International Journal of Mathematics And its Applications
Volume 4, Issue 3–A (2016), 41–61.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Collatz Conjecture for Modulo an Integer

Research Article

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Abstract: A function T_m from a set {1,2,3,...m} into itself defined by $T_m(x) = \frac{x}{2}$, for even x and by $T_m(x) = \frac{3x+1}{2}$ (mod m), for odd x is considered in this article. The asymptotic behaviour of this function is studied in this article for some cases.

MSC: 11B05, 11A07, 11A99, 11Y55.

Keywords: Congruence, Collatz conjecture.

1. Introduction

The Collatz conjecture is a well known conjecture. This is also quoted in the literature as the $3x + 1$ problem and Ulams conjecture. The conjecture is that $T^n(x)$ eventually reaches 1, for any given $x \in N$, for the function $T : N \to N$ defined by

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even} \\ \frac{3x+1}{2}, & \text{if } x \text{ is odd.} \end{cases}$$

Here x and $T(x)$ are all natural numbers. The following discussion is about the same problem with a restriction of starting with "x modulo m" value and "$T(x)$ modulo m" value in A_m={1,2,3,..., m} for a given m. More precisely, let us define a new function $T_m : A_m \to A_m$ defined by

$$T_m(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} (\text{mod } m) & \text{if } x \text{ is odd.} \end{cases}$$

when x, $\frac{x}{2}$, $\frac{3x+1}{2}$(mod m) are in A_m. Suppose $m = 1$. Then $A_m = \{1\}$ and the only possible value of x is 1 and $T_m(1)$ may be considered as 1. So hereafter it is assumed that $m \geq 2$ for a non-trivial situation. It is expected that $T^k_m(x)$ eventually produce the value 1, for any $x \in A_m$ and for some k. But this is not true. A detailed discussion about this one is presented in this article. If there is some $x \neq 1$, for which $T^k_m(x) = x$, for some $k \geq 1$, then it would lead to a cycle, that may not receive the value 1 in subsequent application of the function T_m. So there is a possibility that $T^k_m(x) \neq 1$ for some x and for any k, when there is such a cycle. So these exceptional cases are analyzed in this article so that the favourable

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cases for original conjecture can be identified. Thus different cases are to be discussed to analyze the possibility of having a relation $T_m^k(x) = x$. There are articles which provide theoretical positive results for the original Collatz problem. Some of them are [2, 3, 5, 7, 14, 15]. The article [8] of Everett provides an unexpected result on asymptotic density of the set $\{x : x = 1, 2, 3, 4...; T_m^k(x) < x\}$, for some x. The most interesting result is theorem 1 in [8] which helps to evaluate the asymptotic density of the previous set as 1. There are no other significant articles giving theoretical results. There are a number of articles (for example[1, 4, 9, 11]) which discuss particular cases for the original Collatz problem. There are many survey articles (for example [10]). There are articles which discuss about generalizations and variations of Collatz problem. (for example [5, 6, 10, 12, 13]). One among them is the article [13], which discusses a generalization of Collatz problem in $Z_2[x]$, collection of the polynomials with variable x and coefficients in Z_2. This particular article provides a motivation for a restricted Collatz problem, which restricted to the set $\{A_m = 1, 2, 3, 4...m\}$. The section 2 discusses about the case when x is odd and section 3 discusses about the case when x is even.

2. Cases for Odd Integers

1. Let $x \in A_m$ be arbitrary such that x is odd. Then

$$\begin{aligned} T_m(x) &= \frac{3x+1}{2} (\text{mod } m);\text{ with } T_m(x) \in A_m\ . \\ &= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \end{aligned}$$

Suppose $T_m(x) = x$. Then the following cases will arise.

Case 1: Suppose $\frac{3x+1-2m}{2} = x$, then $x = 2m - 1$. Here $m \geq 2$, and so $x \notin A_m$. So, this is impossible. So, in this case, $T_m(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+1}{2} = x$, then $x = -1$. This is impossible. So in this case $T_m(x) \neq x$, for all $x \geq 2$.

2. Let $x \in A_m$ be arbitrary such that x is odd and $T_m(x)$ is even. Then

$$\begin{aligned} T_m(x) &= \frac{3x+1}{2} (\text{mod } m);\text{ with } T_m(x) \in A_m\ . \\ &= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\ T_m^2(x) &= \begin{cases} \frac{3x+1-2m}{4}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{4}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \end{aligned}$$

Suppose $T_m^2(x) = x$, then the following cases will arise.

Case 1: Suppose $\frac{3x+1-2m}{4} = x$, then $x = 1 - 2m$. Here $m \geq 2$, and so $x \notin A_m$. So this is impossible. So in this case $T_m^2(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+1}{4} = x$, then $x = 1$. Thus $T_m^2(x) = x$ happens in this case only when $x = 1$.

3. Let $x \in A_m$ be arbitrary such that x and $T_m(x)$ are odd. Then

$$\begin{aligned} T_m(x) &= \frac{3x+1}{2} \pmod m; \text{ with } T_m(x) \in A_m. \\ &= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \end{aligned}$$

$$T_m^2(x) = \begin{cases} \frac{9x+5-10m}{4}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x-6m+5}{4} > m \\ \frac{9x-6m+5}{4}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x-6m+5}{4} \leq m \\ \frac{9x-4m+5}{4}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} > m \\ \frac{9x+5}{4}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} \leq m. \end{cases}$$

Suppose $T_m^2(x) = x$, then the following cases will arise.

Case 1: Suppose $\frac{9x+5-10m}{4} = x$, then $x = 2m - 1$. Here $m \geq 2$, and so $x \notin A_m$. So this is impossible. So in this case $T_m^2(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{9x-6m+5}{4} = x$, then $5x+5 = 6m$. Hence $5 \mid m$ and $m = 5y$ for some y. Then $5x+5 = 6(5y)$ and $x = 6y-1$ that is $x = 6(\frac{m}{5}) - 1$. Also here $x \in A_m$, so $1 \leq x$ and $x \leq m$ gives $m \geq 2$ and $m \leq 5$. Thus $2 \leq m \leq 5$ and $m = 5y$, for some y. So the possible value of m is 5, and if $m = 5$, the corresponding value of x is 5. Moreover $\frac{3x+1}{2} > m$ and $\frac{9x-6m+5}{4} \leq m$ are also satisfied for these values $m = 5$ and $x = 5$. Thus in this case if $m = 5$ and $x = 5$, then $T_m^2(x) = x$. That is $T_5^2(5) = 5$.

Case 3: Suppose $\frac{9x+5-4m}{4} = x$, then $5x + 5 = 4m$. So, $5 \mid m$ and $m = 5y$, for some y. Now $5x + 5 = 4(5y)$ and $x = 4y - 1$, that is $x = 4(\frac{m}{5}) - 1$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $1 \leq \frac{4m}{5} - 1$ and $\frac{4m}{5} - 1 \leq m$, so that $3 \leq m$ and $m \geq -5$. Thus $-5 \leq m$ and $m \geq 3$ and $m = 5y$ for some y. So the possible values of m are 5, 10, 15, ... and the corresponding values of x are 3, 7, 11, ... Also here $3x + 1 \leq 2m \Leftrightarrow 3(\frac{4m}{5} - 1) + 1 \leq 2m \Leftrightarrow m \leq 5$ and $9x + 5 > 4m \Leftrightarrow 9(\frac{4m}{5} - 1) + 5 > 4m \Leftrightarrow m \geq 2$. Also here $m = 5y$, for some y. So the possible values of x and m satisfying $T_m^2(x) = x$ are $x = 3$ and $m = 5$. Clearly $T_5(3) = 5$, $T_5^2(3) = 3$.

Case 4: Suppose $\frac{9x+5}{4} = x$, then $x = -1$. So this is impossible. So in this case $T_m^2(x) \neq x$, for every $x \geq 2$.

4. Let $x \in A_m$ be arbitrary such that x is odd and $T_m(x)$ is even and $T_m^2(x)$ is even. Then

$$\begin{aligned} T_m(x) &= \frac{3x+1}{2} \pmod m; \text{ with } T_m(x) \in A_m. \\ &= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\ T_m^2(x) &= \begin{cases} \frac{3x+1-2m}{4}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{4}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\ T_m^3(x) &= \begin{cases} \frac{3x+1-2m}{8}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{8}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \end{aligned}$$

Suppose $T_m^3(x)=x$, then the following cases will arise.

Case 1: Suppose $\frac{3x+1-2m}{8} = x$, then $x = \frac{1-2m}{5}$. Here $m \geq 2$, and so x is negative and then $x \notin A_m$. So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+1}{8} = x$, then $x = \frac{1}{5} < 1$ and $x \notin A_m$, which is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

If $T_m^3(x) < x$, then the following cases are considered.

Case 1: $\frac{3x+1-2m}{8} < x \Leftrightarrow x > \frac{1-2m}{5}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $\frac{3x+1}{8} < x \Leftrightarrow x > \frac{1}{5}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Result 2.1. *If x is odd and $T_m^k(x)$ is even for all k, then $T_m^p(x) \neq x$, for all $x \geq 2$ and $p \geq 2$, $m \geq 2$.*

5. Let $x \in A_m$ be arbitrary such that x is odd, $T_m(x)$ is even and $T_m^2(x)$ is odd. Then

$$\begin{aligned}
T_m(x) &= \frac{3x+1}{2} (\text{mod } m); \text{ with } T_m(x) \in A_m. \\
&= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\
T_m^2(x) &= \begin{cases} \frac{3x+1-2m}{4}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{4}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\
T_m^3(x) &= \begin{cases} \frac{9x+7-14m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+7-6m}{8} > m \\ \frac{9x+7-6m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+7-6m}{8} \leq m \\ \frac{9x+7-8m}{8}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+7}{8} > m \\ \frac{9x+7}{8}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+7}{8} \leq m. \end{cases}
\end{aligned}$$

Suppose $T_m^3(x)$=x, then the following cases will arise.

Case 1: Suppose $\frac{9x+7-14m}{8} = x$, then $x = 14m - 7 > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{9x+7-6m}{8} = x$, then $x = 6m - 7 > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{9x+7-8m}{8} = x$, then $x = 8m - 7 > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{9x+7}{8} = x$, then $x = -7 < 0$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

If $T_m^3(x) < x$, then the following cases are considered.

