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Spaces Over Non-Newtonian Numbers

Research Article

Pranav Sharma[1*] and Sanjay Mishra[1]

1 Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India.

Abstract: We construct the spaces of non-Newtonian numbers using some new operations between them. Then we define generalized metric and generalized norm on the set of non-Newtonian numbers which attains the value as a non-negative non-Newtonian real number (depending on system of measurement) rather than non-negative real. Further, we have studied some order and convergence structures on these spaces.

MSC: 46A19.

Keywords: Non-Newtonian numbers, Ordered vector space, $*_\gamma$-Banach space.

1. Introduction and Preliminaries

Grossman and Katz [5] introduced *non-Newtonian calculus* as an alternative to classical calculus. It provides differentiation and integration tools based on non-Newtonian operations instead of classical operations. Every property in classical calculus has an analogue in non-Newtonian calculus. Non-Newtonian calculus consists of many calculi such as classical, geometric, anageometric, bigeometric calculus etc. In [1] the results with applications of multiplicative calculus corresponding to the well-known properties of derivatives and integrals in classical calculus are presented. Uzer [11] has extended the multiplicative calculus to the complex valued functions. We find the applications of non-Newtonian calculus in the field of Probability, Physics, Image Analysis, Numerical Analysis, Non-Linear Dynamical Systems etc.. The field $\mathbb{R}(N)$ of non-Newtonian real numbers and the the concept of non-Newtonian metric is introduced in [2]. In [4] exponential complex numbers and $*$-complex number systems are introduced. Some sequence spaces defined over the non-Newtonian complex field $\mathbb{C}^*$ and corresponding results for these spaces are proved in [10].

Before proceeding further first we introduce the basic terms used in this paper.

Definition 1.1. *A complete ordered field is a system consisting of a set A with four binary operations $\dot{+}$, $\dot{-}$, $\dot{\times}$, $\dot{\div}$ for A and an order relation $\dot{<}$ for A all of which behave with respect to A exactly as $+, -, \times, \div, <$ behave with respect to the set of Real numbers ($\mathbb{R}$). We call A the realm of the complete ordered field and a complete ordered field is called arithmetic if its realm is a subset of $\mathbb{R}$.*

Definition 1.2. *A bijective function with domain in $\mathbb{R}$ and range a subset of $\mathbb{R}$ is called a generator.*

Example 1.3. *The identity function I, exp, $-exp$, x^3, $-x^3$ etc. are generators (exp denotes the exponential function).*

* E-mail: pranav15851@gmail.com

Definition 1.4. *Let α be a generator with range A. By α-arithmetic we mean the arithmetic whose realm is A and whose operations and ordering are defined as follows:*

$$\begin{aligned}
&\alpha - addition: && y \dot{+} z = \alpha\{\alpha^{-1}(y) + \alpha^{-1}(z)\};\\
&\alpha - subtraction: && y \dot{-} z = \alpha\{\alpha^{-1}(y) - \alpha^{-1}(z)\};\\
&\alpha - multiplication: && y \dot{\times} z = \alpha\{\alpha^{-1}(y) \times \alpha^{-1}(z)\};\\
&\alpha - division: && y(z \neq \dot{0}) \dot{\div} z = \alpha\left\{\frac{\alpha^{-1}(y)}{\alpha^{-1}(z)}\right\};\\
&\alpha - ordering: && y \dot{\leq} z \Leftrightarrow \alpha^{-1}(y) \leq \alpha^{-1}(z), \qquad \forall\, y, z \in A.
\end{aligned}$$

The set $\mathbb{R}_\alpha(N)$ of non-Newtonian real numbers generated by α is defined as: $\mathbb{R}_\alpha(N) = \{\alpha(x) : x \in R\}$ and we say that α-generates α-arithmetic.

Remark 1.5. *All concepts in classical arithmetic have natural counterparts in α-arithmetic $\left(A,\ \dot{+},\ \dot{-},\ \dot{\times},\ \dot{\div},\ \dot{<}\right)$. For instance, α- zero and α- one turn out to be $\alpha(0)$ and $\alpha(1)$ respectively.*

Example 1.6. *Consider $\alpha = -\exp$ as a generator. So, $A = \{-e^x\ \forall\, x \in R\} = (-\infty, 0)$, $\alpha - zero(\ \dot{0}) = -\exp(0) = -1$, $\alpha - positive\ numbers = \left\{x \in A :\ \dot{0} \dot{>} x\right\} = (\infty, -1)$, $\alpha - negative\ numbers = \left\{x \in A :\ \dot{0} \dot{<} x\right\} = (-1, 0)$.*

Remark 1.7. *The α-square of a number $\dot{a} \in A$ denoted $\dot{a}^{\dot{2}}$ will have value:*

$$\dot{a}^{\dot{2}} = \alpha\left\{\alpha^{-1}(\ \dot{a}) \times \alpha^{-1}(\ \dot{a})\right\} = \alpha\left\{\left[\alpha^{-1}(\ \dot{a})\right]^2\right\}.$$

Similarly, $\dot{a}^{\dot{p}} = \alpha\left\{\left[\alpha^{-1}(\ \dot{a})\right]^p\right\}$.

Definition 1.8. *The α-absolute value of a number $\dot{a} \in A$ is defined as $\alpha\left(|\alpha^{-1}(\ \dot{a})|\right)$. For each positive number $\dot{a} \in A$, we have,*

$$\dot{\sqrt{\dot{a}_1^{\dot{2}}}} = \alpha\left(|\alpha^{-1}(\ \dot{a})|\right).$$

Remark 1.9. *In the rest of this paper $\dot{a} \in (A,\ \dot{+},\ \dot{-},\ \dot{\times},\ \dot{\div},\ \dot{<})$, $\ddot{b} \in (B,\ \ddot{+},\ \ddot{-},\ \ddot{\times},\ \ddot{\div},\ \ddot{<})$ and $\dddot{g} \in (G,\ \dddot{+},\ \dddot{-},\ \dddot{\times},\ \dddot{\div},\ \dddot{<})$ will denote the arbitrarily chosen elements from $\alpha-$arithmatic, $\beta-$arithmatic and $\gamma-$arithmatic respectively.*

Definition 1.10. *The isomorphism from $\alpha-$arithmatic to β-arithmatic is the unique function $\imath_{AB}$ that posses the following three properties:*

(1) $\imath_{AB}$ is one-to-one,

(2) $\imath_{AB}$ is onto,

(3) For any numbers $\dot{a}_1,\ \dot{a}_2 \in A$,

- *(a) $\imath_{AB}(\ \dot{a}_1 \dot{+} \dot{a}_2\ ) = \imath_{AB}(\ \dot{a}_1\ ) \ddot{+} \imath_{AB}(\ \dot{a}_2\ )$,*
- *(b) $\imath_{AB}(\ \dot{a}_1 \dot{-} \dot{a}_2\ ) = \imath_{AB}(\ \dot{a}_1\ ) \ddot{-} \imath_{AB}(\ \dot{a}_2\ )$,*
- *(c) $\imath_{AB}(\ \dot{a}_1 \dot{\times} \dot{a}_2\ ) = \imath_{AB}(\ \dot{a}_1\ ) \ddot{\times} \imath_{AB}(\ \dot{a}_2\ )$,*
- *(d) $\imath_{AB}(\ \dot{a}_1 \dot{\div} \dot{a}_2\ ) = \imath_{AB}(\ \dot{a}_1\ ) \ddot{\div} \imath_{AB}(\ \dot{a}_2\ )$, $\dot{a}_2 \neq \dot{0}$,*
- *(e) $\dot{a}_1 \dot{<} \dot{a}_2$ if and only if $\imath_{AB}(\ \dot{a}_1) \ddot{<} \imath_{AB}(\ \dot{a}_2)$.*

Remark 1.11.

