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Fractional Order Model in Generalist Predator–Prey Dynamics

Research Article

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Abstract: In this paper, we construct the fractional order generalist predator–prey dynamics model. The stability of free and positive fixed points is studied. The Adams-Bashforth-Moulton algorithm has been used to solve and simulate the system of differential equations.

Keywords: Generalist predator, Fractional order, Stability, Limit cycle, Numerical method.

1. Introduction

Predator-prey models are arguably the building blocks of the bio- and ecosystems as biomasses are grown out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their specific settings of applications, they can take the forms of resource-consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise [23]. In recent decades, the fractional calculus and Fractional differential equations have attracted much attention and increasing interest due to their potential applications in science and engineering [13, 21]. In this paper, we consider the fractional order model for a model consisting of predator and prey. We give a detailed analysis for the asymptotic and global stability of the model. Adams- Bashforth- Moulton algorithm have been used to solve and simulate the system of fractional differential equations.

2. Model Formulation

The model for generalist predator and prey can be written as a set of coupled nonlinear ordinary differential equations as follows [11]:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{x^2 y}{1+ax^2}, \\ \frac{dy}{dt} &= \frac{bx^2 y}{1+ax^2} + \frac{cy}{1+dy} - ey. \end{aligned} \tag{1}$$

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Where a, b, c, d and e are constant. Fractional order models are more accurate than integer-order models as fractional order models allow more degrees of freedom. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. The presence of memory term in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the process. Fractional differential equations are also regarded as an alternative model to nonlinear differential equations. In consequence, the subject of fractional differential equations is gaining much importance and attention. For some recent work on fractional differential equations, see [13, 21]. Now we introduce fractional order in to the ODE model by Erbach et al. [11]. The new system is described by the following set of fractional order differential equations:

$$\begin{aligned} D_t^\alpha x &= x(1-x) - \frac{x^2 y}{1+ax^2}, \\ D_t^\alpha y &= \frac{bx^2 y}{1+ax^2} + \frac{cy}{1+dy} - ey. \end{aligned} \tag{2}$$

where D_t^α is the Caputo fractional derivative. All the parameters of system (2) are assumed to be non-negative. Furthermore, it can be shown that all state variables of the model are non-negative for all time $t \geq 0$.

Lemma 2.1. *The solutions of the system* (2) *is exist in* $\mathbb{R}_+^2$ *and uniformly bounded .*

Proof. Let $(x(t), y(t))$ be any solution of the system (2) with positive initial conditions. Since

$$D_t^\alpha x \leq x(1-x), \tag{3}$$

by Lemma 9 [14] we have

$$x(t) \leq x(0)E_\alpha\left(t^\alpha\right),$$

where E_α is the Mittag-Leffler function. Let

$$W = x + \frac{1}{b}y,$$

then

$$\begin{aligned} D_t^\alpha W &= D_t^\alpha x + \frac{1}{b} D_t^\alpha y \\ &= x(1-x) + \frac{y}{b}\left(\frac{c}{1+dy} - e\right) \\ &\leq x(1-x) + \frac{ry}{b}\left(\frac{1}{1+dy} - 1\right), \\ &= x(1-x) - \frac{rdy^2}{b\left(1+dy\right)} \\ &\leq x(1-x) \\ &\leq x(0)E_\alpha\left(t^\alpha\right) \end{aligned}$$

where $r = \max\{c, e\}$. By Lemma 9 [14] again, we have

$$0 \leq W(x,y) \leq x(0)E_\alpha\left(t^\alpha\right) t^\alpha E_{\alpha,\alpha+1}\left(t^\alpha\right) = W_1, \tag{4}$$

where E_α is the Mittag-Leffler function. Therefore, all solutions of the model (2) with initial conditions in Ω s.t

$$\Omega = \left\{(X,Y,Z) \in W : 0 \leq W \leq W_1\right\},$$

remain in Ω for all $t > 0$. Thus, region Ω is positively invariant with respect to model (2). ☐

In the following, we will study the dynamics of system (2).