Case 1: $\frac{9x+7-14m}{8} < x \Leftrightarrow x < 7(2m-1)$, which is possible for all $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $\frac{9x+7-6m}{8} < x \Leftrightarrow x < 6m - 7$, which is possible for all $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 3: $\frac{9x+7-8m}{8} < x \Leftrightarrow x < 8m - 7$, which is possible for all $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 4: $\frac{9x+7}{8} < x \Leftrightarrow x < -7$ which is impossible. Also for this case, if $T_m^3(x)$ is even then $T_m^4(x) = \frac{9x+7}{16}$, and $T_m^4(x) < x \Leftrightarrow \frac{9x+7}{16} < x \Leftrightarrow x \geq 1$ which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$, when $T_m^3(x)$ is even.

Result 2.2. *If x is odd, $T_m(x)$ is even and $T_m^2(x)$ is odd and $T_m^k(x)$ is even, for all $k \geq 3$, then $T_m^p(x) \neq x$ for all $p \geq 4$, $x \geq 2$ and $m \geq 2$.*

6. Let $x \in A_m$ be arbitrary such that x and $T_m(x)$ are odd and $T_m^2(x)$ is even. Then

$$\begin{aligned} T_m(x) &= \frac{3x+1}{2} (\text{mod } m); \text{ with } T_m(x) \in A_m. \\ &= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\ T_m^2(x) &= \begin{cases} \frac{9x+5-10m}{4}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+5-6m}{4} > m \\ \frac{9x+5-6m}{4}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+5-6m}{4} \leq m \\ \frac{9x+5-4m}{4}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} > m \\ \frac{9x+5}{4}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} \leq m. \end{cases} \\ T_m^3(x) &= \begin{cases} \frac{9x+5-10m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+5-6m}{4} > m \\ \frac{9x+5-6m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+5-6m}{8} \leq m \\ \frac{9x+5-4m}{8}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} > m \\ \frac{9x+5}{8}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} \leq m. \end{cases} \end{aligned}$$

Suppose $T_m^3(x)$=x, then the following cases will arise.

Case 1: Suppose $\frac{9x+5-10m}{8} = x$, then $x = 10m - 5 > m$, for $m \geq 2$. Hence $x \notin A_m$, so this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{9x+5-6m}{8} = x$, then $x = 6m - 5 > m$, for $m \geq 2$. Hence $x \notin A_m$, so this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{9x+5-4m}{8} = x$, then $x = 4m - 5 > m$, for $m \geq 2$. Hence $x \notin A_m$, So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{9x+5}{8} = x$, then $x = -5 < 0$. Hence $x \notin A_m$, so this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

If $T_m^3(x) < x$, then the following cases are considered.

Case 1: $\frac{9x+5-10m}{8} < x \Leftrightarrow x < 5(2m-1)$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $\frac{9x+5-6m}{8} < x \Leftrightarrow x < 6m - 5$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 3: $\frac{9x+5-4m}{8} < x \Leftrightarrow x < 4m - 5$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 4: $\frac{9x+5}{8} < x \Leftrightarrow x < -5$, which is impossible. Also for this case, if $T_m^3(x)$ is even then $T_m^4(x) = \frac{9x+5}{16}$ and $T_m^4(x) < x \Leftrightarrow \frac{9x+5}{16} < x \Leftrightarrow x \geq \frac{5}{7}$, which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$, when $T_m^3(x)$ is even.

Result 2.3. *If x is odd, $T_m(x)$ is odd and $T_m^k(x)$ is even, for all $k \geq 2$, then $T_m^p(x) \neq x$, for all $p \geq 4$.*

7. Let $x \in A_m$ be arbitrary such that x is odd, $T_m(x)$ is odd and $T_m^2(x)$ is odd, then

$$\begin{aligned} T_m(x) &= \frac{3x+1}{2}(\text{mod } m);\text{ with } T_m(x) \in A_m. \\ &= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases} \\ T_m^2(x) &= \begin{cases} \frac{9x+5-10m}{4}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+5-6m}{4} > m \\ \frac{9x+5-6m}{4}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+5-6m}{4} \leq m \\ \frac{9x+5-4m}{4}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} > m \\ \frac{9x+5}{4}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+5}{4} \leq m. \end{cases} \\ T_m^3(x) &= \begin{cases} \frac{27x+19-38m}{8}, & \text{if } \frac{3x+1}{2} > m \ \frac{9x+5-6m}{4} > m \text{ and } \frac{27x+19-30m}{8} > m \\ \frac{27x+19-30m}{8}, & \text{if } \frac{3x+1}{2} > m \ \frac{9x+5-6m}{4} > m \text{ and } \frac{27x+19-30m}{8} \leq m \\ \frac{27x+19-26m}{8}, & \text{if } \frac{3x+1}{2} > m \ \frac{9x+5-6m}{4} \leq m \text{ and } \frac{27x+19-30m}{8} > m \\ \frac{27x+19-18m}{8}, & \text{if } \frac{3x+1}{2} > m \ \frac{9x+5-6m}{4} \leq m \text{ and } \frac{27x+19-30m}{8} \leq m \\ \frac{27x+19-12m}{8}, & \text{if } \frac{3x+1}{2} \leq m \ \frac{9x+5}{4} > m \text{ and } \frac{27x+19-12m}{8} > m \\ \frac{27x+19-20m}{8}, & \text{if } \frac{3x+1}{2} \leq m \ \frac{9x+5}{4} > m \text{ and } \frac{27x+19-12m}{8} \leq m \\ \frac{27x+19-8m}{8}, & \text{if } \frac{3x+1}{2} \leq m \ \frac{9x+5}{4} \leq m \text{ and } \frac{27x+19}{8} > m \\ \frac{27x+19}{8}, & \text{if } \frac{3x+1}{2} \leq m \ \frac{9x+5}{4} \leq m \text{ and } \frac{27x+19}{8} \leq m. \end{cases} \end{aligned}$$

Suppose $T_m^3(x)=x$, then the following cases will arise.

Case 1: Suppose $\frac{27x+19-38m}{8} = x$, then $x = 2m-1 > m$, for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{27x+19-30m}{8} = x$, then $19x = 30m - 19$ and x is integer. So $19|m$ and $m = 19y$ for some y. Now $19x = 30(19y) - 19$ then

$x = \frac{30m}{19} - 1$.Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 2$ and $m \leq 1$. So there exist no m satisfying these conditions. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{27x+19-26m}{8} = x$, then $19x = 26m - 19$ and x is integer.So $19|m$ and $m = 19y$ for some y. Now $19x = 26(19y) - 19$ then

$x = \frac{26m}{19} - 1$.Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 2$ and $m \leq 2$. Hence the possible value of m is 2. But this m is not of the form $m = 19y$ for some y. So there exist no m satisfying these conditions. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{27x+19-18m}{8} = x$, then $19x = 18m - 19$ and x is an integer. So $19|m$ and $m = 19y$ for some y. Now $19x = 18(19y) - 19$ then $x = 18\frac{m}{19} - 1$.Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 3$ and $m \geq -1$. But here m is of the form $m = 19y$ for some y. Hence the possible values of m are 19,38,57,...and the corresponding values of x are 17,35,53,...

Also here (**i**)$3x + 1 > 2m \Leftrightarrow 3(\frac{18m}{19} - 1) + 1 > 2m \Leftrightarrow m \geq 3$, which is possible. Hence $\frac{3x+1}{2} > m$.

(**ii**) $9x+5 \leq 10m \Leftrightarrow 9(\frac{18m}{19}-1)+5 \leq 10m \Leftrightarrow m \geq 1$, which is possible. Hence $\frac{9x+5}{10} \leq m$.

(**iii**) $27x+19 > 26m \Leftrightarrow 27(\frac{18m}{19}-1)+1 > 26m \Leftrightarrow m \geq -61$, which is possible. Hence $\frac{27x+19}{26} \leq m$.

Result 2.4. *If $m = 19y$, for some y, and x is odd of the form $x = \frac{18m}{19}-1$ then $T_m^3(x) = x$, when $T(x)$ and $T^2(x)$ are odd.*

Case 5: Suppose $\frac{27x+19-12m}{8} = x$, then $19x = 12m-19$ and x is integer. So $19 \mid m$ and $m = 19y$ for some y. Now $19x = 12(19y)-19$ then $x = \frac{12m}{19}-1$.Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 4$ and $m \geq \frac{-19}{7}$. But here m is of the form $m = 19y$ for some y. Hence the possible values of m are 19,38,57,...and the corresponding values of x are 11,23,35,... Also here

(**i**) $3x+1 \leq 2m \Leftrightarrow 3(\frac{12m}{19}-1)+1 \leq 2m \Leftrightarrow m \geq -19$, which is possible. Hence $\frac{3x+1}{2} \leq m$.

(**ii**) $9x+5 > 4m \Leftrightarrow 9(\frac{12m}{19}-1)+5 > 4m \Leftrightarrow m \geq 3$, which is possible. Hence $\frac{9x+5}{10} > m$.

(**iii**) $27x+19 \leq 20m \Leftrightarrow 27(\frac{12m}{19}-1)+1 \leq 20m \Leftrightarrow m \geq (-3)$, which is possible. Hence $\frac{27x+19}{20} \leq m$.

Result 2.5. *If $m = 19y$ for some y and x is odd of the form $x = \frac{12m}{19}-1$, then $T_m^3(x) = x$, when $T(x)$ and $T^2(x)$ are odd.*

Case 6: Suppose $\frac{27x+19-20m}{8} = x$, then $19x = 20m-19$ and x is integer. So $19|m$ and $m = 19y$ for some y. Now $19x = 20(19y)-19$ then $x = \frac{20m}{19}-1$.Here $x \epsilon A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 2$ and $m \leq 19$. But here m is of the form $m = 19y$ for some y. Hence the possible value of m is 19 and the corresponding value of x is 19. But here if $x = 19$ and $m = 19$ then $3x+1 = 57$ and $2m = 38$. Hence $3x+1 \nleq 2m$. So this condition was not satisfied. So in this case $T_m^3(x) \neq x$ for all $x \geq 2$.

