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Commutators of Fractional Integral with Variable Kernel on Variable Exponent Herz and Lebesgue Spaces

Research Article

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Abstract: In this paper, we study the boundedness for commutators of the fractional integral with variable kernel on Lebesgue spaecs with variable exponent. Also, we study the boundedness of the fractional integral operator and their commutator generated by BMO function is obtained on those Herz spaces with two variable exponent $p(\cdot), q(\cdot)$.

MSC: 42B25, 42B30.

Keywords: Fractional integral, variable kernel, commutator, variable exponent, BMO function, Lebesgue spaecs, Herz spaces.

1. Introduction

Let $S^{n-1}(n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $d\sigma(x')$. We say a function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ belongs to $L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r \geq 1)$, if

(i) For any $x, z \in \mathbb{R}^n$, one has $\Omega(x, \lambda z) = \Omega(x,z)$;

(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} := \sup\limits_{x\in\mathbb{R}^n} \left(\int_{s^{n-1}} |\Omega(x,z')|^r d\sigma(z')\right)^{\frac{1}{r}} < \infty$

For $0 \leq \mu < n$ the fractional integral operator with variable kernel $T_{\Omega,\mu}$ is defined by

$$T_{\Omega,\mu}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} f(y)dy, \tag{1}$$

And the commutators of the fractional integral is defined by

$$[b^m, T_{\Omega,\mu}]f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} (b(x)-b(y))^m f(y)dy, \tag{2}$$

Now we need the further assumption for $\Omega(x,z)$. It satisfies

$$\int_{s^{n-1}} \Omega(x,z')d\sigma(z') = 0, \forall x \in \mathbb{R}^n$$

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For $r \geq 1$, we say Kernel function $\Omega(x,z)$ satisfies the L^r- Dini condition, if Ω meets the conditions (i),(ii) and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order r of Ω defined by

$$\omega_r(\delta) = \sup_{x\in\mathbb{R}^n, |\rho|<\delta} (\int_{s^{n-1}} |\Omega(x,\rho z') - \Omega(x,z')|^r d\sigma(z'))^{\frac{1}{r}}$$

where ρ is the a rotation in $\mathbb{R}^n$

$$|\rho| = \sup_{z'\in S^{n-1}} |\rho z' - z'|$$

The corresponding fractional maximal operator with variable kernel is defined by

$$M_{\Omega,\mu} f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\mu}{n}}} \int_Q |\Omega(x, x-y)||f(y)|dy \tag{3}$$

And the commutators of the fractional maximal operator with variable kernel is defined by

$$[b^m, M_{\Omega,\mu}]f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\mu}{n}}} \int_Q |\Omega(y)||f(x,x-y)[b(x)-b(y)]^m|dy \tag{4}$$

Many results about the fractional integral operator with variable kernel have been achieved [1–5]. In 1971, Muckenhoupt and Wheeden [6] had proved the operator $T_{\Omega,\mu}$ was bounded from L^p to L^q. In 1991, Kov*áč*ik and R*á*kosn*í*k [7] introduced variable exponents Lebesgue and Sobolev spaces as a new method for dealing with nonlinear Dirichet boundary value problem. In 2004, Zhang Pu and Zhao Kai [8] introduced the Commutators of fractional integral operators with variable kernel on Hardy spaces. In the last twenty years, more and more researchers have been interested in the theory of the variable exponent function space and its applications [9–15]. In 2012, Wu Huiling and Lan Jiacheng [16] studied the bonudedness property of $T_{\Omega,\mu}$ with a rough kernel on variable exponents Lebesgue spaces. In 2010, M .Izuki [3] discussed the Commutators of fractional integrals on Lebesgue and Herz spaces. Recently, L.Wang and S.Tao [17] introduced the class of Herz spaces with two variable exponents. Throughout this paper $|E|$ denotes the Lebesgue measure, χ_E means he characteristic function of a measurable set $S \subset \mathbb{R}^n$. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Definition of Function Spaces with Variable Exponent

In this section we define the Lebesgue spaces with variable exponent and Herz spaces with two variable exponent, and also define the mixed Lebesgue sequence spaces. Let E be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. We first define the Lebesgue spaces with variable exponent.