3. Equilibrium Point and Stability

In the following, we discuss the stability of the commensurate fractional ordered dynamical system:

$$D_t^{\alpha} x_i = f_i(x_1, x_2), \qquad \alpha \in (0,1), \qquad 1 \le i \le 2. \tag{5}$$

Let $E = (x_1^*, x_2^*)$ be an equilibrium point of system (5) and $x_i = x_i^* + \eta_i$, where η_i is a small disturbance from a fixed point. Then

$$\begin{aligned} D_t^{\alpha} \eta_i &= D_t^{\alpha} x_i \\ &= f_i(x_1^* + \eta_1, x_2^* + \eta_2) \\ &\approx \eta_1 \frac{\partial f_i(E)}{\partial x_1} + \eta_2 \frac{\partial f_i(E)}{\partial x_2}. \end{aligned} \tag{6}$$

System can be written as:

$$D_t^{\alpha} \eta = J\eta, \tag{7}$$

where $\eta = (\eta_1, \eta_2)^T$ and J is the Jacobian matrix evaluated at the equilibrium points. Using Matignon's results [17], it follows that the linear autonomous system (7) is asymptotically stable if $|arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix J at the equilibrium point $E = (x_1^*, x_2^*)$. If $\Phi(x) = x^2 + a_1 x + a_2$, Let $D(\Phi)$ denote the discriminant of a polynomial Φ, then

$$D(\Phi) = -\begin{vmatrix} 1 & a_1 & a_2 \\ 2 & a_1 & 0 \\ 0 & 2 & a_1 \end{vmatrix} = a_1^2 - 4a_2.$$

Following [1–3], we have the proposition.

Proposition 3.1. *One assumes that E exists in R_+^2.*

(1). If the discriminant of $\Phi(x)$ and $D(\Phi)$ is positive and Routh-Hurwitz are satisfied, that is, $D(\Phi) \ge 0$, $a_1 > 0$ and $a_2 > 0$, then E is locally asymptotically stable.

(2). If $D(\Phi) < 0$ and $\left|\tan^{-1}\left(\frac{\sqrt{4a_2 - a_1^2}}{a_1}\right)\right| > \frac{\alpha\pi}{2}$, $\alpha \in [0,1)$ then E is locally asymptotically stable.

To evaluate the equilibrium points let

$$D_t^{\alpha} x = 0, \qquad D_t^{\alpha} y = 0.$$

Then

1. The first trivial equilibrium point is $E_0 = (0,0)$. The point E_0 always exists.

The Jacobian matrix J_0 for system given in (2.2) evaluated at the free equilibrium is as follows:

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & c - e \end{pmatrix}.$$

Theorem 3.2. *The trivial equilibrium point E_0 of system (2) is a saddle point.*

Proof. The trivial equilibrium point E_0 is locally asymptotically stable if all the eigenvalues λ_{0i}, $i = 1,2$ of J_0 satisfy Matignon's conditions. The eigenvalues corresponding to the equilibrium E_0 are $\lambda_{01} = 1$ and $\lambda_{02} = c - e$.

Then we have $\lambda_{01} > 0$. It follows that the node equilibrium point of system (2) is a saddle point, non-empty stable manifolds and an unstable manifold. □

2. The second semi-trivial equilibrium point is $E_1 = (x_1, y_1) = (1, 0)$ when the predator is absent in the prey, in this case $(y = 0)$, therefore the prey is not an exhibition of predation. The point E_1 always exists.