Case 7: Suppose $\frac{27x+19-8m}{8} = x$, then $19x = 8m-19$ and x is integer. So $19|m$ and $m = 19y$ for some y. Now $19x = 8(19y)-19$ then $x = \frac{8m}{19}-1$.Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 5$ and $m \geq \frac{-19}{11}$. But here m is of the form $m = 19y$ for some y. Hence the possible values of m are 19,38,57,...and the corresponding values of x are 7,15,23,31... Also here

(**i**)$3x+1 \leq 2m \Leftrightarrow 3(\frac{8m}{19}-1)+1 \leq 2m \Leftrightarrow m \geq -3$, which is possible. Hence $\frac{3x+1}{2} \leq m$.

(**ii**) $9x+5 \leq 4m \Leftrightarrow 9(\frac{8m}{19}-1)+5 \leq 4m \Leftrightarrow m \geq -19$, which is possible. Hence $\frac{9x+5}{4} \leq m$.

(**iii**) $27x+19 > 8m \Leftrightarrow 27(\frac{8m}{19}-1)+1 > 8m \Leftrightarrow m \geq -3$, which is possible. Hence $\frac{27x+19}{8} > m$.

Result 2.6. *If $m = 19y$ for some y and x is odd of the form $x = \frac{8m}{19}-1$, then $T_m^3(x) = x$ for all $x \geq 2$.*

Case 8: Suppose $\frac{27x+19}{8} = x$, then $x = -1$. So this is not possible. So in this case $T_m^3(x) \neq x$ for all x, when $T(x)$ and $T^2(x)$ are odd.

If $T_m^3(x) < x$, then the following cases are considered.

Case 1: $\frac{27x+19-38m}{8} < x \Leftrightarrow x < (2m-1)$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $\frac{27x+19-30m}{8} < x \Leftrightarrow x < \frac{30m}{19}-1$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 3: $\frac{27x+19-26m}{8} < x \Leftrightarrow x < \frac{26m}{19}-1$, which is not possible for $x \geq 2$ and $x \in A_m$. For this case $T_m^4(x) = \frac{27x+19-26m}{16}$ and $\frac{27x+19-26m}{16} < x \Leftrightarrow x < \frac{26m-19}{11}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 4: $\frac{27x+19-18m}{8} < x \Leftrightarrow x < \frac{18m-19}{19}$ which is not possible for $x \geq 2$ and $x \in A_m$. For this case $T_m^4(x) = \frac{27x+19-18m}{16}$ and $\frac{27x+19-18m}{16} < x \Leftrightarrow x < \frac{18m-19}{11}$, which is not possible for $x \geq 2$ and $x \in A_m$. Now $T_m^5(x) = \frac{27x+19-18m}{32}$ and $\frac{27x+19-18m}{32} < x \Leftrightarrow x < \frac{-18m+19}{5}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^5(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 5: $\frac{27x+19-12m}{8} < x \Leftrightarrow x < \frac{12m-19}{19}$, which is not possible for $x \geq 2$ and $x \in A_m$. For this case $T_m^4(x) = \frac{27x+19-18m}{32}$ and $T_m^5(x) = \frac{27x+19-12m}{32}$ and $\frac{27x+19-12m}{32} < x \Leftrightarrow x < \frac{-12m+19}{5}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^5(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 6: $\frac{27x+19-20m}{8} < x \Leftrightarrow x < \frac{20m-19}{19}$, which is not possible for $x \geq 2$ and $x \in A_m$. For this case $T_m^4(x) = \frac{27x+19-20m}{32}$ and $T_m^5(x) = \frac{27x+19-20m}{32}$ and $\frac{27x+19-20m}{32} < x$ is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^5(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 7: $\frac{27x+19-8m}{8} < x \Leftrightarrow x < \frac{8m-19}{19}$, which is not possible for $x \geq 2$ and $x \in A_m$. For this case $T_m^4(x) = \frac{27x+19-8m}{16}$ and $T_m^5(x) = \frac{27x+19-8m}{32}$ and $\frac{27x+19-8m}{32} < x$ is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^5(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 8: $\frac{27x+19}{8} < x \Leftrightarrow x < -1$, which is not possible for $x \geq 2$ and $x \in A_m$. For this case $T_m^4(x) = \frac{27x+19}{16}$ and $T_m^5(x) = \frac{27x+19}{32}$ and $\frac{27x+19}{32} < x$ is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^5(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Result 2.7. *If x, $T_m(x)$ and $T_m^2(x)$ are odd and $T_m^k(x)$ is even, for all $k \geq 3$, then $T_m^p(x) \neq x$, for all $p \geq 4$.*

8. Let $x \in A_m$ be arbitrary such that x is odd, $T_m(x)$, $T_m^2(x)$ are even and $T_m^3(x)$ is odd. Then

$$\begin{aligned}
T_m(x) &= \frac{3x+1}{2} (\text{mod } m);\text{ with } T_m(x) \in A_m.\\
&= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases}\\
T_m^2(x) &= \begin{cases} \frac{3x+1-2m}{4}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{4}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases}\\
T_m^3(x) &= \begin{cases} \frac{3x+1-2m}{8}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{8}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases}\\
T_m^4(x) &= \begin{cases} \frac{9x+11-22m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+11-6m}{8} > m \\ \frac{9x+11-6m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+11-6m}{8} \leq m \\ \frac{9x+11-6m}{16}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+11}{16} > m \\ \frac{9x+11}{16}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+11}{16} \leq m. \end{cases}
\end{aligned}$$

Suppose $T_m^4(x)$=x, then the following cases will arise.

Case 1: Suppose $\frac{9x+11-22m}{16} = x$, then $x = \frac{11-2m}{7} > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{9x+11-6m}{16} = x$, then $x = \frac{11-6m}{7} > m$ for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{9x+7-16m}{8} = x$, then $x = \frac{11-6m}{7} > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{9x+11}{16} = x$, then $x = \frac{11}{7}$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all x.

If $T_m^4(x) < x$, then the following cases are considered.

Case 1: $\frac{9x+11-22m}{16} < x \Leftrightarrow x < \frac{-22m+11}{7}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $\frac{9x+11-6m}{8} < x \Leftrightarrow x < \frac{-6m+11}{7}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 3: $\frac{9x+11-6m}{16} < x \Leftrightarrow x < \frac{-6m+11}{7}$, which is possible for $x \geq 2$ and $x \in A_m$. Hence $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 4: $\frac{9x+11}{16} < x \Leftrightarrow x > \frac{11}{7}$, which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Result 2.8. *If $x \in A_m$ and x is odd, $T_m(x)$ is even, $T_m^2(x)$ is even and $T_m^3(x)$ is odd then, (i) There exist no $x \in A_m$ such that $T_m^4(x) = x$. (ii) In addition if $T_m^k(x)$ is even, for all $k \geq 4$, there exist no $x \in A_m$ such that $T_m^p(x) \neq x$, for all $p \geq 4$ and $k \geq 3$.*

9. Let $x \in A_m$ be arbitrary such that x is odd, $T_m(x)$ is even and $T_m^2(x)$ is odd and $T_m^3(x)$ are odd. Then

$$T_m(x) = \frac{3x+1}{2} (\text{mod } m); \text{ with } T_m(x) \in A_m.$$
$$= \begin{cases} \frac{3x+1-2m}{2}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{2}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases}$$

$$T_m^2(x) = \begin{cases} \frac{3x+1-2m}{4}, & \text{if } \frac{3x+1}{2} > m \\ \frac{3x+1}{4}, & \text{if } \frac{3x+1}{2} \leq m. \end{cases}$$

$$T_m^3(x) = \begin{cases} \frac{9x+7-14m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+7-6m}{8} > m \\ \frac{9x+7-6m}{8}, & \text{if } \frac{3x+1}{2} > m \text{ and } \frac{9x+7-6m}{8} \leq m \\ \frac{9x+7-8m}{8}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+7}{8} > m \\ \frac{9x+7}{8}, & \text{if } \frac{3x+1}{2} \leq m \text{ and } \frac{9x+7}{8} \leq m. \end{cases}$$

$$T_m^4(x) = \begin{cases} \frac{27x+29-58m}{16}, & \text{if } \frac{3x+1}{2} > m, \frac{9x+7-6m}{8} > m \text{ and } \frac{27x+29-42m}{16} > m \\ \frac{27x+29-42m}{16}, & \text{if } \frac{3x+1}{2} > m, \frac{9x+7-6m}{8} > m \text{ and } \frac{27x+29-42m}{16} \leq m \\ \frac{27x+29-34m}{16}, & \text{if } \frac{3x+1}{2} > m, \frac{9x+7-6m}{8} \leq m \text{ and } \frac{27x+29-18m}{16} > m \\ \frac{27x+29-18m}{16}, & \text{if } \frac{3x+1}{2} > m, \frac{9x+7-6m}{8} \leq m \text{ and } \frac{27x+29-18m}{16} > m \\ \frac{27x+29-40m}{16}, & \text{if } \frac{3x+1}{2} \leq m, \frac{9x+7}{8} > m \text{ and } \frac{27x+15-8m}{16} > m \\ \frac{27x+29-24m}{16}, & \text{if } \frac{3x+1}{2} \leq m, \frac{9x+7}{8} > m \text{ and } \frac{27x+15-8m}{16} \leq m \\ \frac{27x+29-16m}{16}, & \text{if } \frac{3x+1}{2} \leq m, \frac{9x+7}{8} \leq m \text{ and } \frac{27x+29}{16} > m \\ \frac{27x+29}{16}, & \text{if } \frac{3x+1}{2} \leq m, \frac{9x+7}{8} \leq m \text{ and } \frac{27x+29}{16} \leq m. \end{cases}$$

Suppose $T_m^4(x)=x$, then the following cases will arise.