(1) ι_{AB}^{-1} *will be the isomorhism from* $\beta-$*arithmatic to* $\alpha-$*arithmatic.*

(2) In this paper the isomorphisms from $\alpha-$*arithmatic to* $\gamma-$*arithmatic and* $\beta-$*arithmatic to* $\gamma-$*arithmatic are denoted by* ι_{AG} *and* ι_{BG} *respectively.*

Definition 1.12. *The ordered pair* $(\dot{a}, \ddot{b})$ *is called a* ${}_{\alpha}*_{\beta}$*-point. (The subscript* α *and* β *denote the underlying arithmetic).*

In [4], Grossman has introduced the exponential complex numbers and has defined *-complex number system. In [10], non-Newtonian Complex numbers ($\mathbb{C}^*$) are defined as the set of all ${}_{\alpha}*_{\beta}$*-points* and it is proved that ($\mathbb{C}^*$) is a field with the following operations:

(1) $\oplus : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ defined as $\oplus\,({z_1}^*, {z_2}^*) \longmapsto ({z_1}^* \oplus {z_2}^*) = (\dot{a}_1 \dot{+} \dot{a}_2, \ddot{b}_1 \ddot{+} \ddot{b}_2)$ and

(2) $\odot : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ defined as $\odot\,({z_1}^*, {z_2}^*) \longmapsto ({z_1}^* \odot {z_2}^*) = \Big(\alpha(\bar{a}_1\,\bar{a}_2 - \bar{\bar{b}}_1\,\bar{\bar{b}}_2), \beta(\bar{a}_1\,\bar{\bar{b}}_2 + \bar{\bar{b}}_1\,\bar{a}_2)\Big)$.

where $\dot{a}_1, \dot{a}_2 \in A$ and $\ddot{b}_1, \ddot{b}_2 \in B$ with, $\bar{a}_1 = \alpha^{-1}(\dot{a}_1) = \alpha^{-1}(\alpha(a_1)) = a_1 \in \mathbb{R}$ and $\bar{\bar{b}}_1 = \beta^{-1}(\ddot{b}_1) = \beta^{-1}(\beta(b_1)) = b_1 \in \mathbb{R}$.

Remark 1.13. *We denote the set of all* ${}_{\alpha}*_{\beta}$*-points as* ${}_{\alpha}\mathbb{C}^*_{\beta}$.

The main purpose of this paper is to develop a vector spaces of non-Newtonian numbers over different fields of non-Newtonian numbers. The concept of distance is generalized and generalized metric and generalized norm are defined for these spaces. Finally some order and convergence structures are studied on the vector spaces of non-Newtonian complex numbers.

2. Main Results

First we prove that the set of non-Newtonian complex numbers is a Commutative Algebra with identity over field of reals and extend this concept from real field to an arbitrary field of non-Newtonian numbers.

Theorem 2.1. ${}_{\alpha}\mathbb{C}^*_{\beta}$ *is a commutative algebra with identity over* $\mathbb{R}$ *with operations:*

(1) $\oplus : {}_{\alpha}\mathbb{C}^*_{\beta} \times {}_{\alpha}\mathbb{C}^*_{\beta} \to {}_{\alpha}\mathbb{C}^*_{\beta}$ *defined as* $\oplus\,({z_1}^*, {z_2}^*) \longmapsto ({z_1}^* \oplus {z_2}^*) = (\dot{a}_1 \dot{+} \dot{a}_2, \ddot{b}_1 \ddot{+} \ddot{b}_2)$

(2) $\odot : {}_{\alpha}\mathbb{C}^*_{\beta} \times {}_{\alpha}\mathbb{C}^*_{\beta} \to {}_{\alpha}\mathbb{C}^*_{\beta}$ *defined as* $\odot\,({z_1}^*, {z_2}^*) \longmapsto ({z_1}^* \odot {z_2}^*) = \Big(\alpha(\bar{a}_1\,\bar{a}_2 - \bar{\bar{b}}_1\,\bar{\bar{b}}_2), \beta(\bar{a}_1\,\bar{\bar{b}}_2 + \bar{\bar{b}}_1\,\bar{a}_2)\Big)$ *and*

(3) $\otimes : \mathbb{R} \times {}_{\alpha}\mathbb{C}^*_{\beta} \to {}_{\alpha}\mathbb{C}^*_{\beta}$ *defined as* $\otimes\,(r, {z_1}^*) \longmapsto (r \otimes z^*) = \Big(r \otimes (\dot{a}, \ddot{b})\Big) = \Big(\alpha(r) \dot{\times} \dot{a}, \beta(r) \ddot{\times} \ddot{b}\Big) = \Big(\dot{r} \dot{\times} \dot{a}, \ddot{r} \ddot{\times} \ddot{b}\Big)$.

Proof. First we prove that ${}_{\alpha}\mathbb{C}^*_{\beta}$ is a vector space over $\mathbb{R}$ with the above defined operations. It has been proved that $({}_{\alpha}\mathbb{C}^*_{\beta}, \oplus)$ is a commutative group *see* [10]. We prove here the operations of scalar multiplication. For $r, s \in \mathbb{R}, z^* = (\dot{a}, \ddot{b}) \in {}_{\alpha}\mathbb{C}^*_{\beta}$

$$\begin{aligned} r \otimes (s \otimes z^*) &= r \otimes \Big(s \otimes (\dot{a}, \ddot{b})\Big) = r \otimes (\dot{s} \dot{\times} \dot{a}, \ddot{s} \ddot{\times} \ddot{b}) \\ &= (\dot{(s)} \dot{\times} \dot{r} \dot{\times} \dot{a}, \ddot{s} \ddot{\times} \ddot{(r)} \ddot{\times} \ddot{b}) = s \otimes (\dot{r} \dot{\times} \dot{a}, \ddot{r} \ddot{\times} \ddot{b}) = s \otimes (r \otimes z^*). \end{aligned}$$

Thus, multiplication by scalars is associative. For $1 \in \mathbb{R}, \forall z^* = (\dot{a}, \ddot{b}) \in {}_{\alpha}\mathbb{C}^*_{\beta}$ $1 \otimes z^* = 1 \otimes (\dot{a}, \ddot{b}) = (\dot{1} \dot{\times} \dot{a}, \ddot{1} \ddot{\times} \ddot{b}) = (\dot{a}, \ddot{b}) = z^*$. Thus $1 \otimes z^* = z^* \quad \forall z^* \in {}_{\alpha}\mathbb{C}^*_{\beta}$. For $r \in \mathbb{R}$ and $z_1^* = (\dot{a}_1, \ddot{b}_1), z_2^* = (\dot{a}_2, \ddot{b}_2) \in {}_{\alpha}\mathbb{C}^*_{\beta}$,

$$\begin{aligned} r \otimes (z_1^* \oplus z_2^*) &= r \otimes ((\dot{a}_1, \ddot{b}_1) \oplus (\dot{a}_2, \ddot{b}_2)) \\ &= \Big(\dot{r} \dot{\times}(\dot{a}_1 + \dot{a}_2), \ddot{r} \ddot{\times}(\ddot{b}_1 + \dot{b}_2)\Big) \\ &= \Big((\dot{r} \dot{\times} \dot{a}_1, \ddot{r} \ddot{\times} \ddot{b}_1) + (\dot{r} \dot{\times} \dot{a}_2, \ddot{r} \ddot{\times} \dot{b}_2)\Big) \\ &= (r \otimes z_1^*) \oplus (r \otimes z_2^*). \end{aligned}$$