Theorem 3.3. *For the system* (2)*, one have the basic reproduction number*

$$R_0 = \frac{b}{(1+a)(c-e)}.$$

Proof. We will use the next generation method [8] to find the basic reproduction number, for the system (2), rewrite the equations by which classes of the herbivore population y first and then the plant population x secondly, we have

$$\begin{aligned} D_t^\alpha y &= \frac{bx^2 y}{1+ax^2} + \frac{cy}{1+dy} - ey, \\ D_t^\alpha x &= x(1-x) - \frac{x^2 y}{1+ax^2}. \end{aligned} \tag{8}$$

We make matrices f, v, such that the system (8) in the form

$$\frac{d^\alpha X}{dt^\alpha} = f(X) - v(X),$$

where

$$f(X) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{bx^2 y}{1+ax^2} \\ -\frac{x^2 y}{1+ax^2} \end{bmatrix}, \; v(X) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ey - \frac{cy}{1+dy} \\ -x(1-x) \end{bmatrix}.$$

We can get

$$F(X) = \begin{bmatrix} \frac{2bxy}{1+ax^2} - \frac{2abx^3 y}{(1+ax^2)^2} & \frac{bx^2}{1+ax^2} \\ \frac{2xy}{1+ax^2} - \frac{2ax^3 y}{(1+ax^2)^2} & \frac{x^2}{1+ax^2} \end{bmatrix}, V(X) = \begin{bmatrix} 0 & e - \frac{c}{dy+1} + \frac{cdy}{(dy+1)^2} \\ 2x-1 & 0 \end{bmatrix},$$

at the first free equilibrium point $E_1 = (1, 0)$ to get the eigenvalues of $F \cdot V^{-1}$, one solve the equation

$$\left| F \cdot V^{-1} - \lambda I \right| = 0,$$

where λ is the eigenvalues and I is the identity matrix. $F \cdot V^{-1}$ is the next generation matrix for model (8), then $\lambda_1 = \frac{b}{(1+a)(e-c)}$, $\lambda_2 = 0$. It the follows that the spectral radius of matrix $F \cdot V^{-1}$ is $\rho\left(F \cdot V^{-1}\right) = \max\left(\lambda_i\right), \, i = 1, 2$. According to Theorem 2 in [8], the basic reproduction number of model (8) is $R_0 = \frac{b}{(1+a)(e-c)}$. □

The Jacobian matrix J_1 for system given in (2) evaluated at the free equilibrium is as follows:

$$J_1 = \begin{pmatrix} -1 & \frac{-1}{1+a} \\ 0 & 1 - R_0 \end{pmatrix}.$$

Theorem 3.4. *The semi-trivial equilibrium point E_1 of system* (2) *is an asymptotically stable if $R_0 > 1$.*

Proof. The eigenvalues of J_1 which corresponding to the equilibrium E_1 are $\lambda_{01} = -1$ and $\lambda_{02} = 1 - R_0$. Then we have $\lambda_{01} < 0$ and $\lambda_{02} < 0$ if $R_0 > 1$, it follows that E_1 is an asymptotically stable if $R_0 > 1$. □

3. By (2), the third semi-trivial equilibrium point is $E_2 = \left(0, \frac{c-e}{ed}\right)$ which is exist if $c > e$.

The Jacobian matrix J_2 for system given in (2) evaluated at E_2 is as follows:

$$J_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{eb}{c(1+a)R_0} \end{pmatrix}.$$

Theorem 3.5. *The semi-trivial equilibrium point E_1 of system (2) is a saddle point.*

Proof. The eigenvalues of J_2 which corresponding to the equilibrium E_2 are $\lambda_{01} = 1$ and $\lambda_{02} = \frac{eb}{c(1+a)R_0}$. Then we have $\lambda_{01} > 0$, it follows that E_1 is a saddle point. □

4. The least one interior equilibrium point is $E_3 = (x_*, y_*)$, where x_* is the real root of the equation

$$f(Z) = ad(ae-b)\left(Z^5 - Z^4\right) - [(ae-b)(d-1)+$$

$$a(de+c)]\, Z^3 - d(2ae-b)Z^2 + [e\,(d-1)+c]\, Z - de = 0,$$

and $y_* = \frac{(1-x_*)\left(1+ax_*^2\right)}{x_*}$, E_3 must be have non negative component, then we have the condition $0 < x_* < 1$.
We now discuss the asymptotic stability of a positive interior equilibrium point E_3 of the system given by (2). The Jacobian matrix J_3 evaluated at a positive equilibrium E_3 is given as:

$$J_3 = \begin{pmatrix} 1 - 2x_* - \frac{2x_* y_*}{\left(1+ax_*^2\right)^2} & \frac{-x_*^2}{1+ax_*^2} \\ \frac{2x_* y_*}{\left(1+ax_*^2\right)^2} & \frac{bx_*^2}{1+ax_*^2} + \frac{c}{(1+dy_*)^2} - e \end{pmatrix}.$$