Case 1: Suppose $\frac{27x+29-58m}{16} = x$, then $x = \frac{29(2m-1)}{11} > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{27x+29-42m}{16} = x$, then $x = \frac{42m-29}{11} > m$ for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{27x+29-34m}{16} = x$ then $x = \frac{34m-29}{11} > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{27x+29-18m}{16}=x$, then $x=\frac{18m-29}{7}>m$. Hence $x\notin A_m$, which is impossible. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 5: Suppose $\frac{27x+29-40m}{16}=x$, then $x=\frac{40m-29}{11}>m$. Hence $x\notin A_m$, which is impossible. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 6: Suppose $\frac{27x+29-24m}{16}=x$, then $x=\frac{24m-29}{7}>m$. Hence $x\notin A_m$, which is impossible. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 7: Suppose $\frac{27x+29-16m}{16}=x$, then $x=\frac{16m-29}{7}>m$. Hence $x\notin A_m$, which is impossible. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 8: Suppose $\frac{27x+29}{16}=x$, then $x=\frac{-29}{9}$. Hence $x\notin A_m$, which is impossible. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Also for this case, if $T_m^4(x)$ is even, then:

$$T_m^5(x)=\begin{cases}\frac{27x+29-58m}{32}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+7-6m}{8}>m \text{ and } \frac{27x+29-42m}{16}>m\\ \frac{27x+29-42m}{32}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+7-6m}{8}>m \text{ and } \frac{27x+29-42m}{16}\leq m\\ \frac{27x+29-34m}{32}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+7-6m}{8}\leq m \text{ and } \frac{27x+29-18m}{16}>m\\ \frac{27x+29-18m}{32}, & \text{if } \frac{3x+1}{2}>m\ ,\frac{9x+7-6m}{8}\leq m \text{ and } \frac{27x+29-18m}{16}>m\\ \frac{27x+29-40m}{32}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+7}{8}>m \text{ and } \frac{27x+15-8m}{16}>m\\ \frac{27x+29-24m}{32}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+7}{8}>m \text{ and } \frac{27x+15-8m}{16}\leq m\\ \frac{27x+29-16m}{32}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+7}{8}\leq m \text{ and } \frac{27x+29}{16}>m\\ \frac{27x+29}{32}, & \text{if } \frac{3x+1}{2}\leq m,\ \frac{9x+7}{8}\leq m \text{ and } \frac{27x+29}{16}\leq m.\end{cases}$$

and clearly $T_m^5(x)<x$, for all x.

Result 2.9. *If $x\in A_m$ is odd, $T_m(x)$ is even, $T_m^2(x)$ is odd ,$T_m^3(x)$ is odd and $T_m^k(x)$ is even, for all $k\geq 3$, then $T_m^p(x)\neq x$, for all $p\geq 5$.*

10. Let $x\in A_m$ be arbitrary such that x is odd, $T_m(x)$ is odd, $T_m^2(x)$ is even and $T_m^3(x)$ is odd. Then

$$T_m^4(x)=\begin{cases}\frac{27x+23-46m}{16}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+5-6m}{4}>m \text{ and } \frac{27x+23-30m}{16}>m\\ \frac{27x+23-30m}{16}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+5-6m}{4}>m \text{ and } \frac{27x+23-30m}{16}\leq m\\ \frac{27x+23-34m}{16}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+5-6m}{4}\leq m \text{ and } \frac{27x+19-18m}{16}>m\\ \frac{27x+29-18m}{16}, & \text{if } \frac{3x+1}{2}>m\ ,\ \frac{9x+7-6m}{8}\leq m \text{ and } \frac{27x+29-18m}{16}>m\\ \frac{27x+29-40m}{16}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+7}{8}>m \text{ and } \frac{27x+15-8m}{16}>m\\ \frac{27x+29-24m}{16}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+7}{8}>m \text{ and } \frac{27x+15-8m}{16}\leq m\\ \frac{27x+29-16m}{16}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+7}{8}\leq m \text{ and } \frac{27x+19}{8}>m\\ \frac{27x+23}{16}, & \text{if } \frac{3x+1}{2}\leq m\ ,\ \frac{9x+5}{4}\leq m \text{ and } \frac{27x+29}{8}\leq m.\end{cases}$$

Suppose $T_m^4(x)$=x, then the following eight cases will arise.

Case 1: Suppose $\frac{27x+23-46m}{16}=x$, then $x=\frac{46m-23}{16}>m$, for $m\geq 2$. Hence $x\notin A_m$. So this is impossible. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 2: Suppose $\frac{27x+23-30m}{16} = x$, then $x = \frac{30m-23}{11} > m$, for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{27x+23-34m}{16} = x$, then $x = \frac{34m-23}{11} > m$, for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{27x+23-18m}{16} = x$, then $x = \frac{18m-23}{11}$, for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^3 4x) \neq x$, for all $x \geq 2$.

Case 5: Suppose $\frac{27x+19-39m}{16} = x$, then $x = \frac{39m-19}{11} > m$, for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 6: Suppose $\frac{27x+23-12m}{16} = x$,, then $x = \frac{12m-23}{11} > m$, for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 7: Suppose $\frac{27x+23-34m}{16} = x$, then $x = \frac{34m-23}{11} > m$, for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 8: Suppose $\frac{27x+23}{16} = x$, then $x = \frac{-23}{11}$, for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Also here if $T_m^4(x)$ is even, then

$$T_m^5(x) = \begin{cases} \frac{27x+23-46m}{32}, & \text{if } \frac{3x+1}{2} > m \text{ , } \frac{9x+5-6m}{4} > m \text{ and } \frac{27x+23-30m}{16} > m \\ \frac{27x+23-30m}{32}, & \text{if } \frac{3x+1}{2} > m \text{ , } \frac{9x+5-6m}{4} > m \text{ and } \frac{27x+23-30m}{16} \leq m \\ \frac{27x+23-34m}{32}, & \text{if } \frac{3x+1}{2} > m \text{ , } \frac{9x+5-6m}{4} \leq m \text{ and } \frac{27x+19-18m}{16} > m \\ \frac{27x+23-18m}{32}, & \text{if } \frac{3x+1}{2} > m \text{ , } \frac{9x+7-6m}{8} \leq m \text{ and } \frac{27x+29-18m}{16} > m \\ \frac{27x+12-39m}{32}, & \text{if } \frac{3x+1}{2} \leq m \text{ , } \frac{9x+7}{8} > m \text{ and } \frac{27x+15-8m}{16} > m \\ \frac{27x+12-12m}{32}, & \text{if } \frac{3x+1}{2} \leq m \text{ , } \frac{9x+7}{8} > m \text{ and } \frac{27x+15-8m}{16} \leq m \\ \frac{27x+23-16m}{32}, & \text{if } \frac{3x+1}{2} \leq m \text{ , } \frac{9x+7}{8} \leq m \text{ and } \frac{27x+19}{8} > m \\ \frac{27x+23}{32}, & \text{if } \frac{3x+1}{2} \leq m \text{ , } \frac{9x+5}{4} \leq m \text{ and } \frac{27x+29}{8} \leq m \end{cases}$$

Clearly here $T_m^5(x) < x$, for all $x \in A_m$.

Result 2.10. *If $x \in A_m$ and x, $T_m(x)$ are odd, $T_m^2(x)$ is even and $T_m^3(x)$ is odd then, (i) there exist no $x \in A_m$ such that $T_m^4(x) = x$. (ii) In addition if $T_m^k(x)$ is even, for all $k \geq 4$, there exist no $x \in A_m$ such that $T_m^p(x) \neq x$, for all $p \geq 4$ and $k \geq 3$.*

11. Let $x \in A_m$ be arbitrary such that x, $T_m(x)$, $T_m^2(x)$, $T_m^3(x)$ are odd, then

$$T_m^4(x) = \begin{cases} \frac{81x+65-130m}{16}, & \text{if } \frac{3x+1}{2} > m,\ \frac{9x+5-6m}{4} > m,\ \frac{27x+19-30m}{16} > m \text{ and } \frac{81x+65-114m}{16} > m \\ \frac{81x+65-114m}{16}, & \text{if } \frac{3x+1}{2} > m,\ \frac{9x+5-6m}{4} > m,\ \frac{27x+19-30m}{16} > m \text{ and } \frac{81x+65-114m}{16} \leq m \\ \frac{81x+65-106m}{16}, & \text{if } \frac{3x+1}{2} > m, \frac{9x+5-6m}{4} > m,\ \frac{27x+19-30m}{16} \leq m \text{ and } \frac{81x+65-90m}{16} > m \\ \frac{81x+65-90m}{16}, & \text{if } \frac{3x+1}{2} > m,\ \frac{9x+5-6m}{4} > m,\ \frac{27x+19-30m}{16} \leq m \text{ and } \frac{81x+65-90m}{16} \leq m \\ \frac{81x+65-104m}{16}, & \text{if } \frac{3x+1}{2} > m, \frac{9x+5-6m}{4} \leq m,\ \frac{27x+19-18m}{16} > m \text{ and } \frac{81x+65-78m}{16} > m \\ \frac{81x+65-78m}{16}, & \text{if } \frac{3x+1}{2} > m,\ \frac{9x+5-6m}{4} \leq m,\ \frac{27x+19-19m}{16} > m \text{ and } \frac{81x+65-65m}{16} \leq m \\ \frac{81x+65-70m}{16}, & \text{if } \frac{3x+1}{2} > m,\ \frac{9x+5-6m}{4} \leq m,\ \frac{27x+19-19m}{16} \leq m \text{ and } \frac{81x+65-65m}{16} > m \\ \frac{81x+65-130m}{16}, & \text{if } \frac{3x+1}{2} > m,\ \frac{9x+5-6m}{4} \leq m,\ \frac{27x+19-18m}{16} \leq m \text{ and } \frac{81x+65-54m}{16} \leq m \\ \frac{81x+65-52m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} > m,\ \frac{27x+19-12m}{8} > m \text{ and } \frac{81x+65-36m}{16} > m \\ \frac{81x+65-36m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} > m,\ \frac{27x+19-12m}{8} > m \text{ and } \frac{81x+65-36m}{16} \leq m \\ \frac{81x+65-96m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} > m,\ \frac{27x+19-12m}{8} \leq m \text{ and } \frac{81x+65-36m}{16} > m \\ \frac{81x+65-60m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} > m,\ \frac{27x+19-12m}{8} \leq m \text{ and } \frac{81x+65-36m}{16} \leq m \\ \frac{81x+65-40m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} \leq m,\ \frac{27x+19}{8} > m \text{ and } \frac{81x+65-24m}{16} > m \\ \frac{81x+65-24m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} \leq m,\ \frac{27x+19}{8} > m \text{ and } \frac{81x+65-24m}{16} \leq m \\ \frac{81x+65-16m}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} \leq m,\ \frac{27x+19}{8} \leq m \text{ and } \frac{81x+65}{16} > m \\ \frac{81x+65}{16}, & \text{if } \frac{3x+1}{2} \leq m,\ \frac{9x+5}{4} \leq m,\ \frac{27x+19}{8} \leq m \text{ and } \frac{81x+65}{16} \leq m \end{cases}$$

Suppose $T_m^4(x){=}x$, then the following cases will arise.