Thus Multiplication by scalars is distributive with respect to vector addition. For $r, s \in \mathbb{R}$ and $z_1^* = (\,\dot{a}_1,\ \ddot{b}_1) \in {}_{\alpha}\mathbb{C}_{\beta}^*$,

$$\begin{aligned} (r+s)\ \otimes\ z_1^* &= (r+s)\ \otimes\ \Big((\,\dot{a}_1,\ \ddot{b}_1)\Big) \\ &= \Big(((\,\dot{r} \dot{\times} \dot{a}_1)\ \dot{+}(\,\dot{s} \dot{\times} \dot{a}_1))\,,\Big((\,\ddot{r} \ddot{\times} \ddot{b}_1)\ \ddot{+}(\,\ddot{s} \ddot{\times} \ddot{b}_1)\Big)\Big) \\ &= \Big(r\ \otimes\ (\,\dot{a}_1,\ \ddot{b}_1)\Big)\ \oplus\ \Big(s\ \otimes\ (\,\dot{a}_1,\ \ddot{b}_1)\Big) = (r\ \otimes\ z^*) + (s\ \otimes\ z^*). \end{aligned}$$

Thus Multiplication by a vector is distributive with respect to scalar addition. This proves that ${}_{\alpha}\mathbb{C}_{\beta}^*$ is a Vector space over $\mathbb{R}$. Further we have , $(\,{}_{\alpha}\mathbb{C}_{\beta}^* - \{0^*\},\ \odot\,)$ is a commutative group so,

(1) $(z_1\ \odot\ z_2)\ \odot\ z_3 = z_1\ \odot\ (z_2\ \odot\ z_3)$,

(2) $z_1\ \odot\ (z_2\ \oplus\ z_3) = (z_1\ \odot\ z_2)\ \otimes\ (z_1\ \odot\ z_3)$,

(3) $(z_1\ \oplus\ z_2)\ \odot\ z_3 = (z_1\ \odot\ z_3)\ \otimes\ (z_2\ \odot\ z_3)$ and

(4) Multiplicative identity exists.

Now we prove that $r\ \otimes\ (z_1\ \odot\ z_2) = (r\ \otimes\ z_1)\ \odot\ z_2 = z_1\ \odot\ (r\ \otimes\ z_2)$. For $r, s \in \mathbb{R}$ and $z_1^* = (\,\dot{a}_1,\ \ddot{b}_1) \in {}_{\alpha}\mathbb{C}_{\beta}^*$,

$$\begin{aligned} r\ \otimes\ (z_1\ \odot\ z_2) &= r\ \otimes\ \Big(\alpha(\,\bar{\dot{a}}_1\ \bar{\dot{a}}_2 -\ \bar{\ddot{b}}_1\ \bar{\ddot{b}}_2), \beta(\,\bar{\dot{a}}_1\ \bar{\ddot{b}}_2 +\ \bar{\ddot{b}}_1\ \bar{\dot{a}}_2)\Big) \\ &= \Big(\alpha(r)\ \dot{\times}\alpha(\,\bar{\dot{a}}_1\ \bar{\dot{a}}_2 -\ \bar{\ddot{b}}_1\ \bar{\ddot{b}}_2), \beta(r)\ \ddot{\times}\beta(\,\bar{\dot{a}}_1\ \bar{\ddot{b}}_2 +\ \bar{\ddot{b}}_1\ \bar{\dot{a}}_2)\Big) \\ &= \Big(\alpha(\bar{\dot{r}}\ \bar{\dot{a}}_1\ \bar{\dot{a}}_2 - \bar{\ddot{r}}\ \bar{\ddot{b}}_1\ \bar{\ddot{b}}_2), \beta(\bar{\dot{r}}\ \bar{\dot{a}}_1\ \bar{\ddot{b}}_2 + \bar{\ddot{r}}\ \bar{\ddot{b}}_1\ \bar{\dot{a}}_2)\Big) \\ &= \Big(\,\dot{r}\ \dot{\times}\ \dot{a}_1,\ \ddot{r}\ \ddot{\times}\ \ddot{b}_1\Big)\ \odot\ z_2 = (r\ \otimes\ z_1)\ \odot\ z_2. \end{aligned}$$

Similarly, we can show that $r \otimes (z_1\ \odot\ z_2) = z_1 \odot (r \otimes z_2)$. Thus, we have $r \otimes (z_1\ \odot\ z_2) = (r \otimes z_1) \odot z_2 = z_1 \odot (r \otimes z_2)$. Hence, ${}_{\alpha}\mathbb{C}_{\beta}^*$ is a commutative Algebra with identity over $\mathbb{R}$ with operations defined as above. □

In the same manner we can also prove that ${}_{\alpha}\mathbb{C}_{\beta}^*$ is a commutative algebra with identity over non-Newtonian real numbers $\mathbb{R}_{\gamma}(\mathrm{N})$ (with an arbitrary realm γ) and scalar product:

$$\begin{aligned} &\otimes\ : \mathbb{R}_{\gamma}(\mathrm{N}) \times {}_{\alpha}\mathbb{C}_{\beta}^* \to {}_{\alpha}\mathbb{C}_{\beta}^* \text{ defined as} \\ &\otimes\ (r, z^*) \longmapsto r\ \otimes\ z^* = \Big(r\ \otimes\ (\,\dot{a},\ \ddot{b})\Big) = \Big(\alpha(\gamma^{-1}(r))\ \dot{\times}\ \dot{a}, \beta(\gamma^{-1}(r))\ \ddot{\times}\ \ddot{b}\Big) \text{ for } r \in G. \end{aligned}$$

Remark 2.2.

(1) For a fixed α, β we can produce infinitely many vector spaces over different fields (infinitely many γ as generator).

(2) For $\alpha = \beta = \gamma = I$(identity function) this space will be the complex vector space over the field of reals..

Theorem 2.3. *The set ${}_{\alpha}\mathbb{C}_{\beta}^*$ of all ${}_{\alpha}*_{\beta}$-points of non-Newtonian numbers is a vector space over field $({}_{\gamma}\mathbb{C}_{\delta}^*,\ \otimes\ ,\ \odot\ )$ with operations:*

(1) $\oplus\ :\ {}_{\alpha}\mathbb{C}_{\beta}^* \times\ {}_{\alpha}\mathbb{C}_{\beta}^* \to\ {}_{\alpha}\mathbb{C}_{\beta}^*$ defined as $\oplus\, ({z_1}^*, {z_2}^*) \longmapsto ({z_1}^*\ \oplus\ {z_2}^*) = (\,\dot{a}_1 \dot{+} \dot{a}_2,\ \ddot{b}_1 \ddot{+} \ddot{b}_2)$

(2) $\circ : {}_{\gamma}\mathbb{C}_{\delta}^* \times\ {}_{\alpha}\mathbb{C}_{\beta}^* \longrightarrow {}_{\alpha}\mathbb{C}_{\beta}^*$ defined as $\circ\ \Big((\,\ddot{g}\ ,\ \ddddot{d}\,), (\,\dot{a}\ ,\ \ddot{b})\Big) \longmapsto \Big((\,\ddot{g}\ ,\ \ddddot{d}\,)\ \circ\ (\,\dot{a}\ ,\ \ddot{b})\Big) = (\alpha(ga), \beta(db))\,.$

Proof. $({}_{\alpha}\mathbb{C}_{\beta}^*,\ \oplus\ )$ is a commutative group and the scalar product operations can be verified in a similar way (as above). □

Example 2.4.

(1) If $\alpha = \beta = I$ (identity map) then we get a family of vector space of complex numbers over the field of non-Newtonian complex numbers.

(2) If $\gamma = \delta = I$ (identity map) then we get a family of vector space of non-Newtonian complex numbers over the field of complex numbers.

In the same way we can generate different families of vector spaces. In non-Newtonian calculus the distance concept is generalized to $*$-distance concept between the ${}_\alpha *_\beta$-points. We further generalize the concept of $*$-distance to $*_\gamma$-distance, where the distances between ${}_\alpha *_\beta$-points depends on the arithmetic used to measure the distances. We define the $*_\gamma$-distance (distance in γ-*arithmetic*) between two ${}_\alpha *_\beta$-points $z_1^* = (\dot{a}_1, \ddot{b}_1)$, $z_2^* = (\dot{a}_2, \ddot{b}_2) \in {}_\alpha\mathbb{C}_\beta^*$ as,

$${}_\gamma d^* : ({}_\alpha\mathbb{C}_\beta^*) \times ({}_\alpha\mathbb{C}_\beta^*) \rightarrow \dddot{[}\ \ddot{0}, \infty\ \dddot{)} = G' \subset G \text{ as } {}_\gamma d^*(z_1^*, z_2^*) = \dddot{\sqrt{}}\left[\imath_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\dddot{2}} \dddot{+} \left[\imath_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\dddot{2}}.$$

Remark 2.5.