Definition 2.1 ([1])**.** *Let $p(\cdot) : E \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by $L^{p(\cdot)}(E) = \{f$ is measurable : $\int_E (\frac{|f(x)|}{\eta})^{p(x)} dx < \infty$ for some constant $\eta > 0\}$. The space $L^{p(\cdot)}_{loc}(E)$ is defined by $L^{p(\cdot)}_{loc}(E) = \{f$ is measurable : $f \in L^{p(\cdot)}(K)$ for all compact $K \subset E\}$. The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by*

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\{\eta > 0 : \int_E (\frac{|f(x)|}{\eta})^{p(x)} dx \leq 1\}$$

We denote $p_- = essinf\ \{p(x) : x \in E\}$, $p_+ = esssup\{p(x) : x \in E\}$. Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Let M be the Hardy - Littlewood maximal operator. We denote $\mathfrak{B}(E)$ to be the set of all function $p(\cdot) \in \mathcal{P}(E)$ satisfying the M is bounded on $L^{p(\cdot)}(E)$.

Definition 2.2 ([18]). *Let $p(\cdot), q(\cdot) \in \mathcal{P}(E)$. The mixed Lebesgue sequence space with variable exponent $l^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{f_j\}_{j=0}^{\infty}$ of the measurable functions on $\mathbb{R}^n$ such that*

$$\left\|\{f_j\}_{j=0}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\eta > 0 : Q_{l^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j=0}^{\infty}\right) \le 1\right\} < \infty$$

$$Q_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=0}^{\infty}) = \sum_{j=0}^{\infty} inf\left\{\mu_j; \int_{R^n}\left(\frac{|f_j(x)|}{\mu_j^{\frac{1}{q(x)}}}\right)^{p(x)} dx \le 1\right\}$$

Noticing $q_+ < \infty$, we see that

$$Q_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=0}^{\infty}) = \sum_{j=0}^{\infty}\left\||f_j|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$$

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}, C_k = B_k \backslash B_{k-1}, \chi_k = \chi_{C_k}$, $k \in \mathbb{Z}$.

Definition 2.3 ([17]). *Let $\alpha \in \mathbb{R}, q(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with two variable exponent $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n) = \{f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)} < \infty\}$$

Where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)} = \left\|\{2^{k\alpha}|f\chi_k|\}_{k=0}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\eta > 0 : \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1\right\}.$$

Remark 2.4 ([17]).

(1) If $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying $(q_1)_+ \le (q_2)_+$, then $\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n) \subset \dot{K}_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.

(2) If $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $(q_1)_+ \le (q_2)_+$, then $\frac{q_2(\cdot)}{q_1(\cdot)} \in \mathcal{P}(\mathbb{R}^n)$ and $\frac{q_2(\cdot)}{q_1(\cdot)} \ge 1$. Thus, by Lemma 3.7 and Remark 2.2, for any $f \in \dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{p_h} \le \left\{\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{p_h}\right\}^{p_*} \le 1$$

Where

$$p_h = \begin{cases} (\frac{q_2(\cdot)}{q_1(\cdot)})_-, & \frac{2^{k\alpha}|f\chi_k|}{\eta} \le 1 \\ (\frac{q_2(\cdot)}{q_1(\cdot)})_+, & \frac{2^{k\alpha}|f\chi_k|}{\eta} > 1 \end{cases}$$

$$p_* = \begin{cases} \min_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty} a_h \le 1 \\ \max_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty} a_h > 1 \end{cases}$$

This implies that $\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n) \subset \dot{K}_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.

Remark 2.5. *Let $h \in \mathbb{N}, a_h \ge 0, 1 \le p_h < \infty$. Then*

$$\sum_{h=0}^{\infty} a_h \le \left(\sum_{h=0}^{\infty} a_h\right)^{p_*}$$

Where

$$p_* = \begin{cases} \min_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty} a_h \le 1 \\ \max_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty} a_h > 1 \end{cases}$$

3. Properties of Variable Exponent

In this section we state some properties of variable exponent belonging to the class $\mathfrak{B}(\mathbb{R}^n)$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$.