Case 1: Suppose $\frac{81x+65-130m}{16} = x$, then $x = (2m-1) > m$ for $m \geq 2$. Hence $x \notin A_m$. So this is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{81x+65-114m}{16} = x$, then $x = \frac{114m-65}{65} > m$ for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{81x+65-106m}{16} = x$, then $x = \frac{106m-65}{65} > m$ for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{81x+65-90m}{16} = x$, then $x = \frac{90m-65}{65} > m$ for $m \geq 3$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 5: Suppose $\frac{81x+65-104m}{16} = x$, then $x = \frac{104m-65}{65} > m$ for $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 6: Suppose $\frac{81x+65-78m}{16} = x$, then $x = \frac{78m-65}{65}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 2$ and $m \leq 5$. Hence the possible value of m are 2,3,4,5...and the corresponding value of x are not integer except $m = 5$ and $x = 5$. So in this case $T_m^4(x) = x$, for $x = 5$ and $m = 5$. So in this case for $x = 5$ and $m = 5$, $T_m^4(x) = x$.

Case 7: Suppose $\frac{81x+65-70m}{16} = x$, then $x = \frac{14m-13}{13}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 2$ and $m \leq 13$.Hence the possible value of m are 2,3,4,5....and the corresponding value of x are not integer except $m = 13$ and $x = 13$. But $\frac{27x-18m+19}{8} \nleq x$, for $x = 13, m = 13$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 8: Suppose $\frac{81x+65-54m}{16} = x$, then $x = \frac{54m-65}{65}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 3$ and $m \geq -5$. Hence the possible value of m are 3,4,5...and the corresponding value of x are of the form $x = \frac{54m-65}{65}$. Also here

(**i**) $3x + 1 > 2m \Leftrightarrow 3(\frac{54m}{65} - 1) + 1 > 2m$ which is possible. Hence $\frac{3x+1}{2} > m$.

(**ii**) $9x + 5 \leq 10m \Leftrightarrow 9(\frac{54m}{65} - 1) + 5 \leq 10m \Leftrightarrow m \geq \frac{-570}{64}$, which is possible. Hence $\frac{9x+5}{10} \leq 10m$.

(**iii**) $27x + 19 \leq 26m \Leftrightarrow 27(\frac{54m}{65} - 1) + 19 \leq 26m \Leftrightarrow m \geq \frac{(-1736)}{232}$, which is possible. Hence $\frac{27x+19}{26} \leq 26m$.

Result 2.11. *If $x \in A_m$ is odd, $T_m(x)$ is even, $T_m^2(x)$ is odd and $T_m^3(x)$ is odd and if $x = \frac{54m-65}{65}$ and $m = 65p$, then $T_m^4(x) \neq x$, for all $p \geq 1$.*

Case 9: Suppose $\frac{81x+65-52m}{16} = x$, then $x = \frac{52m-65}{65}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 3$ and $m \geq -5$. Hence the possible values of m are 3,4,5...and the corresponding values of x are of the form $x = \frac{52m-65}{65}$. Also here

(**i**) $3x + 1 \leq 2m \Leftrightarrow 3(\frac{54m}{65} - 1) + 1 > 2m$ which is possible. Hence $\frac{3x+1}{2} > m$.

(**ii**) $9x + 5 \leq 10m \Leftrightarrow 9(\frac{52m}{65} - 1) + 1 \leq 2m \Leftrightarrow m \leq \frac{130}{26} \Leftrightarrow m \leq 5$. Hence the range of m is $3 \leq m \leq 5$. But in this range the value of x is not an integer. Hence in this case $T_m^4(x) \neq x$ for all x.

Case 10: Suppose $\frac{81x+65-36m}{16} = x$, th en $x = \frac{36m-65}{65}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 4$ and $m \geq \frac{-65}{29}$. Also here

(**i**) $3x + 1 \leq 2m \Leftrightarrow 3(\frac{36m}{65} - 1) + 1 \leq 2m \Leftrightarrow m \geq \frac{-130}{12}$ which is possible. Hence $\frac{3x+1}{2} \leq m$.

(**ii**) $9x + 5 > 4m \Leftrightarrow 9(\frac{36m}{65} - 1) + 1 > 4m \Leftrightarrow m \geq 8$, which is possible. Hence $\frac{9x+5}{4} > m$.

(**iii**) $27x + 19 \leq 20m \Leftrightarrow 27(\frac{36m}{65} - 1) + 19 \leq 20m \Leftrightarrow m \leq 1$, which is not possible. Hence in this case $T_m^4(x) \neq x$ for all x.

Case 11: Suppose $\frac{81x+65-96m}{16} = x$, then $x = \frac{96m-65}{65} > m$, which is not possible. Hence in this case $T_m^4(x) \neq x$ for all x.

Case 12: Suppose $\frac{81x+65-60m}{16} = x$, then $x = \frac{12m-13}{13}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 3$ and $m \geq -13$. Also here

(**i**) $3x + 1 \leq 2m \Leftrightarrow 3(\frac{12m}{13} - 1) + 1 \leq 2m \Leftrightarrow m \leq 1$ which is not possible. Hence in this case $T_m^4(x) \neq x$ for all x.

Case 13: Suppose $\frac{81x+65-40m}{16} = x$, then $x = \frac{8m-13}{13}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 4$ and $m \geq \frac{-13}{5}$. Also here

(**i**) $3x + 1 \leq 2m \Leftrightarrow m \geq -13$ which is possible. Hence $\frac{3x+1}{2} \leq m$.

(**ii**) $9x + 5 \leq 4m \Leftrightarrow 9(\frac{8m}{13} - 1) + 5 \leq 4m \Leftrightarrow m \leq 2$, which is not possible. Hence in this case $T_m^4(x) \neq x$ for all x.

Case 14: Suppose $\frac{81x+65-24m}{16} = x$, then $x = \frac{24m-65}{65}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $m \geq 6$ and $m \geq \frac{-65}{39}$. Also here

(**i**) $3x + 1 \leq 2m \Leftrightarrow 3(\frac{24m}{65} - 1) + 1 \leq 2m \Leftrightarrow m \geq \frac{-130}{58}$ which is possible. Hence $\frac{3x+1}{2} \leq m$.

(**ii**) $9x + 5 \leq 4m \Leftrightarrow 9(\frac{24m}{65} - 1) + 5 \leq 4m \Leftrightarrow m \geq \frac{-260}{44}$, which is possible. Hence $\frac{9x+5}{4} \leq m$.

(**iii**) $27x + 19 \leq 8m \Leftrightarrow 81(\frac{24m}{65} - 1) + 65 \leq 40m \Leftrightarrow m \geq \frac{-1040}{656}$, which is possible. Hence $\frac{27x+19}{8} \leq m$.

(**iv**) $81x + 65 \leq 40m \Leftrightarrow 81(\frac{24m}{65} - 1) + 65 \leq 40m \Leftrightarrow m \geq \frac{-1040}{656}$, which is possible. Hence the possible values of m are 6,7,8,... and the corresponding possible values of x are $x = 24p - 1$, $p = 1, 2, 3, \ldots$.

Result 2.12. *If $x \in A_m$ is odd, $T_m(x)$ is even, $T_m^2(x)$ is odd and $T_m^3(x)$ is odd and if $x = \frac{24m-65}{65}$ and $m = 65p$ then $T_m^4(x) \neq x$, for all $x \geq 2$ and $p \geq 1$.*

Case 15: Suppose $\frac{81x+65-16m}{16} = x$, then $x = \frac{16m-65}{65}$ .Here $x \in A_m$, so $1 \leq x$ and $x \leq m \Rightarrow m \geq 9$ and $m \geq -2$. Also here

(**i**) $3x+1 \leq 2m \Leftrightarrow 3(\frac{16m}{65} - 1) + 1 \leq 2m \Leftrightarrow m \geq 2$, which is possible. Hence $\frac{3x+1}{2} \leq m$.

(**ii**) $9x+5 \leq 4m \Leftrightarrow 9(\frac{16m}{65} - 1) + 1 \leq 4m \Leftrightarrow m \geq \frac{-325}{116}$, which is possible. Hence $\frac{9x+5}{4} \leq m$.

(**iii**) $27x+19 \leq 8m \Leftrightarrow 81(\frac{16m}{65} - 1) + 19 \leq 8m \Leftrightarrow m \geq \frac{-520}{88}$, which is possible. Hence $\frac{27x+19}{8} \leq m$.

(**iv**) $81x+65 > 16m \Leftrightarrow 81(\frac{16m}{65} - 1) + 16 > 16m \Leftrightarrow m > 4$.

Result 2.13. *If $x \in A_m$ is odd , $T_m(x)$ is even, $T_m^2(x)$,$T_m^3(x)$ are odd and if $x = \frac{16m-65}{65}$ and $m = 65p$ then $T_m^4(x) \neq x$ for all $x \geq 2$ and $p \geq 1$.*

Case 16: Suppose $\frac{81x+65}{16} = x$, then $x = -1$ and x not in A_m. Hence in this case $T_m^4(x) \neq x$ for all x.