(1) Informally the distance is measured using an arbitrary $\gamma - ruler$.

(2) $*_\gamma$-distance is not a metric in the usual sense but is a generalized metric. Where the metric attains the value in more general ordered field (as the range of this function is γ-positive real numbers in place of the set of positive real numbers. If we take $\gamma = -\exp$ then $\ddot{0} = -\exp(0) = -1$).

Example 2.6.

(1) For $\gamma = I$ (identity function), define ${}_I d^* : ({}_\alpha\mathbb{C}_\beta^*) \times ({}_\alpha\mathbb{C}_\beta^*) \rightarrow [0, \infty) =$ non-negative reals, as

$${}_I d^*(z_1^*, z_2^*) = \gamma^{-1}\left(\dddot{\sqrt{}}\left[\imath_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+} \left[\imath_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}}\right).$$

In this case the $*_I$-distance satisfies the property of metric.

(2) For $\gamma = \beta$, define ${}_\beta d^* : ({}_\alpha\mathbb{C}_\beta^*) \times ({}_\alpha\mathbb{C}_\beta^*) \rightarrow \ddot{[}\ \ddot{0}, \infty\ \ddot{)} = B' \subset B$ as

$${}_\beta d^*(z_1^*, z_2^*) = \ddot{\sqrt{}}\left[\imath_{AB}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \ddot{+} \left[(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}}.$$

In this case the $*_\gamma$-distance coincides with the distance as used in [10].

Definition 2.7. *We define $*_\gamma$-absolute value ${}_\gamma|.|^*$ of $z^* \in {}_\alpha\mathbb{C}_\beta^*$ in the following manner:*

$${}_\gamma|.|^* : {}_\alpha\mathbb{C}_\beta^* \longrightarrow \dddot{[}\ \dddot{0}, \infty\ \dddot{)} \in G \text{ as } {}_\gamma|z|^* = {}_\gamma d^*(z^*, 0^*), \quad \text{where} \quad z^*, 0^* \in {}_\alpha\mathbb{C}_\beta^*.$$

Definition 2.8 (Generalized Metric)**.** *We call a function ${}_\gamma d^* : ({}_\alpha\mathbb{C}_\beta^*) \times ({}_\alpha\mathbb{C}_\beta^*) \rightarrow \dddot{[}\ \dddot{0}, \infty\ \dddot{)} = G' \subset G$, a $*_\gamma$-metric if for $z_1^*, z_2^*, z_3^* \in {}_\alpha\mathbb{C}_\beta^*$ it satisfies the following:*

(i) ${}_\gamma \overset{*}{d}(z_1^*, z_2^*) \ \dddot{\geq}\ \dddot{0}$,

(ii) ${}_\gamma \overset{*}{d}(z_1^*, z_2^*) = \dddot{0}$ *if and only if* $z_1^* = z_2^*$,

(iii) ${}_\gamma \overset{*}{d}(z_1^*, z_2^*) = {}_\gamma \overset{*}{d}(z_2^*, z_1^*)$,

(iv) ${}_{\gamma}\overset{*}{d}(z_1^*, z_2^*) \dddot{+} {}_{\gamma}\overset{*}{d}(z_2^*, z_3^*) \;\dddot{\geq}\; {}_{\gamma}\overset{*}{d}(z_1^*, z_2^*)$.

Theorem 2.9. *${}_{\gamma}d^*$ is a $*_{\gamma}$-metric on ${}_{\alpha}\mathbb{C}_{\beta}^*$ for, ${}_{\gamma}d^*(z_1^*, z_2^*) = \dddot{\sqrt{}}\left[1_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+}\left[1_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}}$.*

Proof. As ${}_{\gamma}d^* : ({}_{\alpha}\mathbb{C}_{\beta}^*) \times ({}_{\alpha}\mathbb{C}_{\beta}^*) \to A' \subset A$ where A' is the set of positive numbers in γ-*arithmetic* and this implies ${}_{\gamma}d^*(z_1^*, z_2^*) \ddot{\geq} \ddot{0}$, where $\ddot{0}$ is the zero element of G. For $z_1^* = (\dot{a}_1, \ddot{b}_1)$, $z_2^* = (\dot{a}_2, \ddot{b}_2) \in {}_{\alpha}\mathbb{C}_{\beta}^*$

$$
\begin{aligned}
{}_{\gamma}d^*(z_1^*, z_2^*) = \ddot{0} \;\Leftrightarrow\; & \dddot{\sqrt{}}\left[1_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+}\left[1_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}} = \ddot{0},\\
\Leftrightarrow\; & \gamma\left(\sqrt{(a_1-a_2)^2 + (b_1-b_2)^2}\right) = \gamma(0),\\
\Leftrightarrow\; & a_1 - a_2 = 0, b_1 - b_2 = 0 \;\Leftrightarrow\; \dot{a}_1 = \dot{a}_2,\ \ddot{b}_1 = \ddot{b}_2 \quad\Leftrightarrow\quad z_1^* = z_2^*.
\end{aligned}
$$

So ${}_{\gamma}d^*(z_1^*, z_2^*) = \ddot{0}$ iff $z_1^* = z_2^*$. For $z_1^* = (\dot{a}_1, \ddot{b}_1)$, $z_2^* = (\dot{a}_2, \ddot{b}_2) \in {}_{\alpha}\mathbb{C}_{\beta}^*$

$$
\begin{aligned}
{}_{\gamma}d^*(z_1^*, z_2^*) &= \dddot{\sqrt{}}\left[1_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+}\left[1_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}}\\
&= \gamma\left(\sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}\right) = {}_{\gamma}d^*(z_2^*, z_1^*).
\end{aligned}
$$

So, ${}_{\gamma}d^*(z_1^*, z_2^*) = {}_{\gamma}d^*(z_2^*, z_1^*)$. Consider $z_1^* = (\dot{a}_1, \ddot{b}_1)$, $z_2^* = (\dot{a}_2, \ddot{b}_2)$, $z_3^* = (\dot{a}_3, \ddot{b}_3) \in {}_{\alpha}\mathbb{C}_{\beta}^*$. Then,

$$
\begin{aligned}
{}_{\gamma}d^*(z_1^*, z_2^*) \dddot{+} {}_{\gamma}d^*(z_2^*, z_3^*) &= \dddot{\sqrt{}}\left[1_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+}\left[1_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}} \dddot{+} \dddot{\sqrt{}}\left[1_{AC}^{-1}(\dot{a}_2 \dot{-} \dot{a}_3)\right]^{\ddot{2}} \dddot{+}\left[1_{BC}^{-1}(\ddot{b}_2 \ddot{-} \ddot{b}_3)\right]^{\ddot{2}}\\
&= \gamma\left(\sqrt{(a_1-a_2)^2 + (b_1-b_2)^2} + \sqrt{(a_2-a_3)^2 + (b_2-b_3)^2}\right)\\
&\dddot{\geq} \gamma\left(\sqrt{(a_1-a_3)^2 + (b_1-b_3)^2}\right) = {}_{\gamma}d^*(z_1^*, z_3^*).
\end{aligned}
$$

So ${}_{\gamma}d^*$ is a $*_{\gamma}$-*metric* on ${}_{\alpha}\mathbb{C}_{\beta}^*$. ∎