The characteristic equation of J_3 is

$$\lambda^2 - \text{trace}(J_3)\lambda + \det(J_3) = 0, \tag{9}$$

where

$$\begin{aligned}
\text{trace}(J_3) &= 1 - 2x_* - \frac{2x_* y_*}{(ax_*^2+1)^2} + \frac{bx_*^2}{ax_*^2+1} + \frac{c}{(dy_*+1)^2} - e, \\
\det(J_3) &= -(-2a^2d^2ex_*^5y_*^2 + a^2d^2ex_*^4y_*^2 + 2abd^2x_*^5y_*^2 - 4a^2dex_*^5y_* - \\
&\quad abd^2x_*^4y_*^2 + 2adex_*^4y_* + 4abdx_*^5y_* - 4ad^2ex_*^3y_*^2 + 2a^2cx_*^5 - \\
&\quad 2a^2ex_*^5 - 2abdx_*^4y_* + 2ad^2ex_*^2y_*^2 + 2bd^2x_*^3y_*^2 - a^2cx_*^4 + \\
&\quad a^2ex_*^4 + 2abx_*^5 - 8adex_*^3y_* - bd^2x_*^2y_*^2 - 2d^2ex_*y_*^3 - abx_*^4 + \\
&\quad 4adex_*^2y_* + 4bdx_*^3y_* - 2d^2ex_*y_*^2 + 4acx_*^3 - 4aex_*^3 - \\
&\quad 2bdx_*^2y_* + d^2ey_*^2 - 4dex_*y_*^2 - 2acx_*^2 + 2aex_*^2 + 2bx_*^3 - \\
&\quad 4dex_*y_* - bx_*^2 + 2cx_*y_* + 2dey_* - 2ex_*y_* + 2cx_* - \\
&\quad 2ex_* - c + e)/\left[(ax_*^2+1)^2(dy_*+1)^2\right].
\end{aligned}$$

For the characteristic equation (9) have the roots

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[\text{trace}(J_3) \pm \sqrt{\text{trace}^2(J_3) - 4\det(J_3)}\right].$$

Theorem 3.6. *The positive equilibrium E_3 of system (2) is locally asymptotically stable if and only if all the following conditions are satisfied:*

(1). $\det(J_3) > 0$ *and*

(2). $\text{trace}(J_3) < 2\sqrt{\det(J_3)}\cos(\frac{\alpha\pi}{2})$.

Proof. It is clear that E_3 is locally asymptotically stable if λ_1, λ_2 has negative real part implies that $\det(J_3) > 0$, then $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$, $j = 1, 2$, if and only if the conditions (1) and (2) hold. □

Theorem 3.7. *With respect to system (2), the following statements can be obtained.*

(a). If $\mathrm{trace}(J_3) \le 0$*, the equilibrium* E_3 *is locally asymptotically stable, for any* $\alpha \in (0, 1)$*,*

(b). If $0 < \mathrm{trace}(J_3) < 2\sqrt{\det(J_3)}$*, the equilibrium* E_3 *is locally asymptotically stable if and only if* $\alpha \in (0, \alpha^*)$*, where* $\alpha^* = \frac{2}{\pi}\left|\cos^{-1}\left(\frac{\mathrm{trace}(J_3)}{2\sqrt{\det(J_3)}}\right)\right|$ *and*

(c). If $\mathrm{trace}(J_3) \ge 2\sqrt{\det(J_3)}$*, the equilibrium* E_3 *is unstable for any* $\alpha \in (0, 1)$*.*