Result 2.14. *If $x \in A_m$ is odd, $T_m(x)$ is even, $T_m^2(x)$ is odd and $T_m^3(x)$ is odd and if $T_m^5(x)$,$T_m^6(x)$ are even then $T_m^7(x) < x$ for all x and hence if $T_m^k(x)$ are even for all $k \geq 5$, then $T_m^p(x) \neq x$, for all $x \geq 2$ and for all $p \geq 7$.*

Theorem 2.15. *If x is odd and $x \in A_m$, then the following table provides some failure cases of the expected statement $T_m^k(x) = 1$ for some k, corresponding to "modulo m problem".*

Serial number	x	$T_m(x)$	$T_m^2(x)$	$T_m^3(x)$	RESULT
1	odd	-	-	-	$T_m(x) \neq x$, for all x
2	odd	even	-	-	(a) $T_m^2(x) \neq x$, for all x (b) $T_m^2(x) = x$ only if x=1
3	odd	odd	-	-	(a) $T_m^2(x) \neq x$, for all x (b) $T_5^2(x) = 5$ (c) $T_5^2(3) = 3$
4	odd	even	even	-	$T_m(x) \neq x$, for all x and $T_m^3(x) < x$ for all x if $T_m^k(x)$ is even, for all k$\geq$ 3, then $T_m^p(x) \neq x$, for all $x, p \geq$ 4.
5	odd	even	odd	-	$T_m(x) \neq x$,for all x and $T_m^3(x) < x$, for all x if $T_m^k(x)$ is even, for all k$\geq$ 3, then $T_m^p(x) \neq x$, for all $x, p \geq$ 4.
6	odd	odd	even	-	$T_m(x) \neq x$, for all x and $T_m^3(x) < x$ for all x if $T_m^k(x)$ is even, for all k$\geq$ 3, then $T_m^p(x) \neq x$, for all $x, p \geq$ 4.
7	odd	odd	odd	-	(a) if $m = 19y$, for some y and $x = \frac{18m}{19} - 1$, then $T_m^3(x) = x$ (b) if $m = 19y$, for some y and $x = \frac{12m}{19} - 1$, then $T_m^3(x) = x$ (c) if $m = 19y$, for some y and $x = \frac{8m}{19} - 1$, then $T_m^3(x) = x$ (d) if $T_m^p(x)$ is even, for all $p \geq 3$,
8	odd	even	even	odd	(a) $T_m^4(x) \neq x$,for all x. (b) $T_m^4(x) < x$ (c) if $T_m^p(x)$ is even, for all $p \geq 5$, then $T_m^k(x) \neq x$, for all x ,if $k \geq 5$.
9	odd	even	odd	odd	(a) $T_m^4(x) \neq x$,for all x. (b) $T_m^5(x) < x$, only if $T_m^4(x)$ is even. (c) if $T_m^p(x)$ is even, for all $p \geq 5$, then $T_m^k(x) \neq x$, for all x ,if $k \geq 5$.
10	odd	odd	even	odd	(a) $T_m^4(x) \neq x$,for all x. (b) $T_m^5(x) < x$, only if $T_m^4(x)$ is even. (c) if $T_m^p(x)$ is even, for all $p \geq 5$, then $T_m^k(x) \neq x$, for all x ,if $k \geq 5$.
11	odd	odd	odd	odd	(a) if $x = 5$ and $m = 5$ then $T_m^4(x) = x$ (b) if $m = 65y$, for $y = 1, 2, 3.....$ and $x = \frac{54m}{65} - 1$, then $T_m^4(x) = x$ (c) if $m = 65y$, for $y = 1, 2, 3.....$ and $x = \frac{24m}{65} - 1$, then $T_m^4(x) = x$ (d) if $m = 65y$, for $y = 1, 2, 3.....$ and $x = \frac{16m}{65} - 1$, then $T_m^4(x) = x$

3. Cases for Even Integers

1. Let $x \in A_m$ be arbitrary such that x is even. Then $T_m(x) = \frac{x}{2} < m$. So in this case $T_m(x) \neq x$, for all $x \geq 2$.
2. Let $x \in A_m$ be arbitrary and x is even and $T_m(x)$ is even. Then $T_m(x) = \frac{x}{2} < m$, $T_m^2(x) = \frac{x}{2} < m$. So in this case

$T_m^2(x) \neq x$, for all $x \geq 2$.

3. Let $x \in A_m$ be arbitrary and x is even and $T_m(x)$ is odd. Then

$$\begin{aligned} T_m(x) &= \frac{x}{2} < m \\ T_m^2(x) &= \begin{cases} \frac{3x+2-4m}{4}, & \text{if } \frac{3x+2}{4} > m \\ \frac{3x+2}{4}, & \text{if } \frac{3x+2}{4} \leq m. \end{cases} \end{aligned}$$

Suppose $T_m^2(x) = x$, then the following cases arise.

Case 1: Suppose $\frac{3x+2-4m}{4} = 4x$, then $3x + 2 - 4m = 4x$, $2 - 4m = x$,

$x = 2(1 - 2m) < 0$, for $m \geq 2$. Hence $x \notin A_m$, which is a contradiction. So in this case $T_m^2(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+2}{4} = x$, then $x = 2$, which is a trivial case.

4. Let $x \in A_m$ be arbitrary and x is even, $T_m(x)$ is even, and $T_m^2(x)$ is even.

$$\begin{aligned} T_m(x) &= \frac{x}{2} < m. \\ T_m^2(x) &= \frac{x}{4} < m. \\ T_m^3(x) &= \frac{x}{8} < m. \end{aligned}$$

So in this case $T_m^3(x) \neq x$, for all $x \geq 2$. Here $T_m^3(x) < x$, for all $m \geq 2$. So if $T_m^k(x)$ is even, for all $k \geq 3$ then $T_m^p(x) \neq x$, for all $x \geq 2$ and $p \geq 4$.

5. Let $x \in A_m$ be arbitrary and x is even, $T_m(x)$ is even, and $T_m^2(x)$ is odd. Then

$$\begin{aligned} T_m(x) &= \frac{x}{2} < m \\ T_m^2(x) &= \frac{x}{4} < m \\ T_m^3(x) &= \begin{cases} \frac{3x-8m+4}{8}, & \text{if } \frac{3x+4}{8} > m \\ \frac{3x+4}{8}, & \text{if } \frac{3x+4}{8} < m. \end{cases} \end{aligned}$$

Suppose $T_m^3(x) = x$, then the following cases arise.

Case 1: Suppose $\frac{3x+4-8m}{8} = x$, then $x = \frac{4(1-2m)}{5} < 0$, for $m \geq 2$. Here $x \notin A_m$, which is a contradiction. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+4}{8} = x$,then $5x = 4$, $x = \frac{4}{5} \notin A_m$, so that in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Suppose $T_m^3(x) < x$, then the following cases are considered.

Case 1: $\frac{3x+4-8m}{8} < x \Leftrightarrow 3x + 4 - 8m < 8x \Leftrightarrow -8m + 4 < 5x \Leftrightarrow x > \frac{-8m+4}{5}$ which is possible for $x > 2$ and $m \geq 2$. Here $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $\frac{3x+4}{8} < x \Leftrightarrow 3x + 4 < 8x \Leftrightarrow 4 < 5x \Leftrightarrow x > \frac{4}{5}$ which is possible for $x \geq 2$ and $m \geq 2$. Here $T_m^3(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Result 3.1. *Let $x \in A_m$ such that x is even, $T_m(x)$ is even and $T_m^2(x)$ is odd. Then $T_m^3(x) < x$, for all x and $m \geq 2$. Also if $T_m^k(x)$ is even, for all $k \geq 3$, then $T_m^p(x) \neq x$, for all $x \geq 2$, for all $p \geq 3$.*

6. Let $x \in A_m$ be arbitrary such that x is even, $T_m(x)$ is odd, and $T_m^2(x)$ is odd, then.

$$T_m(x) = \frac{x}{2} < m$$

$$T_m^2(x) = \begin{cases} \frac{3x+2-4m}{4}, & \text{if } \frac{3x+2}{4} > m \\ \frac{3x+2}{4}, & \text{if } \frac{3x+2}{4} \leq m \end{cases}$$

$$T_m^3(x) = \begin{cases} \frac{9x+10-20m}{8}, & \text{if} \frac{3x+2}{4} > m \text{ and } \frac{9x+10-12m}{8} > m \\ \frac{9x+10-12m}{8}, & \text{if} \frac{3x+2}{4} > m \text{ and } \frac{9x+10-12m}{8} \leq m \\ \frac{9x+10-8m}{8}, & \text{if } \frac{3x+2}{4} \leq m \text{ and } \frac{9x+10}{8} > m \\ \frac{9x+10}{8}, & \text{if } \frac{3x+2}{4} \leq m \text{ and } \frac{9x+10}{8} \leq m \end{cases}$$

Suppose $T_m^3(x) = x$, then the following cases arise.

Case 1: Suppose $T_m^3(x) = x$, then $x = 20m-10$ and$x = 10(2m-1) > m$, for $m \geq$2. Hence $x \notin A_m$ which is a contradiction. Hence in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{9x+10-12m}{8} = x$, then $x = 2(6m-5) > m$, for $m \geq 2$. Hence $x \notin A_m$ which is a contradiction. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $\frac{9x+10-8m}{8} = x$, then $9x + 10 - 8m = 8x$ and $x = 8m - 10 > m$, for $m \geq 2$. Hence $x \notin A_m$, which is a contradiction. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{9x+10}{8} = x$, then $x = -10 \notin A_m$, which is a contradiction. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Suppose $T_m^3(x)$ is even, then

$$T_m^4(x) = \begin{cases} \frac{9x+10-12m}{16}, & \text{if } \frac{9x+10-12m}{8} > m \\ \frac{9x+10-12m}{16}, & \text{if } \frac{9x+10-12m}{8} \leq m \\ \frac{9x+10-8m}{16}, & \text{if } \frac{9x+10}{16} > m \\ \frac{9x+10}{16}, & \text{if } \frac{9x+10}{16} \leq m \end{cases}$$

Suppose $T_m^4(x) < x$, then the following cases are considered.

Case 1: $9x - 12m + 10 < 16x \Leftrightarrow x > \frac{-20m+10}{7}$, which is possible for all $x \geq 2$ and $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $9x - 12m + 10 < 16x \Leftrightarrow x > \frac{-12m+10}{7}$, which is possible for all $x \geq 2$ and $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 3: $9x - 8m + 10 < 16x \Leftrightarrow x > \frac{-8m+10}{7}$, which is possible for all $x \geq 2$ and $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 4: $9x - 8m + 10 < 16x \Leftrightarrow x > \frac{-8m+10}{7}$, which is possible for all $x \geq 2$ and $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$ and $m \geq 2$.

Case 5: $9x+10<8x \Leftrightarrow x<-10$, which is possible for all x and $m \geq 2$. So in this case $T_m^4(x)<x$, for all $x \geq 2$ and $m \geq 2$.