Remark 2.10. *For $\gamma = \alpha$ and $\gamma = \beta$ we get ${}_{\alpha}d^*$ and ${}_{\beta}d^*$ is a $*_{\gamma}$-metric on ${}_{\alpha}\mathbb{C}_{\beta}^*$.*

Theorem 2.11. *For any $g \dddot{>} \ddot{0} \in G$ and $z_1^* \in {}_{\alpha}\mathbb{C}_{\beta}^*$. Let ${}_{\gamma}B^*(z_1^*, g) = \left\{z_2^* \in {}_{\alpha}\mathbb{C}_{\beta}^* : {}_{\gamma}d^*(z_1^*, z_2^*) \dddot{<} g\right\}$ and ${}_{\gamma}B^*(z_1^*, g)$ denotes the $*_{\gamma}$-open ball centered at z_1 and radius g. Here,*

$$
{}_{\gamma}d^*(z_1^*, z_2^*) = \dddot{\sqrt{}}\left[1_{AC}^{-1}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+}\left[1_{BC}^{-1}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}}.
$$

We denote $\mathfrak{B} = \left\{ {}_{\gamma}B^*(z_1^*, g);\ \forall\ z_1^* \in {}_{\alpha}\mathbb{C}_{\beta}^*,\ g \dddot{>} \ddot{0}\right\}$ the family of all $*_{\gamma}$-open balls. Then, $\mathfrak{B}$ is a base for a topology on ${}_{\alpha}\mathbb{C}_{\gamma}^*$.

Proof. For any point $z_1 \in {}_{\alpha}\mathbb{C}_{\beta}^*$, $z_1 \in {}_{\gamma}B^*(z_1^*, \ddot{1})$, and ${}_{\gamma}B^*(z_1^*, \ddot{1}) \in \mathfrak{B}$. Hence the sets in $\mathfrak{B}$ covers ${}_{\alpha}\mathbb{C}_{\beta}^*$. Let $z_1, z_2, z_3, z_4 \in {}_{\alpha}\mathbb{C}_{\beta}^*$ and g_1, g_2, g_3 be γ-positive numbers in G. Suppose,

$$
z_3 \in \{{}_{\gamma}B^*(z_1^*, g_1) \cap {}_{\gamma}B^*(z_2^*, g_2)\} \text{ and } g_3 = min\{g_1 \dddot{-} {}_{\gamma}d^*(z_1, z_3), , g_2 \dddot{-} {}_{\gamma}d^*(z_2, z_3)\}.
$$

Since $z_3 \in {}_{\gamma}B^*(z_1, g_1)$ and $z_1 \in {}_{\gamma}B^*(z_2, g_2)$ so, ${}_{\gamma}d^*(z_1, z_3) \dddot{<} g_1$ and ${}_{\gamma}d^*(z_2, z_3) \ddot{<} g_2$. For any point $z_4 \in B(z_3, g_3)$, we have ${}_{\gamma}d^*(z_3, z_4) \dddot{<} g_3$, by $*_{\gamma}$-triangle inequality

$$
\begin{aligned}
{}_{\gamma}d^*(z_1, z_4) &\dddot{\leq} {}_{\gamma}d^*(z_1, z_2) \dddot{+} {}_{\gamma}d^*(z_2, z_4) \dddot{<} {}_{\gamma}d^*(z_1, z_3) \dddot{+} g_3\\
&\dddot{\leq} {}_{\gamma}d^*(z_1, z_3) \dddot{+} (g_1 \dddot{-} {}_{\gamma}d^*(z_1, z_3)) = g_1.\\
{}_{\gamma}d^*(z_2, z_4) &\dddot{\leq} {}_{\gamma}d^*(z_2, z_3) \dddot{+} {}_{\gamma}d^*(z_3, z_4) \dddot{<} {}_{\gamma}d^*(z_2, z_3) \dddot{+} g_3\\
&\dddot{\leq} {}_{\gamma}d^*(z_2, z_3) \dddot{+}(g_2 \dddot{-} {}_{\gamma}d^*(z_2, z_3)) = g_2.
\end{aligned}
$$

Hence, $z_4 \in {}_{\gamma}B^*(z_1, g_1)$ and $z_4 \in {}_{\gamma}B^*(z_2, g_2)$. So, ${}_{\gamma}B^*(z_3, g_3) \subseteq \left({}_{\gamma}B^*(z_1, g_1) \cap {}_{\gamma}B^*(z_2, g_2)\right)$. Thus, $\mathfrak{B}$ is a base for a topology and hence generates a topology $\mathfrak{T}$ on ${}_{\alpha}\mathbb{C}^*_{\beta}$ as follows: $\mathfrak{T} = \{\bigcup\{\mathfrak{B}\} \cup \phi\}$, where $\bigcup\{\mathfrak{B}\}$ represents the all unions of elements of $\mathfrak{B}$. □

Definition 2.12. *A sequence (z^*_n) in $({}_{\alpha}\mathbb{C}^*_{\beta}, {}_{\gamma}d^*)$ is said to be $*_{\gamma}$-convergent to z^* if for every $\varepsilon (\in G) \ddot{>} \ddot{0}$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that, ${}_{\gamma}d^*(z^*_n, z^*) \ddot{<} \varepsilon \quad \forall\, n > n_0$. We denote this as $\quad z^*_n \xrightarrow{*} z^*$.*

Definition 2.13. *A sequence (z^*_n) in $({}_{\alpha}\mathbb{C}^*_{\beta}, {}_{\gamma}d^*)$ is said to be $*_{\gamma}$-Cauchy sequence if for every $\varepsilon (\in G) \ddot{>} \ddot{0}$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that, ${}_{\gamma}d^*(z^*_n, z^*_m) \dddot{<} \varepsilon \quad \forall\, m, n > n_0$ and a space X is said to be $*_{\gamma}$-complete if every $*_{\gamma}$-Cauchy sequence in X is $*_{\gamma}$-convergent.*

Theorem 2.14. *$({}_{\alpha}\mathbb{C}^*_{\beta}, {}_{\gamma}d^*)$ is $*_{\gamma}$-complete with,*

$$ {}_{\gamma}d^*(z^*_1, z^*_2) = \dddot{\sqrt{\left[\imath^{-1}_{AC}(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \dddot{+} \left[\imath^{-1}_{BC}(\ddot{b}_1 \ddot{-} \ddot{b}_2)\right]^{\ddot{2}}}}. $$

Proof. Consider an arbitrary $*_{\gamma}$-*Cauchy sequence* $z^* = (z^*_n) \in ({}_{\alpha}\mathbb{C}^*_{\beta}, {}_{\gamma}d^*)$. For every $\varepsilon (\in G) \ddot{>} \ddot{0}$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that, ${}_{\gamma}d^*(z^*_n, z^*_m) \ddot{<} \varepsilon\ \forall\, m, n > n_0$ so,

$$ {}_{\gamma}d^*(z^*_m, z^*_n) = \dddot{\sqrt{\left[\imath^{-1}_{AC}(\dot{a}_m \dot{-} \dot{a}_n)\right]^{\ddot{2}} \dddot{+} \left[\imath^{-1}_{BC}(\ddot{b}_m \ddot{-} \ddot{b}_n)\right]^{\ddot{2}}}} \dddot{<} \varepsilon, $$