Proposition 3.1 ([1]). *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies*

$$|p(x) - p(y)| \leq \frac{-C}{log(|x-y|)}, |x-y| \leq 1/2$$
$$|p(x) - p(y)| \leq \frac{C}{log(e+|x|)}, |y| \geq |x|$$

then, we have $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.

Proposition 3.2 ([3]). *Suppose that $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $0 < \mu \leq \frac{n}{(p_1)_+}$. And define the variable exponent $p_2(\cdot)$ by : $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$. Then we have that*

$$\|[b, I_\mu]f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{BMO}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$$

For all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $b \in BMO$.

Now, we need recall some lemmas.

Lemma 3.3 ([1]).

(1). Given $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ have that for all function f and g,

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

(2). If $p(\cdot), q(\cdot), r(\cdot) \in \mathbb{R}^n$, define $p(\cdot)$ by : $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{1}{r(\cdot)}$. Then there exists a constant C such that for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$, $g \in L^{r(\cdot)}(\mathbb{R}^n)$, we have

$$\|fg\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{r(\cdot)}(\mathbb{R}^n)}$$

Lemma 3.4. *Suppose that $0 < \mu < n$, $1 < s' < \frac{n}{\mu}$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$. Then we have*

$$|M_{\Omega,\mu,b}f(x)| \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \left(M_{\mu s',b}(|f|^{s'})(x)\right)^{\frac{1}{s'}}.$$

Proof. Applying Hölder's inequality, we get that

$$\begin{aligned}
|M_{\Omega,\mu,b}f(x)| &= \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\mu}{n}}} \int_Q |\Omega(y)||f(x, x-y)b(x) - b(y)|dy \\
&\leq \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\mu}{n}}} \left(\int_Q |\Omega(y)|^s dy\right)^{\frac{1}{s}} \left(|f(x, x-y)b(x) - b(y)|^{s'} dy\right)^{\frac{1}{s'}} \\
&\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\mu s'}{n}}} \times \left(|f(x, x-y)b(x) - b(y)|^{s'} dy\right)^{\frac{1}{s'}} \\
&\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \left(M_{\mu s',b}(|f|^{s'})(x)\right)^{\frac{1}{s'}}
\end{aligned}$$

The proof of Lemma 3.2 is finished. □

Lemma 3.5 ([19]). *Suppose that $x \in \mathbb{R}^n$, the variable function $\tilde{q}(x)$ is defined by $\frac{1}{p(x)} = \frac{1}{q} + \frac{1}{\tilde{q}(x)}$, then for all measurable function f and g, we have*

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|g(x)\|_{L^q(\mathbb{R}^n)}\|f(x)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3.6 ([20]). *Suppose that $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ and $0 < p^- \leq p^+ < \infty$.*

(1). For any cube and $|Q| \leq 2^n$, all the $\chi \in Q$, then : $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p(x)}$

(2). For any cube and $|Q| \geq 1$, then $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p_\infty}$ where $p_\infty = \lim_{x\to\infty} p(x)$.

Lemma 3.7 ([21]). *If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2, C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subset $S \subset R$*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \ \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \ \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Lemma 3.8 ([14]). *If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, there exist a constant $C > 0$ such that for any balls B in $\mathbb{R}^n$. We have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 3.9 ([17]). *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.If $f \in L^{p(\cdot)q(\cdot)}$, then*

$$\min(\|f\|^{q+}_{L^{p(\cdot)q(\cdot)}} \ , \ \|f\|^{q-}_{L^{p(\cdot)q(\cdot)}}) \leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}} \leq \max(\|f\|^{q+}_{L^{p(\cdot)q(\cdot)}} \ , \ \|f\|^{q-}_{L^{p(\cdot)q(\cdot)}}).$$

Lemma 3.10 ([9]). *Let $\in BMO$, n is a positive integer, and there constants $C > 0$, such that for any $l, j \in \mathbb{Z}$ with $l > j$*

(1). $C^{-1}\|b\|_*^n \leq \sup\limits_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)^n\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*^n$

(2). $\|(b - b_{B_j})^n\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(l - j)^n\|b\|_*^n\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 3.11 ([10]). *For any $\varepsilon > 0$ with $0 < \mu - \varepsilon < \mu + \varepsilon$, we have*

$$|T_{\Omega,\mu,b}f(x)| \leq C[M_{\Omega,\mu+\varepsilon,b}f(x)]^{\frac{1}{2}}[M_{\Omega,\mu-\varepsilon,b}f(x)]^{\frac{1}{2}}.$$

4. Main Theorems and Their Proof

Theorem 4.1. *Suppose that $b \in$ BMO, $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \beta \leq \frac{n}{(p_1)_+}$ and define the variable exponent $p_2(\cdot)$ by : $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$, Then we have that*

$$\|[b, T_{\Omega,\mu}]f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$$

For all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$.