Result 3.2. *Let $x \in A_m$. If x is even and $T_m(x)$ is odd, $T_m^2(x)$ is odd and $T_m^k(x)$ is even $k \geq 3$ then $T_m^p(x) \neq x$, for all $x \geq 2$ and $m \geq 2$ and $p \geq 4$.*

7. Let $x \in A_m$ such that x is even, $T_m(x)$ is odd, $T_m^2(x)$ is even. Then

$$T_m(x) = \frac{x}{2}$$

$$T_m^2(x) = \begin{cases} \frac{3x+2-4m}{4}, & \text{if } \frac{3x+2}{4} > m \\ \frac{3x+2}{4}, & \text{if } \frac{3x+2}{4} \leq m \end{cases}$$

$$T_m^3(x) = \begin{cases} \frac{3x+2-4m}{8}, & \text{if } \frac{3x+2}{4} > m \\ \frac{3x+2}{8}, & \text{if } \frac{3x+2}{4} \leq m \end{cases}$$

Suppose $T_m^3(x)=x$, then the following cases may arise.

Case 1: Suppose $\frac{3x+2-4m}{8}=x$ then $x=\frac{2-4m}{5}<0$, for all $m \geq 2$. Hence $x \notin A_m$, which is impossible. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+2}{8}=x$. Then $x=\frac{2}{5} \notin A_m$, which is a contradiction. So in this case $T_m^3(x) \neq x$, for all $x \geq 2$.

Suppose $T_m^3(x)<x$,then the following case are considered.

Case 1: $3x+2-4m<8x \Leftrightarrow \frac{-4m+2}{5}$, which is possible for all $x \in A_m$ and $m \geq 2$. So in this case $T_m^3(x)<x$, for all $x \geq 2$ and $m \geq 2$.

Case 2: $3x+2<8x \Leftrightarrow \frac{2}{5}$, which is possible for all $x \in A_m$ and $m \geq 2$. So in this case $T_m^3(x)<x$, for all $x \geq 2$ and $m \geq 2$.

Result 3.3. *Let $x \in A_m$. If x is even,$T_m(x)$ is odd, $T_m^2(x)$ is even and $T_m^k(x)$ is even, $k \geq 3$ then $T_m^p(x) \neq x$, for all $x \geq 2$ and $m \geq 2$ and $p \geq 4$.*

8. Let $x \in A_m$ be arbitrary such that x is even, $T_m(x)$ is even, $T_m^2(x)$ is even, and $T_m^3(x)$ is odd. Then,

$$T_m^4(x) = \begin{cases} \frac{3x-16m+8}{16}, & \text{if } \frac{3x+8}{m} > m \\ \frac{3x+8}{16}, & \text{if } \frac{3x+8}{16} \leq m \end{cases}$$

Suppose $T_m^4(x)=x$, then the following cases may arise.

Case 1: Suppose $3x-16m+8=16x$, then $x=\frac{-16m+8}{13}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$ which imply $1 \leq \frac{-16m+8}{13}$ and $\frac{-16m+8}{13} \leq m$ so that $m \leq \frac{-5}{16}$ and $m \geq \frac{8}{29}$. This is not possible. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $\frac{3x+8}{16}=x$ then $x=\frac{8}{13}$ which is not an integer. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$

Suppose $T_m^4(x)<x$ then the following cases are considered.

Case 1: $3x-16+8<16x \Leftrightarrow x \geq \frac{-16m+8}{13}$, which is possible for all $x \geq 2$. So in this case $T_m^4(x)<x$, for all $x \geq 2$.

Case 2: $3x+8<16x \Leftrightarrow x>\frac{8}{13}$, which is possible. So in this case $T_m^4(x)<x$, for all $x \geq 2$.

Result 3.4. *Let $x \in A_m$. If x is even, $T_m(x)$ is even, $T_m^2(x)$ is even and $T_m^k(x)$ is even, $k \geq 3$ then $T_m^p(x) \neq x =$, for all $x \geq 2$ and $m \geq 2$ and $p \geq 4$.*

9. Let $x \in A_m$ be arbitrary and x is even, $T_m(x)$ even, $T_m^2(x)$ is odd and $T_m^3(x)$ is odd. Then

$$T_m^4(x) = \begin{cases} \frac{9x+20-40m}{16}, & \text{if } \frac{9x+20-24m}{16} > m \text{ and } \frac{3x+4}{8} > m \\ \frac{9x+20-24m}{16}, & \text{if } \frac{9x+20-24m}{16} \leq m \text{ and } \frac{3x+4}{8} > m \\ \frac{9x+20-24m}{16}, & \text{if } \frac{9x+20}{16} > m \text{ and } \frac{3x+4}{8} \leq m \\ \frac{9x+20}{16}, & \text{if } \frac{9x+20}{16} \leq m \text{ and } \frac{3x+4}{8} \leq m \end{cases}$$

Suppose $T_m^4(x) = x$, then the following cases may arise.

Case 1: Suppose $9x - 40m + 20 = 16x$, then $x = \frac{-40m+20}{7}$, which is not possible for $m \geq 2$, since $x \leq m$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $9x - 24m + 20 = 16x$, then $x = \frac{-24m+20}{7}$, which is not possible for $m \geq 2$, since $x \leq m$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $9x - 16m + 20 = 16x$, then $x = \frac{-16m+20}{7}$, which is not possible for $m \geq 2$, since $x \leq m$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $9x + 20 = 16x$, then $x = \frac{20}{7} \notin A_m$, for any $m \geq 2$. So in this case also $T_m^4(x) \neq x$, for all $x \geq 2$.

Suppose $T_m^4(x) < x$, then the following cases are considered.

Case 1: $9x - 40m + 20 < 16x \Leftrightarrow x > \frac{-40m+20}{7}$, which is possible for $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Case 2: $\Leftrightarrow 9x - 24m + 20 < 16x \Leftrightarrow x > \frac{-24m+20}{7}$, which is possible for $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Case 3: $9x - 16m + 20 < 16x \Leftrightarrow x > \frac{-16m+20}{7}$, which is possible for $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Case 4: $\frac{9x+20}{16} < x \Leftrightarrow x > \frac{20}{7}$, which is possible for $m \geq 2$. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Result 3.5. *(i) If $x \in A_m$ is even, $T_m(x)$ is even, $T_m^2(x)$ is odd, $T_m^3(x)$ is odd, then $T_m^4(x) \neq x$, for all $x \geq 2$. (ii) In addition if $T_m^k(x)$ is even, $k \geq 4$, then $T_m^p(x) \neq x$, for all $p \geq 5$.*

10. Let $x \in A_m$ be arbitrary such that x is even, $T_m(x)$ is odd, $T_m^2(x)$ is odd, $T_m^3(x)$ is odd. Then

$$T_m^4(x) = \begin{cases} \frac{27x+38-76m}{16}, & \text{if } \frac{27x+38-60m}{16} > m, \frac{9x-12m+10}{8} > m, \frac{3x+2}{4} > m \\ \frac{27x+38-60m}{16}, & \text{if } \frac{27x+38-60m}{16} \leq m, \frac{9x-12m+10}{8} > m, \frac{3x+2}{4} > m \\ \frac{27x+38-52m}{16}, & \text{if } \frac{27x+38-36m}{16} > m, \frac{9x-12m+10}{8} \leq m, \frac{3x+2}{4} > m \\ \frac{27x+38-36m}{16}, & \text{if } \frac{27x+38-36m}{16} \leq m, \frac{9x-12m+10}{8} \leq m, \frac{3x+2}{4} > m \\ \frac{27x+38-40m}{16}, & \text{if } \frac{27x+38-24m}{16} > m, \frac{9x+10}{8} > m, \frac{3x+2}{4} \leq m \\ \frac{27x+38-24m}{16}, & \text{if } \frac{27x+38-24m}{16} \leq m, \frac{9x+10}{8} > m, \frac{3x+2}{4} \leq m \\ \frac{27x+38-16m}{16}, & \text{if } \frac{27x+38}{16} > m, \frac{9x+10}{8} \leq m, \frac{3x+2}{4} \leq m \\ \frac{27x+38}{16}, & \text{if } \frac{27x+38}{16} \leq m, \frac{9x+10}{8} \leq m, \frac{3x+2}{4} \leq m \end{cases}$$

Suppose $T_m^4(x) = x$ then following cases may arise.

Case 1: Suppose $27x - 76m + 38 = 16x$, then $x = \frac{76m-38}{11}$. Here $x \in A_m$, so that $1 \leq x$ and $x \leq m$, which imply $1 \leq \frac{76m-38}{11}$ and $\frac{76m-38}{11} \leq m$ which imply $m \geq \frac{49}{76}$ and $m \leq \frac{38}{65} \leq 1$, which is not possible, since $m \geq 2$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $27x-60m+38=16x$, then $x=\frac{60m-38}{11}$. Here $x\in A_m$, so that $1\leq x$ and $x\leq m$, which imply $1\leq \frac{60m-38}{11}$ and $\frac{60m-38}{11}\leq m$ which imply $m\geq \frac{49}{60}$ and $m\leq \frac{38}{49}$, which is not possible, since $m\geq 2$. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 3: Suppose $27x-52m+38=16x$, then $x=\frac{52m-38}{11}$. Here $x\in A_m$, so that $1\leq x$ and $x\leq m$ which imply $1\leq \frac{52m-38}{11}$ and $\frac{52m-38}{11}\leq m$ which imply $m\geq \frac{49}{52}$ and $m\leq \frac{38}{41}\leq 1$, which is not possible, since $m\geq 2$. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 4: Suppose $27x-36m+38=16x$, then $x=\frac{36m-38}{11}$. Here $x\in A_m$, so that $1\leq x$ and $x\leq m$, which imply $1\leq \frac{36m-38}{11}$ and $\frac{36m-38}{11}\leq m$ which imply $m\geq \frac{49}{36}$ and $m\leq \frac{38}{25}\leq 2$, which is not possible, since $m\geq 2$. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 5: Suppose $\frac{27x-40m+38}{16}=x$, then $x=\frac{40m-38}{11}$. Here $x\in A_m$, so that $1\leq x$ and $x\leq m$, which imply $1\leq \frac{40m-38}{11}$ and $\frac{40m-38}{11}\leq m$ which imply $m\geq \frac{49}{40}$ and $m\leq \frac{38}{29}\leq 2$, which is not possible, since $m\geq 2$. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 6: Suppose $27x-24m+38=16x$, then $x=\frac{24m-38}{11}$. Here $x\in A_m$, so that $1\leq x$ and $x\leq m$, which imply $1\leq \frac{24m-38}{11}$ and $\frac{24m-38}{11}\leq m$ which imply $m\geq \frac{49}{24}>2$ and $m\leq \frac{38}{13}\leq 3$ and $2<m\leq 3$, which is not possible, since $m\geq 2$ and m is an integer. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 7: Suppose $27x-16m+38=16x$, then $x=\frac{16m-38}{11}$. Here $x\in A_m$, so that $1\leq x$ and $x\leq m$, which imply $1\leq \frac{16m-38}{11}$ and $\frac{16m-38}{11}\leq m$ which imply $m\geq \frac{49}{16}\geq 4$ and $m\leq \frac{38}{5}\leq 8$ for $4\leq m\leq 7$ the value of x is not an integer. So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Case 8: Suppose $\frac{27x+38}{16}=x$, then $x=\frac{38}{11}$ which is not possible, So in this case $T_m^4(x)\neq x$, for all $x\geq 2$.