$\left[\imath^{-1}_{AC}(\dot{a}_m \dot{-} \dot{a}_n)\right]^{\ddot{2}} \dddot{+} \left[\imath^{-1}_{BC}(\ddot{b}_m \ddot{-} \ddot{b}_n)\right]^{\ddot{2}} \dddot{<} \varepsilon^{\ddot{2}}$, by $\gamma-$squaring $\Rightarrow \left[\imath^{-1}_{AC}(\dot{a}_m \dot{-} \dot{a}_n)\right]^{\ddot{2}} \dddot{<} \varepsilon^{\ddot{2}}$ and $\left[\imath^{-1}_{BC}(\ddot{b}_m \ddot{-} \ddot{b}_n)\right]^{\ddot{2}} \dddot{<} \varepsilon^{\ddot{2}} \Rightarrow |a_m - a_n| < \varepsilon'$ and $|b_m - b_n| < \varepsilon'$ where $\varepsilon' = \gamma^{-1}(\varepsilon^{\ddot{2}})$. $\forall\, m, n > n_0$. From this we get that (a_n) and (b_n) are Cauchy sequences with real terms. Hence, (a_n) and (b_n) are convergent sequences. Thus, for every $\varepsilon' > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon'}{2}\ \forall\, n \geq n_1$ and $|b_n - b| < \frac{\varepsilon'}{2}\ \forall\, n \geq n_2$. So;

$$ \begin{aligned} {}_{\gamma}d^*(z^*_n, z*) &= \dddot{\sqrt{\left[\imath^{-1}_{AC}(\dot{a}_n \dot{-} \dot{a})\right]^{\ddot{2}} \dddot{+} \left[\imath^{-1}_{BC}(\ddot{b}_n \ddot{-} \ddot{b})\right]^{\ddot{2}}}} \\ &\dddot{\leq} \gamma\left(\sqrt{(a_1 - a_2)^2} + \sqrt{(b_1 - b_2)^2}\right) = \gamma\left(|(a_n - a)| + |(b_n - b)|\right) \\ &\dddot{\leq} \gamma\left(\frac{\varepsilon'}{2} + \frac{\varepsilon'}{2}\right) = \gamma(\varepsilon') = \varepsilon. \end{aligned} $$

Therefore, every $*_{\gamma}$-*Cauchy sequence* is $*_{\gamma}$-*Convergent* in $({}_{\alpha}\mathbb{C}^*_{\beta}, {}_{\gamma}d^*)$ and hence $({}_{\alpha}\mathbb{C}^*_{\beta}, {}_{\gamma}d^*)$ is $*_{\gamma}$-*complete.* □

Definition 2.15 (Generalized norm)**.** *A function ${}_{\gamma}\|.\|^* : {}_{\alpha}\mathbb{C}^*_{\beta} \to \dddot{[}\ \ddot{0}, \infty \dddot{)}\ (= G' \subset (G, \dddot{+}, \dddot{-}, \dddot{\times}, \dddot{\div}, \dddot{<}))$ is called $*_{\gamma}$-norm if it satisfies the following axioms:*

(1) ${}_{\gamma}\|z^*_1\|^* = \ddot{0} \Leftrightarrow z^*_1 = \ddot{0}$,

(2) ${}_{\gamma}\|g \otimes z^*_1\|^* = {}_{\gamma}|g| \dddot{\times}\ {}_{\gamma}\|z^*_1\|^*$,

(3) ${}_{\gamma}\|z^*_1 \oplus z^*_2\|^* \dddot{\leq} {}_{\gamma}\|z^*_1\|^* \dddot{+} {}_{\gamma}\|z^*_2\|^*$. $\forall\, z^*_1, z^*_2 \in {}_{\alpha}\mathbb{C}^*_{\beta}$ and $g \in G'$:,

Remark 2.16. *Any vector space together with a $*_{\gamma}$-norm is called $*_{\gamma}$-normed space and a complete $*_{\gamma}$-normed space is a $*_{\gamma}$-Banach space.*

Theorem 2.17. *For $z^* \in {}_{\alpha}\mathbb{C}^*_{\beta}$ define ${}_{\gamma}\|.\|^* : {}_{\alpha}\mathbb{C}^*_{\beta} \longrightarrow G' \subset G$ as*

$$ {}_{\gamma}\|z^*\|^* = \dddot{\sqrt{\left[\imath^{-1}_{AC}(\dot{a}_1)\right]^{\ddot{2}} \dddot{+} \left[\imath^{-1}_{BC}(\ddot{b}_1)\right]^{\ddot{2}}}} $$

Then, ${}_{\gamma}\|.\|$ is a $*_{\gamma}$-norm.

Proof. The proof of (i) and (iii) will be on the same lines as in [10]. Now, we prove the condition (ii),

$$_{\gamma}\|g \otimes z_1^*\|^* = {}_{\gamma}\Big\|g \otimes (\dot{a}_1, \ddot{b}_1)\Big\|^* = {}_{\gamma}\Big\|\Big(\alpha\left(\gamma^{-1}(g) \times a_1\right), \beta\left(\gamma^{-1}(g) \times b_1\right)\Big)\Big\|^* = {}_{\gamma}|g| \,\ddot{\times}\, \gamma\Big(\sqrt{a_1^2 + b_1^2}\Big) = {}_{\gamma}|g| \,\ddot{\times}\, {}_{\gamma}\|z_1^*\|^*.$$

Thus, $_{\gamma}\|g \otimes z_1^*\|^* = {}_{\gamma}|g| \,\ddot{\times}\, {}_{\gamma}\|z_1^*\|^*$. Hence, the vector space $_{\alpha}\mathbb{C}_{\beta}^*$ (over any field) together with a $_{\gamma}\|.\|^*$ is a $*_{\gamma}$-normed space. ☐

Remark 2.18. *We call a $_{\gamma}\|.\|^*$ over a vector space non-decreasing if $_{\gamma}\|g \otimes z^*\|^* \ddot{\leq}_{\gamma}\|z^*\|^*$ for $\ddot{0} \,\ddot{\leq}\, \ddot{1}$. We can easily prove that the norm $_{\gamma}\|z^*\|^* = \ddot{\sqrt{\left[1_{AC}^{-1}(\dot{a}_1)\right]^{\ddot{2}} \ddot{+} \left[1_{BC}^{-1}(\ddot{b}_1)\right]^{\ddot{2}}}}$ is a non decreasing norm.*

Theorem 2.19. *The vector space $_{\alpha}\mathbb{C}_{\beta}^*$ (over any field) together with $_{\gamma}\|.\|^*$ is a $*_{\gamma}$-Banach space, where $_{\gamma}\|.\|^*$ is defined as follows:*

$$_{\gamma}\|z^*\|^* = \ddot{\sqrt{\left[1_{AC}^{-1}(\dot{a}_1)\right]^{\ddot{2}} \ddot{+} \left[1_{BC}^{-1}(\ddot{b}_1)\right]^{\ddot{2}}}}, 0^* \left(= (\dot{0}, \ddot{0})\right) \in {}_{\alpha}\mathbb{C}_{\beta}^*.$$

Proof. The norm $_{\gamma}\|.\|^*$ induces a $*_{\gamma}$-metric $_{\gamma}d^*$ on $_{\alpha}\mathbb{C}_{\beta}^*$ defined as $_{\gamma}d^*(z_1^*, z_2^*) = {}_{\gamma}\|z_1^* - z_2\|^*$. With this $*_{\gamma}$-metric the space is complete and hence the result. ☐

Definition 2.20. *Let F be an ordered field. An ordered F-vector space [6] is an ordered set $(V, \leq)$ where V is a vector space over F and $\leq$ satisfies the following conditions:*

(i) For all $u, v, w \in V$ such that $u \in v$, we have $u + w \leq v + w$ and

(ii) For all $u \in V$ and all $\lambda \in F$ such that $0 \leq u$ and $0 \leq \lambda$, we have $0 \leq \lambda u$.

Further if S the set of all infinite sequences in V. A binary relation $\to$ between S and Y is called a convergence on V if it satisfies the following axiom:

(i) If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$,

(ii) If $x_n \to x$ and $\lambda \in F$, then $\lambda x_n \to \lambda x$ and

(iii) If $\lambda_n \to \lambda$ in F, then $\lambda_n x \to \lambda x$.

The pair $(V, \to)$ is said to be a vector space with convergence and $\leq$ is called vector ordering on ordered vector space $(V(F), \leq)$ with convergence, if it is compatible with the convergence structure on V.

We define the ordering [3] for non-Newtonian Complex numbers and prove some results on this basis.