Proof. The conclusions (a) and (c) are obvious. For the statement (b), due to $0 < \mathrm{trace}(J_3) < 2\sqrt{\det(J_3)}$, the equation (9) has two complex roots λ_1, λ_2, and their real part is $\frac{\mathrm{trace}(J_3)}{2} > 0$. Then $|\arg(\lambda_j)| = \cos^{-1}\left(\frac{\mathrm{trace}(J_3)}{2\sqrt{\det(J_3)}}\right)$, $j = 1, 2$. Besides, according to the condition $\cos^{-1}\left(\frac{\mathrm{trace}(J_3)}{2\sqrt{\det(J_3)}}\right) = \frac{\alpha^*\pi}{2}$, $\alpha \in (0, \alpha^*)$ if and only if $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$, $j = 1, 2$, it is concluded that Theorem 3.6 is true. □

According to the statement of Theorem 3.5 and Theorem 3.6, it can be concluded that the positive equilibrium is locally asymptotically stable if and only if $\alpha \in (0, \alpha^*)$. At $\alpha = \alpha^*$ the Hopf bifurcation is expected to take place. As increases above the critical value α^* the positive equilibrium is unstable and a limit cycle is expected to appear in the proximity of E_3 due to the Hopf bifurcation phenomenon.

4. Numerical Methods and Simulation

Since most of the fractional-order differential equations do not have exact analytic solutions, approximation and numerical techniques must be used. Several analytical and numerical methods have been proposed to solve the fractional order differential equations. For numerical solutions of system (2), one can use the generalized Adams-Bashforth-Moulton method. To give the approximate solution by means of this algorithm, consider the following nonlinear fractional differential equation [5, 6, 12, 15]

$$\begin{aligned} D_t^{\alpha} y(t) &= f(t, y(t)), \qquad 0 \le t \le T, \\ y^{(k)}(0) &= y_0^k, \qquad k = 0, 1, 2, \ldots, m-1, \quad \text{where } m = [\alpha], \end{aligned}$$

This equation is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds. \tag{10}$$

Diethelm et al. used the predictor-correctors scheme [5, 6], based on the Adams-Bashforth-Moulton algorithm to integrate Equation (10). By applying this scheme to the the fractional order generalist predator–prey dynamics model, and setting $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, 2, ..., N \in Z^+$, Equation (10) can be discretized as follows [5, 6, 15]:

$$\begin{aligned} x_{n+1} &= x_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left[x_{n+1}^p(1 - x_{n+1}^p) - \frac{\left(x_{n+1}^p\right)^2 y_{n+1}^p}{1 + a\left(x_{n+1}^p\right)^2}\right] \\ &\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=1}^{n} a_{j,n+1}\left[x_j(1 - x_j) - \frac{x_j^2 y_j}{1 + a x_j^2}\right], \\ y_{n+1} &= y_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left[\frac{b\left(x_{n+1}^p\right)^2 y_{n+1}^p}{1 + a\left(x_{n+1}^p\right)^2} + \frac{c y_{n+1}^p}{1 + d y_{n+1}^p} - e y_{n+1}^p\right] \\ &\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=1}^{n} a_{j,n+1}\left[\frac{b x_j^2 y_j}{1 + a x_j^2} + \frac{c y_j}{1 + d y_j} - e y_j\right], \end{aligned}$$

where

$$x_{n+1}^{p} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} \left[x_j(1-x_j) - \frac{x_j^2 y_j}{1+ax_j^2} \right],$$
$$y_{n+1}^{p} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} \left[\frac{bx_j^2 y_j}{1+ax_j^2} + \frac{cy_j}{1+dy_j} - ey_j \right],$$

$$a_{j,n+1} = \begin{cases} n^{\alpha-1} - (n-\alpha)(n+1), & j=0 \\ (n-j-2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1. \end{cases}$$
$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha} \left[(n-j+1)^{\alpha} - (n-j)^{\alpha} \right], \qquad 0 \le j \le n.$$

5. Conclusion

In the present paper, a fractional order generalist predator-prey dynamics is proposed and dynamical behavior of this system has been extensively investigated. We establish conditions under which equilibria of the fractional system exist and we derive conditions for stability of the positive equilibria. The numerical solutions and simulations are also given to verify the feasibility of the results. The trajectory of the system (3) with the initial condition close to the positive equilibrium E_3, as indicated in Figures 1 and 2, converges to an asymptotically stable limit cycle. The theoretical and numerical results presented in this paper show that the fractional order generalist predator-prey dynamics model may exhibit rich dynamical behavior.