Proof. Let $0 < \varepsilon < \min(\mu, n - \mu)$, and $r(\cdot) : \mathbb{R}^n \longrightarrow [1, +\infty)$. And let

$$\frac{1}{p_1(\cdot)} - \frac{1}{\frac{r(\cdot)p_2(\cdot)}{2}} = \frac{\mu - \varepsilon}{2}$$

$$\frac{1}{p_1(\cdot)} - \frac{1}{\frac{r'(\cdot)p_2(\cdot)}{2}} = \frac{\mu + \varepsilon}{2}$$

By Lemma 3.11, we have

$$|T_{\Omega,\mu,b}f(x)| \leq C[M_{\Omega,\mu+\varepsilon,b}f(x)]^{\frac{1}{2}}[M_{\Omega,\mu-\varepsilon,b}f(x)]^{\frac{1}{2}}$$

Applying the generalized Hölder's Inequality, we get that

$$\|T_{\Omega,\mu,b}f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|(M_{\Omega,\mu+\varepsilon,b}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r'(\cdot)}}\|(M_{\Omega,\mu-\varepsilon,b}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r(\cdot)}}$$

By Lemma 3.3 and Proposition 3.2, we obtain that

$$\begin{aligned}\|(M_{\Omega,\mu-\varepsilon,b}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r(\cdot)}} &\le C\|M_{(\mu-\varepsilon)s',b}(|f|^{s'})\|^{\frac{1}{2}}_{L^{\frac{p_2(\cdot)r(\cdot)}{2s'}}} \le C\|b\|_*^{\frac{1}{2}}\||f|^{s'}\|^{\frac{1}{2}}_{L^{\frac{p_1(x)}{2s'}}}\\ &\le C\|b\|_*^{\frac{1}{2}}\|f\|^{\frac{1}{2}}_{L^{p_1(\cdot)}(\mathbb{R}^n)} \end{aligned}\tag{5}$$

As the same way, we can concluded that

$$\|(M_{\Omega,\mu+\varepsilon,b}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r'(\cdot)}} \le C\|b\|_*^{\frac{1}{2}}\|f\|^{\frac{1}{2}}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\tag{6}$$

Therefore by (5) and (6), we have that

$$\|[b,T_{\Omega,\mu}]f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

□

Theorem 4.2. *Let $b\in BMO(\mathbb{R}^n)$, $m\in\mathbb{N}$. Suppose that $0<\mu<n$, $\mu-n\delta_2<\alpha<n\delta_1+\frac{n}{r}$, $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})(r>p_2^+)$, $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $(q_2)_-\ge(q_1)_+$. If $p_1(\cdot)\in\mathfrak{B}(\mathbb{R}^n)$ satisfy $0<\mu\le\frac{n}{(p_1)_+}$ and define the variable exponent $p_2(x)$ by $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\mu}{n}$. Then the commutators $[b^m,T_{\Omega,\mu}]$ is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $b\in$ BMO, $f\in\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. We write

$$f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_k=\sum_{j=-\infty}^{\infty}f_j(x)$$

From definition of $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$

$$\|[b^m,T_{\Omega,\mu}](f)\chi_k\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b^m,T_{\Omega,\mu}](f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le 1\right\}$$

Since

$$\begin{aligned}\left\|\left(\frac{2^{k\alpha}|[b^m,T_{\Omega,\mu}](f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} &\le \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{\infty}[b^m,T_{\Omega,\mu}](f)\chi_k|}{\eta_{21}+\eta_{22}+\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\ &\le \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f)\chi_k|}{\eta_{31}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}+\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f)\chi_k|}{\eta_{32}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\ &\quad+\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,T_{\Omega,\mu}](f)\chi_k|}{\eta_{33}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\end{aligned}$$