Theorem 2.21. *Let $z_1^* = (\dot{a}_1, \ddot{b}_1), z_2^* = (\dot{a}_2, \ddot{b}_2) \in {}_{\alpha}\mathbb{C}_{\beta}^*$. We define*

$$z_1^* \leq^* z_2^* \Leftrightarrow (\dot{a}_1 \dot{<} \dot{a}_2) \cup \left[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)\right] \text{ and } z_1^* \geq^* z_2^* \Leftrightarrow (\dot{a}_1 \dot{>} \dot{a}_2) \cup \left[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\geq} \ddot{b}_2)\right].$$

Then, $_{\alpha}\mathbb{C}_{\beta}^*$ is totally ordered set with $\leq^*$ as ordering relation.

Proof. Reflexive: $z_1^* \leq z_1^*$ so $\leq^*$ is reflexive.

Symmetric: Consider, $\{(z_1^* \geq^* z_2^*) \cap (z_1^* \leq^* z_2^*)\}$

$$\begin{aligned}
&\Leftrightarrow \left\{(\dot{a}_1 \dot{>} \dot{a}_2) \cup \left[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\geq} \ddot{b}_2)\right]\right\} \cap \left\{(\dot{a}_1 \dot{<} \dot{a}_2) \cup \left[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)\right]\right\} \\
&\Leftrightarrow \{(\dot{a}_1 \dot{>} \dot{a}_2) \cap (\dot{a}_1 \dot{<} \dot{a}_2)\} \cup \left\{[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\geq} \ddot{b}_2)]\right\} \cap \left\{[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)]\right\} \\
&\qquad \cup \left\{(\dot{a}_1 \dot{>} \dot{a}_2) \cap [(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)]\right\} \cup \left\{[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)]\right\} \cap (\dot{a}_1 \dot{<} \dot{a}_2) \\
&\Leftrightarrow \left\{[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\geq} \ddot{b}_2)]\right\} \cap [(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)] \\
&\Leftrightarrow (\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 = \ddot{b}_2) \Leftrightarrow z_1^* = z_2^*.
\end{aligned}$$

So, $[(z_1^* \geq^* z_2^*)$ and $(z_1^* \leq^* z_2^*)] \Leftrightarrow (z_1^* = z_2^*)$.

Transitive: For $z_1^*, z_2^*, z_3^* \in {}_{\alpha}\mathbb{C}_{\beta}^*$, consider $z_1^* \leq^* z_2^*$ and $z_2^* \leq^* z_3^*$.

$$\begin{aligned}
&\Leftrightarrow \Big\{[\dot{a}_1 \dot{<} \dot{a}_2] \cup [(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)]\Big\} \cap \Big\{[\dot{a}_2 \dot{<} \dot{a}_3] \cup [(\dot{a}_2 = \dot{a}_3) \cap (\ddot{b}_2 \ddot{\leq} \ddot{b}_3)]\Big\}\\
&\Leftrightarrow \{[\dot{a}_1 \dot{<} \dot{a}_2] \cap [\dot{a}_2 \dot{<} \dot{a}_3]\} \cup \Big\{[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)] \cap [(\dot{a}_2 = \dot{a}_3) \cap (\ddot{b}_2 \ddot{\leq} \ddot{b}_3)]\Big\}\\
&\qquad \cup \Big\{[\dot{a}_1 \dot{<} \dot{a}_2] \cap (\dot{a}_2 = \dot{a}_3) \cap (\ddot{b}_2 \ddot{\leq} \ddot{b}_3)]\Big\} \cup \Big\{[(\dot{a}_1 = \dot{a}_2) \cup (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)] \cap [\dot{a}_2 \dot{<} \dot{a}_3]\Big\}\\
&\Leftrightarrow \{[\dot{a}_1 \dot{<} \dot{a}_3]\} \cup \Big\{[\dot{a}_1 \dot{<} \dot{a}_3) \cap (\ddot{b}_2 \ddot{\leq} \ddot{b}_3)]\Big\} \cup \Big\{[(\dot{a}_1 \dot{<} \dot{a}_3) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)]\Big\} \cup \Big\{(\dot{a}_1 = \dot{a}_3) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_3)\Big\}\\
&\Leftrightarrow \{[\dot{a}_1 \dot{<} \dot{a}_3]\} \cup \Big\{(\dot{a}_1 = \dot{a}_3) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_3)\Big\} \Leftrightarrow z_1^* \leq^* z_3^*.
\end{aligned}$$

Comparability: For every $z_1^*, z_2^* \in {}_{\alpha}\mathbb{C}_{\beta}^*$ atleast one of the relations $z_1^* \leq^* z_2^*$ or $z_1^* \geq^* z_2^*$ holds. So, ${}_{\alpha}\mathbb{C}_{\beta}^*$ is totally ordered set with $\leq^*$ as ordering relation. □

Theorem 2.22. *$({}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}_{\gamma}(N)), \leq^*)$ is an ordered $(\mathbb{R}_{\gamma}(N))$-Vector space.*

Proof. First we prove that $z_1^* + z_3^* \leq^* z_2^* + z_3^*$ iff $z_1^* \leq^* z_2^*$. We have, $z_1^* + z_3^* \leq^* z_2^* + z_3^* \Leftrightarrow \Big\{(\dot{a}_1 \dot{+} \dot{a}_3 \dot{<} \dot{a}_2 \dot{+} \dot{a}_3) \cup \Big[(\dot{a}_1 \dot{+} \dot{a}_3 = \dot{a}_2 \dot{+} \dot{a}_3) \cap (\ddot{b}_1 \ddot{+} \ddot{b}_3 \ddot{\leq} \ddot{b}_2 \ddot{+} \ddot{b}_3)\Big]\Big\} \Leftrightarrow \Big\{(\dot{a}_1 \dot{<} \dot{a}_2) \cup \Big[(\dot{a}_1 = \dot{a}_2) \cap (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)\Big]\Big\} \Leftrightarrow z_1^* \leq^* z_2^*$. Next, we prove that for $g > 0$, and $g \odot z_1^* \leq^* g \odot z_2^* \Leftrightarrow z_1^* \leq^* z_2^*$. Suppose $g > 0$, we have $g \odot z_1^* \leq^* g \odot z_2^*$

$$\begin{aligned}
&\Leftrightarrow \Big(\alpha\Big((\gamma^{-1}(g))\,\dot{a}_1\Big) \dot{<} \alpha\Big((\gamma^{-1}(g))\,\dot{a}_2\Big)\Big)\\
&\qquad \cup \Big[\Big(\alpha\Big((\gamma^{-1}(g))\,\dot{a}_1\Big) = \alpha\Big((\gamma^{-1}(g))\,\dot{a}_2\Big)\Big) \cap \Big(\beta\Big((\gamma^{-1}(g))\,\ddot{b}_1\Big) \ddot{<} \beta\Big((\gamma^{-1}(g))\,\ddot{b}_2\Big)\Big)\Big]\\
&\Leftrightarrow (\dot{a}_1 \dot{<} \dot{a}_2) \cap [(\dot{a}_1 = \dot{a}_2) \cup (\ddot{b}_1 \ddot{\leq} \ddot{b}_2)], \Leftrightarrow z_1^* \leq^* z_2^*.
\end{aligned}$$

□

Theorem 2.23. *${}_{\gamma}\|.\|^*$ produces a convergence on ${}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}_{\gamma}(N))$ in the following sense. A sequence $(z_n)^* \in {}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}(N))$ is convergent if ${}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}_{\gamma}(N))$ contains a z such that $\lim^*_{n\to\infty} {}_{\gamma}\|(z_n)^* - z^*\|^* = \ddot{0}$ $\Big($We say $\Big({}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}(N)), {}_{\gamma}\|z^*\|^*\Big)$ is a vector space with convergence$\Big)$. Further $\leq^*$ is a vector ordering on $({}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}_{\gamma}(N)), {}_{\gamma}\|.\|^*)$.*