The transformation of a classical model into a fractional one makes it very sensitive to the order of differentiation α : a small change in α may result in a big change in the final result. From the numerical results Figures follows, it is clear that the approximate solutions depend continuously on the fractional derivative α, and that their dynamics becomes more and more complex by varying the fractional order $n \in (0,1)$. The approximate solutions $x(t)$ and $y(t)$ are displayed in Figure 1 for $a = 100$, $b := 55$, $c = 1$, $d = 0.9$ and $e = 0.5$.

The results show that the the trajectory of system (2) converges to the equilibrium $E_3$for $\alpha = 0.75$. When $\alpha = 0.79$ the trajectory of system (2) converges to an asymptotically stable limit cycle. From Theorem 3.7, it is known that when $\alpha < \alpha^*$, the trajectories converge to the equilibrium point, as shown in Figure 1; whereas when α is increased to exceed α^*, the origin loses its stability, and a Hopf-type bifurcation occurs, as shown in Figure 2 and 3. Illustrates the approximate solutions $x(t)$ and $y(t)$, when $a = 80$, $b := 4.2$, $c = 0.7$, $d = 0.6$, $e = 0.1$; and $(x_0, y_0) = (0.14, 15.84)$.

In this case, $\text{trace}(J_3) = 0.02296643976$, $\det(J_3) = 0.0157137163$ and the values of the basic reproductive number $R_0 = 0.08641975308$, the equilibrium point $E_3 = (0.1476008626, 15.84021599)$ and $\alpha^* = 0.9415998952$. The results show that the the trajectory of system (2) converges to the equilibrium $E_3$for $\alpha = 0.9$ as shown in Figure.1. When $\alpha = 0.95$ the trajectory of system (2) converges to an asymptotically stable limit cycle, as shown in Figure 2.

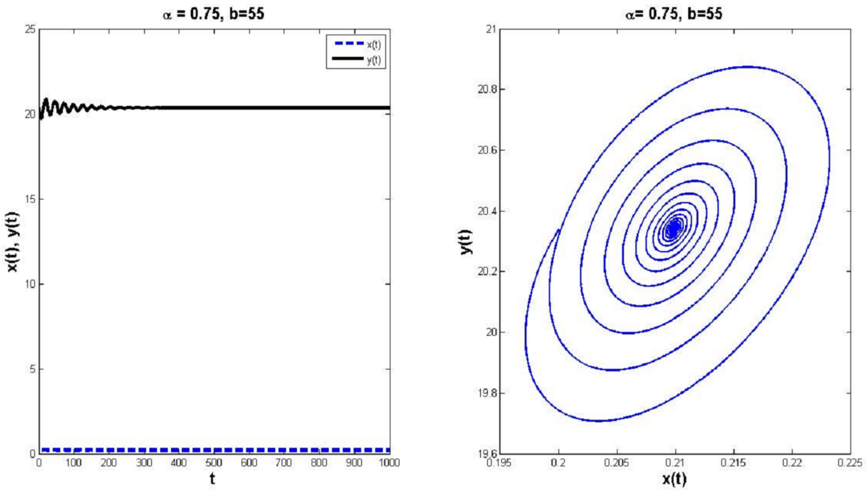

Figure 1.

