudden loading after that it decreases and oscillates for all G. This ensures my consideration.

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