Where

$$\eta_{21}=\left\|\left\{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

$$\eta_{22}=\left\|\left\{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

$$\eta_{23}=\left\|\left\{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,T_{\Omega,\mu}](f)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

And $\eta=\eta_{21}+\eta_{22}+\eta_{23}$, thus

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le C$$

That is

$$\|[b^m,T_{\Omega,\mu}](f)\chi_k\|_{\dot K^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}\le C\eta\le C[\eta_{21}+\eta_{22}+\eta_{23}]$$

Hence $\eta_{21},\ \eta_{22},\ \eta_{23}\le C\|b\|_*^m\|f\|_{\dot K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$. Denote $\eta_1=\|f\|_{\dot K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$. Now we consider η_{22}. Applying Lemma 3.9, we get

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}&\le C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right\|_{L^{p_2(\cdot)}}^{(q_2^1)k}\\&\le C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{k\alpha}|[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right\|_{L^{p_2(\cdot)}}\right)^{(q_2^1)k}\end{aligned}$$

Where

$$(q_2^1)k=\begin{cases}(q_2)_-\quad, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le 1\\(q_2)_+\quad, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

By the Theorem 4.1, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le C\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{k+1}\left\|\frac{2^{k\alpha}|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}^{(q_2^1)k}$$

Since $f\in\dot K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$, then we have $\left\|\frac{2^{j\alpha}|f\chi_j|}{\eta_1}\right\|_{L^{p(\cdot)}}\le 1$, and

$$\sum_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}|f\chi_j|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\le 1$$

Again by Lemma 3.9 and Remark 2.5, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}&\le C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)k}{(q_1)_+}}\\&\le C\left[\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right]^{q_*}\le C\end{aligned}$$

Hence $(p_1)_+,(p_2)_-\le(q_2^1)k$, and $q_*=\min_{k\in\mathbb{N}}\frac{(q_2^1)k}{(q_1)_+}$, this implies that

$$\eta_{22}\le C\eta_1\|b\|_*^m\le C\|b\|_*^m\|f\|_{\dot K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\tag{7}$$

Now, we estimate of η_{21}. Let $x \in C_k$ $j \le k-2$, then $|x-y| \sim |x|$, we show that

$$\begin{aligned}|[b^m, T_{\Omega,\mu}]f_j(x)| &\le C\int_{C_j}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}\right||(b(x)-b(y))|^m|f_j(y)|dy,\\ &\le 2^{-k(n-\mu)}\int_{C_j}|\Omega(x,x-y)||(b(x)-b(y))|^m|f_j(y)|dy\\ &\le C2^{-k(n-\mu)}\left[|b(x)-b_j|^m\int_{C_j}|\Omega(x,x-y)||f_j(y)|dy+\int_{C_j}|\Omega(x,x-y)||(b(x)-b_j)|^m|f_j(y)|dy\right]\end{aligned}$$

Applying the generalized Hölder's Inequality, we know that

$$|[b^m, T_{\Omega,\mu}]f_j(x)| \le C2^{-k(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\left[|(b(x)-b_j)|^m\|\Omega(x,x-y)\chi_j\|_{L^{p_1'(\cdot)}}+\|\Omega(x,x-y)(b(y)-b_j)^m\chi_j\|_{L^{p_1'(\cdot)}}\right] \quad (8)$$

Define the variable exponent $\frac{1}{\tilde{p}_1(\cdot)}=\frac{1}{r}+\frac{1}{\widetilde{p_1'}(\cdot)}$ by Lemma 3.5, then we have

$$\|\Omega(x,x-y)\chi_j\|_{L^{p_1'}} \le \|\Omega(x,x-y)\|_{L^r}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \le \left[\int_{2^{k-1}}^{2^k} r^{n-1}dr(\int_{s^{n-1}}|\Omega(x,y')|^r d\sigma(y')\right]^{\frac{1}{r}}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}}$$

According to Lemma 3.6 and the formula $\frac{1}{\widetilde{p}_1'(x)}=\frac{1}{p_1'(x)}-\frac{1}{r}$, then we get

$$\|\Omega(x,x-y)\chi_j\|_{L^{p_1'}} \le 2^{\frac{kn}{r}}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \le C2^{\frac{kn}{r}}2^{\frac{-jn}{r}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \le C2^{(k-j)\frac{n}{r}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \quad (9)$$