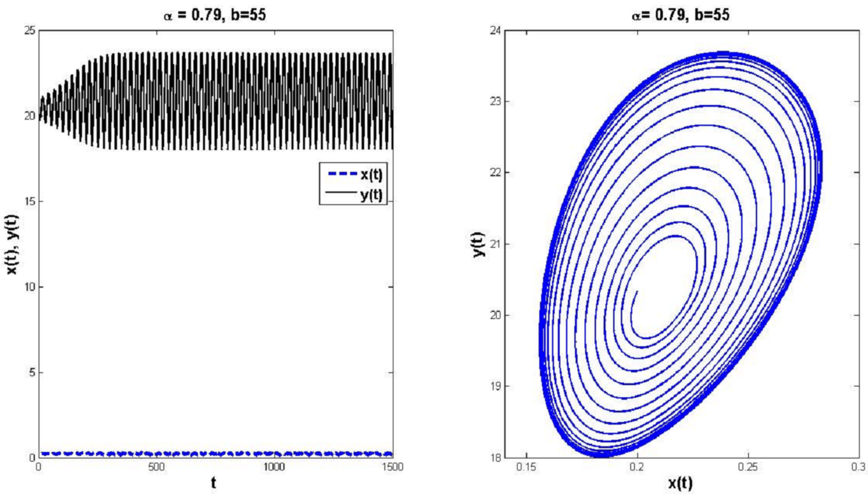

Figure 2.

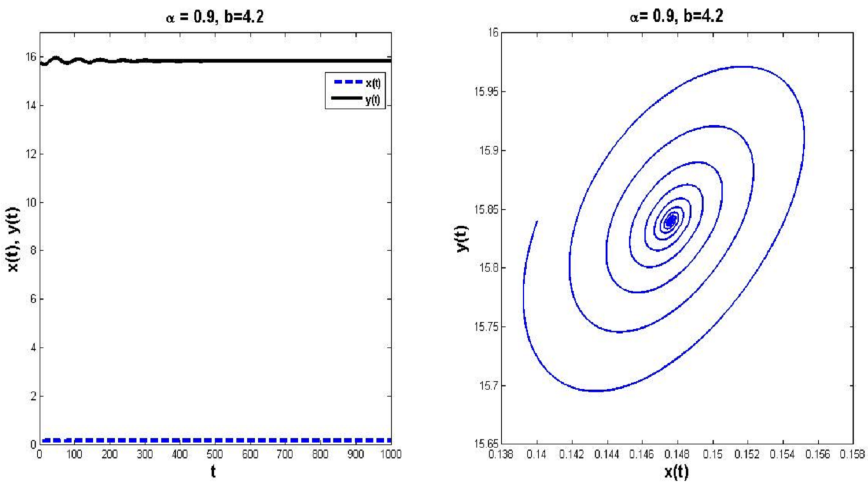

Figure 3.

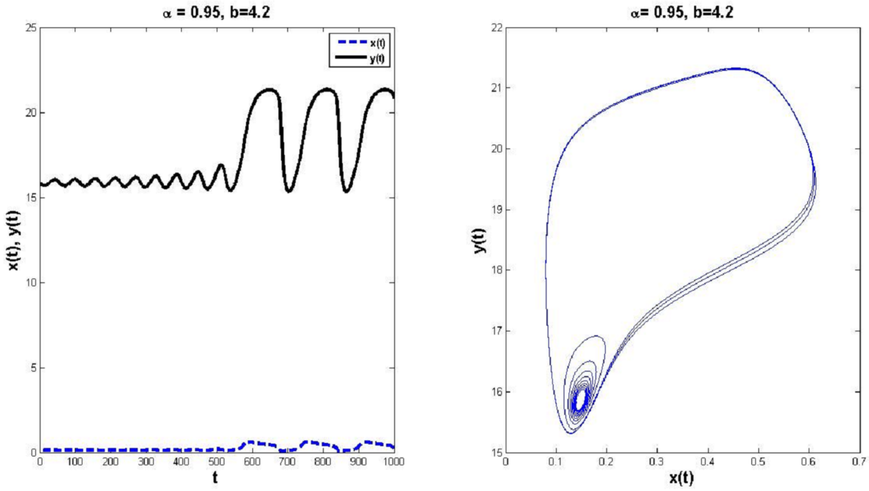

Figure 4.

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