Proof. First part can be verified using the properties of the $*_{\gamma}$-norm. We prove that $\leq^*$ is a vector ordering on $({}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}_{\gamma}(N)), \leq^*)$. Consider convergent sequences $Z_1 = ({z_1}^*)_n$, $Z_2 = ({z_2}^*)_n$, $Z_3 = ({z_3}^*)_n \in {}_{\alpha}\mathbb{C}_{\beta}^*(\mathbb{R}_{\gamma}(N))$ with limits ${z_1}^*$, ${z_2}^*$, ${z_3}^*$ respectively. Let $({z_1}^*)_n \leq^* ({z_2}^*)_n \;\; \forall n$ and $({z_3}^*)_n = \Big(({z_2}^*)_n \ominus ({z_1}^*)_n\Big)$. As $({z_1}^*)_n \leq^* ({z_2}^*)_n \;\; \forall n$, so $({z_3}^*)_n \geq^* 0^*$. Using properties of the norm we get $({z_3}^*) = \lim_{n\to\infty}({z_1}^*)_n \geq^* 0^*$ and hence $0^* \leq^* \lim({z_3}^*)_n = \lim({z_2}^*)_n \ominus \lim({z_1}^*)_n$. Thus, ${z_1}^* \leq^* {z_2}^*$. □

Further we define the extended non-Newtonian real number system (realm G)[8]: The extended non newtonian real number system (realm G) denoted $(\mathbb{R}_{\gamma}^{ext}(N))$ consists of the non newtonian real numbers $(\mathbb{R}_{\gamma}(N))$ and two symbols $+\infty$ and $-\infty$. We preserve the original order in $(\mathbb{R}_{\gamma}(N))$, and define $(-\infty) \ddot{<} g \ddot{<} (+\infty)$ for every $g \in (\mathbb{R}_{\gamma}^{ext}(N))$. $(\mathbb{R}_{\gamma}^{ext}(N))$ does not form a field, but following conventions are made:

(1) For $g \in G$, $g \dddot{+} \infty = +\infty$, $g \dddot{-} \infty = -\infty$, $g \dddot{\div} (+\infty) = g \dddot{\div} (-\infty) = \ddot{0}$,

(2) If $g \dddot{>} \ddot{0}$ then $g \dddot{\times} (+\infty) = (+\infty)$, $g \dddot{\times} (-\infty) = (-\infty)$ and

(3) If $g \dddot{<} \ddot{0}$ then $g \dddot{\times} (+\infty) = (-\infty)$, $g \dddot{\times} (-\infty) = (+\infty)$.

Definition 2.24 (Generalized modular)**.** *A function,* ${}_{\gamma}\rho^{*} : {}_{\alpha}\mathbb{C}^{*}_{\beta}(\mathbb{R}) \to \ddot{[}\ \ddot{0}, \infty\ \ddot{]} = G'$ $\Big($*where* G' *denote the* γ*-positive numbers in* $(\mathbb{R}^{ext}_{\gamma}(N))\Big)$ *is called* $*_{\gamma}$*-modular if it satisfy the following conditions:*

(1) ${}_{\gamma}\rho^{*}(z_1^{*}) = \ddot{0}$ *if and only if* $z_1^{*} = 0^{*}$,

(2) ${}_{\gamma}\rho^{*}(g \otimes z_1^{*}) = {}_{\gamma}\rho^{*}(z_1^{*})$ *provided* $g = \ddot{1}$,

(3) ${}_{\gamma}\rho^{*}(g_1 \otimes z_1^{*} + g_2 \otimes z_2^{*}) \ddot{\leq} {}_{\gamma}\rho^{*}(z_1^{*}) + {}_{\gamma}\rho^{*}(z_2^{*})$, *provided* $g_1,\ g_2 \ddot{\leq} \ddot{0},\ g_1 \ddot{+} g_2 = \ddot{1}$.

(4) ${}_{\gamma}\rho^{*}(g_n \otimes z_1^{*}) \to \ddot{0}$ *if* $g_n \to \ddot{0}$ *and* ${}_{\gamma}\rho^{*}(z_1^{*}) \ddot{<} \infty$.

Theorem 2.25. *Let* $\mathbb{C}^{*}_{\gamma\rho^{*}}$ *denotes the set of all* $z^{*} \in {}_{\alpha}\mathbb{C}^{*}_{\beta}$ *such that,* ${}_{\gamma}\rho^{*}(g \otimes z^{*}) \ddot{<} \infty$ *for some* $g \in \ddot{[}\ \ddot{0}, \infty\ \ddot{]} (= G'_{ext})$*. The set* $\mathbb{C}^{*}_{\gamma\rho^{*}}$ *is a subspace of* ${}_{\alpha}\mathbb{C}^{*}_{\beta}$*.*

Proof. Let $z^{*} \in \mathbb{C}^{*}_{\gamma\rho^{*}}$ and $g_1 (\in G)$ be a scalar. So, ${}_{\gamma}\rho^{*}(g \otimes z^{*}) = {}_{\gamma}\rho^{*}\left[\left(\frac{g}{\gamma|t|^{*}}\right) \ddot{\times}_{\gamma}|t|^{*} \otimes z^{*}\right] = {}_{\gamma}\rho^{*}(g \otimes z^{*}) = {}_{\gamma}\rho^{*}\left[\left(\frac{g}{\gamma|t|^{*}}\right) \ddot{\times} t \otimes z^{*}\right] \ddot{<} \infty$. (Here $\left(\frac{g}{\gamma|t|^{*}}\right)$ means $g\ \ddot{\div}\ {}_{\gamma}|t|^{*}$) So, ${}_{\gamma}\rho^{*}(t \otimes z^{*}) \in \mathbb{C}^{*}_{\gamma\rho^{*}}$. Let $z_1^{*}, z_2^{*} \in \mathbb{C}^{*}_{\gamma\rho^{*}}$, then there are $g_1, g_2 \in G'_{ext}$ such that, ${}_{\gamma}\rho^{*}(g_1 \otimes z_1^{*}) \ddot{<} \infty$ and ${}_{\gamma}\rho^{*}(g_2 \otimes z_2^{*}) \ddot{<} \infty$. Let $k (\in G) = min(g_1, g_2)$ then we have, ${}_{\gamma}\rho^{*}\left(\frac{k}{2} \otimes (z_1^{*} \oplus z_2^{*})\right) \ddot{\leq} {}_{\gamma}\rho^{*}(k \otimes z_1^{*}) \ddot{\leq} {}_{\gamma}\rho^{*}(g_1 \otimes z_1^{*}) \ddot{+} {}_{\gamma}\rho^{*}(g_1 \otimes z_1^{*}) \ddot{<} \infty$. Hence, $z_1^{*} \oplus z_2^{*} \in \mathbb{C}^{*}_{\gamma\rho^{*}}$. ☐

Remark 2.26. ${}_{\gamma}\rho^{*}(.)$ *produces a convergence on* $\mathbb{C}^{*}_{\gamma\rho^{*}}$ *as; A sequence* $(z_n)^{*} \in \mathbb{C}^{*}_{\gamma\rho^{*}}$ *is convergent if* $\mathbb{C}^{*}_{\gamma\rho^{*}}$ *contains a* z^{*} *such that,* $\lim^{*}_{n\to\infty}\ {}_{\gamma}\rho^{*} g \otimes \left((z_n)^{*} - z^{*}\right)^{*} = \ddot{0}$ *for some* $g \in G$ *(we say* $(\mathbb{C}^{*}_{\gamma\rho^{*}},\ {}_{\gamma}\rho^{*}(.))$ *is a vector space with convergence). Further this convergence is equivalent to the convergence generated by the norm* ${}_{\gamma}\|.\|^{*}$ *on* $\mathbb{C}^{*}_{\gamma\rho^{*}}$*.*

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