Similarly, by Lemma 3.10, we have

$$\begin{aligned}\|\Omega(x,x-y)(b(y)-b_j)^m\chi_j\|_{L^{p_1'(\cdot)}} &\le \|\Omega(x,x-y)\|_{L^r}\|(b(y)-b_j)^m\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}}\\ &\le C2^{\frac{kn}{r}}(k-j)^m\|b\|_*^m\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \le C2^{(k-j)\frac{n}{r}}(k-j)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \quad (10)\end{aligned}$$

By (9), (10), and Lemma 3.8, we get

$$\begin{aligned}\|[b^m, T_{\Omega,\mu}]f_j,\chi_k\|_{L^{p_2(\cdot)}} &\le C2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}\|f_j\|_{L^{p_1(\cdot)}}\Big[\|(b(x)-b_j)^m\chi_k\|_{L^{p_2(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\\ &\quad+(k-j)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\Big]\\ &\le C2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}(k-j)^m\|b\|_*^m\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}} \quad (11)\end{aligned}$$

By (11), and using Lemmas 3.7-3.9, and

$$\left\|\left(\frac{2^{j\alpha}|f\chi_j|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \le 1.$$

Note that $\|\chi_{B_k}\|_{L^{p_2(\cdot)}} \le C2^{-k\mu}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}$, then (see [22])

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}(k-j)^m\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]^{(q_2^2)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}(k-j)^m2^{(k-j)\frac{n}{r}}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\right]^{(q_2^2)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r})}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha)}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1)_+}}_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)k}$$

Where

$$(q_2^2)k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

Since $f\in\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$, then we have $\left\|\frac{2^{j\alpha}|f\chi_j|}{\eta_1}\right\|_{L^{p(\cdot)}}\leq 1$, and

$$\sum_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}|f\chi_j|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\leq 1$$

Now if $(q_1)_+<1$, and $\alpha<n\delta_1+\frac{n}{r}$, then we have $\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha)}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{\frac{(q_2^2)k}{(q_1)_+}}$$

$$\leq C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha)}\right]^{q_*}\leq C$$

Where $q_*=\min_{k\in\mathbb{N}}\frac{(q_2^2)k}{(q_1)_+}$. If $(q_1)_+\geq 1$, and $\alpha<n\delta_1+\frac{n}{r}$, then we have $\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha)\frac{(q_2)_+}{2}}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{\frac{(q_2^1)k}{(q_1)_+}}$$

$$\times\left[\sum_{j=-\infty}^{k-2}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha)\frac{((q_1)_+)'}{2}}\right]^{\frac{(q_2^2)_+}{((q_1)_+)'}}$$

$$\leq C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}(k-j)^m2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha)\frac{(q_1)_+}{2}}\right]^{q_*}\leq C$$

Where $q_*=\min_{k\in\mathbb{N}}\frac{(q_2^2)k}{(q_1)_+}$, this implies that

$$\eta_{21}\leq C\eta_1\|b\|_*^m\leq C\|b\|_*^m\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\qquad(12)$$

Finally, we estimate η_{23}. Let $x \in C_k$ $j \geq k+2$, then $|x-y| \sim |y|$, we get

$$
\begin{aligned}
|[b^m, T_{\Omega,\mu}]f_j(x)| &\leq C\int_{C_j}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}\right| |(b(x)-b(y))|^m |f_j(y)|dy, \\
&\leq 2^{-j(n-\mu)}\int_{C_j}|\Omega(x,x-y)||(b(x)-b(y))|^m|f_j(y)|dy \\
&\leq C2^{-j(n-\mu)}\left[|b(x)-b_j|^m\int_{C_j}|\Omega(x,x-y)||f_j(y)|dy + \int_{C_j}|\Omega(x,x-y)||(b(x)-b_j)|^m|f_j(y)|dy\right]
\end{aligned}
$$

Applying the generalized Hölder's Inequality, we know that

$$
|[b^m, T_{\Omega,\mu}]f_j(x)| \leq C2^{-j(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\left[|(b(x)-b_j)|^m\|\Omega(x,x-y)\chi_j\|_{L^{p_1'(\cdot)}} + \|\Omega(x,x-y)(b(y)-b_j)^m\chi_j\|_{L^{p_1'(\cdot)}}\right] \tag{13}
$$

Define the variable exponent $\frac{1}{\tilde{p}_1(\cdot)} = \frac{1}{r} + \frac{1}{\widetilde{p_1'}(\cdot)}$ by Lemma 3.5, then we have

$$
\|\Omega(x,x-y)\chi_j\|_{L^{p_1'}} \leq \|\Omega(x,x-y)\|_{L^r}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \leq \left[\int_{2^{j-2}}^{2^j} r^{n-1}dr\int_{s^{n-1}}|\Omega(x,y')|^r d\sigma(y')\right]^{\frac{1}{r}}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}}
$$

According to Lemma 3.6 and the formula $\frac{1}{\widetilde{p}_1'(x)} = \frac{1}{p_1'(x)} - \frac{1}{r}$, then we get

$$
\|\Omega(x,x-y)\chi_j\|_{L^{p_1'}} \leq C2^{\frac{jn}{r}}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|\chi_{B_j}\|_{L^{\widetilde{p_1'}(\cdot)}} \leq C2^{\frac{jn}{r}}2^{\frac{-jn}{r}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \leq C\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \tag{14}
$$

Similarly, by Lemma 3.10, we can concluded that

$$
\begin{aligned}
\|\Omega(x,x-y)(b(y)-b_j)^m\chi_j\|_{L^{p_1'(\cdot)}} &\leq \|\Omega(x,x-y)\|_{L^r}\|(b(y)-b_j)^m\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \\
&\leq C\ 2^{\frac{jn}{r}}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}(k-j)^m\|b\|_*^m\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \leq C\ (k-j)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}
\end{aligned} \tag{15}
$$

By (14), (15) and again Lemma 3.10, then we get

$$
\begin{aligned}
\|[b^m, T_{\Omega,\mu}]f_j,\chi_k\|_{L^{p_2(\cdot)}} &\leq C2^{-k(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\left[\|(b(x)-b_j)^m\chi_k\|_{L^{p_2(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} + (k-j)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right] \\
&\leq C2^{-j(n-\mu)}(k-j)^m\|b\|_*^m\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}
\end{aligned} \tag{16}
$$

By (16), Lemmas 3.7-3.9, and $\left\|\left(\frac{2^{j\alpha}|f\chi_j|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \leq 1$. Note that $\|\chi_{B_k}\|_{L^{p_2(\cdot)}} \leq C2^{-k\mu}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}$, then (see [22])

$$
\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m, T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m, T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}
$$

$$
\begin{aligned}
&\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-j(n-\mu)}(k-j)^m\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]^{(q_2^3)k} \\
&\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-j(n-\mu)}(k-j)^m\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}2^{-k\mu}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^3)k}
\end{aligned}
$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}(k-j)^m 2^{j\mu}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}2^{-jn}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}2^{-k\mu}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^3)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}(k-j)^m 2^{(j-k)\mu}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\right]^{(q_2^3)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}(k-j)^m 2^{(k-j)(n\delta_2-\mu)}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^3)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}(k-j)^m 2^{(k-j)(n\delta_2-\mu+\alpha)}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1)_+}}_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^3)k}$$

Where

$$(q_2^3)k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\\[2ex] (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

Furthermore, since $(q_2)_-\geq(q_1)_+$ and $\alpha>\mu-n\delta_2$, as the same way to estimate η_{21}, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,T_{\Omega,\mu}](f_j)\chi_k|}{\eta_1\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}(k-j)^m 2^{(k-j)(n\delta_2-\mu+\alpha)}\left\|\left(\frac{|2^{\alpha j}f_j\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{q_*}\leq C$$

Where $q_*=\min_{k\in\mathbb{N}}\frac{(q_2^3)k}{(q_1)_+}$, this implies that

$$\eta_{23}\leq C\eta_1\|b\|_*^m\leq C\|b\|_*^m\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\tag{17}$$

By (7), (12) and (12) the proof of Theorem 2.2 is finished. □

Competing interests

The authors declare that they have no competing interests.

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