Result 3.6. *If $x\in A_m$ is even, and $T_m(x)$ is odd, $T_m^2(x)$ is odd, $T_m^3(x)$ is odd, then $T_m^4(x)\neq x$, for all $x\geq 2$.*

Let $x\in A_m$ is even such that $T_m(x)$ is odd, $T_m^2(x)$ is odd, $T_m^3(x)$ is odd, and if $T_m^4(x)$ is even then

$$T_m^5(x)=\begin{cases}\frac{27x-76m+38}{32}, & \text{if } \frac{27x-60m+38}{16}>m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}>m\\ \frac{27x-60m+38}{32}, & \text{if } \frac{27x-60m+38}{16}\leq m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}>m\\ \frac{27x-52m+38}{32}, & \text{if } \frac{27x-36m+38}{16}>m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}>m\\ \frac{27x-36m+38}{32}, & \text{if } \frac{27x-36m+38}{16}\leq m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}>m\\ \frac{27x-40m+38}{32}, & \text{if } \frac{27x-24m+38}{16}>m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}\leq m\\ \frac{27x-24m+38}{32}, & \text{if } \frac{27x-24m+38}{16}\leq m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}\leq m\\ \frac{27x-16m+38}{32}, & \text{if } \frac{27x+38}{16}>m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}\leq m\\ \frac{27x+38}{32}, & \text{if } \frac{27x+38}{16}\leq m,\ \frac{9x-12m+10}{8}>m,\ \frac{3x+2}{4}\leq m\end{cases}$$

Suppose $T_m^5(x)<x$ then following cases may arise.

Case 1: $\frac{27x-76m+38}{32}<x\Leftrightarrow x>\frac{-76m+38}{5}$, which is possible for $m\geq 2$. So in this case $T_m^5(x)<x$, for all $x\geq 2$.

Case 2: $\frac{27x-60m+38}{32}<x\Leftrightarrow x>\frac{-60m+38}{5}$, which is possible for $m\geq 2$. So in this case $T_m^5(x)<x$, for all $x\geq 2$.

Case 3: $\frac{27x-52m+38}{32}<x\Leftrightarrow x>\frac{-52m+38}{5}$, which is possible $m\geq 2$. So in this case $T_m^5(x)<x$, for all $x\geq 2$.

Case 4: $\frac{27x-36m+38}{32}<x\Leftrightarrow x>\frac{-36m+38}{5}$, which is possible $m\geq 2$. So in this case $T_m^5(x)<x$, for all $x\geq 2$.

Case 5: $\frac{27x-40m+38}{32}<x\Leftrightarrow x>\frac{-40m+38}{5}$, which is possible $m\geq 2$. So in this case $T_m^5(x)<x$, for all $x\geq 2$.

Case 6: $\frac{27x-24m+38}{32} < x \Leftrightarrow x > \frac{-24m+38}{5}$, which is possible $m \geq 2$. So in this case $T_m^5(x) < x$, for all $x \geq 2$.

Case 7: $\frac{27x-16m+38}{32} < x \Leftrightarrow x > \frac{-16m+38}{5}$, which is possible $m \geq 2$. So in this case $T_m^5(x) < x$, for all $x \geq 2$.

Case 8: $27x + 38 < 32x \Leftrightarrow x > \frac{38}{5}$, which is possible $m \geq 2$. So in this case $T_m^5(x) < x$, for all $x \geq 2$.

Result 3.7. *(i) If $x \in A_m$ is even, $T_m(x)$ is odd ,$T_m^2(x)$ is odd, $T_m^3(x)$ is odd, then $T_m^4(x) \neq x$ for all $x \geq 2$. (ii) In addition if $T_m^k(x)$ are even, $k \geq 4$, then $T_m^p(x) \neq x$, for all $p \geq 5$.*

11. Let $x \in A_m$ be arbitrary such that x is even, $T_m(x)$ is odd, $T_m^2(x)$ is even and $T_m^3(x)$ is odd, then

$$T_m^4(x) = \begin{cases} \frac{9x-28m+14}{16}, & \text{if } \frac{9x-12m+14}{16} > m \text{ and } \frac{3x+2}{4} > m \\ \frac{9x-12m+14}{16}, & \text{if } \frac{9x-12m+14}{16} \leq m \text{ and } \frac{3x+2}{4} > m \\ \frac{9x-16m+14}{16}, & \text{if } \frac{9x+14}{16} > m \text{ and } \frac{3x+2}{4} \leq m \\ \frac{9x+14}{16}, & \text{if } \frac{9x+14}{16} \leq m \text{ and } \frac{3x+2}{4} \leq m \end{cases}$$

Suppose $T_m^4(x) = x$, then the following case may arise.

Case 1: Suppose $9x - 28m + 14 = 16x$,then $x = \frac{-28m+14}{7}$, which is not possible, for all $m \geq 2$ and $x \leq m$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 2: Suppose $9x - 12m + 14 = 16x$, then $x = \frac{-12m+14}{7}$, which is not possible, for all $m \geq 2$ and $x \leq m$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 3: Suppose $9x - 16m + 14 = 16x$, then $x = \frac{-16m+14}{7}$, which is not possible, for all $m \geq 2$ and $x \leq m$. So in this case $T_m^4(x) \neq x$, for all $x \geq 2$.

Case 4: Suppose $\frac{9x+14}{16} = x$, then $x = 2$, which is trivial.So in this case $T_m^4(x) \neq x$ for all $x \geq 2$.

Suppose $T_m^4(x) < x$, then the following cases are considered.

Case 1: $\Leftrightarrow 9x - 28m + 14 < 16x \Leftrightarrow x > \frac{-28m+14}{7}$, which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Case 2: $9x - 12m + 14 < 16x \Leftrightarrow x > \frac{-12m+14}{7}$, which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Case 3: $9x - 16m + 14 < 16x \Leftrightarrow x > \frac{-16m+14}{7}$, which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Case 4: $9x + 14 < 16x \Leftrightarrow x > \frac{14}{7} = 2$, which is possible. So in this case $T_m^4(x) < x$, for all $x \geq 2$.

Result 3.8. *(i) If $x \in A_m$ is even $T_m(x)$ is odd, $T_m^2(x)$ is even, $T_m^3(x)$ is odd, then $T_m^4(x) < x$ for all $x \geq 2$. (ii) In addition if $T_m^k(x)$ is Even for all $k \geq 5$, then $T_m^p(x) \neq x$ for all $x \geq 2$ and $p \geq 5$.*

Theorem 3.9. *If x is even and $x \in A_m$, then the following table provides some failure cases of the expected statement $T_m^k(x) = 1$, for some k corresponding to "modulo m problem".*

Serial Number	x	$T_m(x)$	$T_m^2(x)$	$T_m^3(x)$	RESULT
1	even	-	-	-	$T_m(x) \neq x$ for all x
2	even	even	-	-	(a) $T_m^2(x) \neq x$ for all x
3	even	odd	-	-	(a)$T_m^2(x) \neq x$ for all x
4	even	even	even	-	$T_m^3(x) \neq x$ for all x and $T_m^3(x) < x$ for all x if $T_m^k(x)$ is even, for all $k \geq 3$, then $T_m^p(x) \neq x$ for all x ,p$\geq$ 4.
5	even	even	odd	-	$T_m^3(x) \neq x$for all x and $T_m^3(x) < x$ for all x if $T_m^k(x)$ is even, for all $k \geq 3$, then $T_m^p(x) \neq x$ for all x ,p$\geq$ 4.

Serial Number	x	$T_m(x)$	$T^2_m(x)$	$T^3_m(x)$	RESULT
6	even	odd	odd	-	$T^3_m(x) \neq x$for all x and $T^3_m(x) < x$ for all x if $T^k_m(x)$ is even, for all $k \geq 3$, then $T^p_m(x) \neq x$ for all x ,p$\geq$ 4.
7	even	odd	even	-	$T^3_m(x) \neq x$for all x and $T^3_m(x) < x$ for all x if $T^k_m(x)$ is even, for all $k \geq 3$, then $T^p_m(x) \neq x$ for all x ,p$\geq$ 4.
8	even	even	even	odd	(a) $T^4_m(x) \neq x$,for all x. (b) $T^4_m(x) < x$ (c) if $T^p_m(x)$ is even, for all $p \geq 5$, then $T^k_m(x) \neq x$ for all x ,if $k \geq 5$.
9	even	even	odd	odd	(a) $T^4_m(x) \neq x$,for all x. (b) $T^4_m(x) < x$, (c) if $T^p_m(x)$ is even, for all $p \geq 5$, then $T^k_m(x) \neq x$ for all x ,if $k \geq 5$.
10	even	odd	odd	odd	(a) $T^4_m(x) \neq x$,for all x. (b) $T^4_m(x) < x$. (c) if $T^p_m(x)$ is even, for all $p \geq 5$, then $T^k_m(x) \neq x$ for all x ,if $k \geq 5$.
11	even	odd	even	odd	(a) $T^4_m(x) \neq x$,for all x. (b) $T^4_m(x) < x$. (c) if $T^p_m(x)$ is even, for all $p \geq 5$, then $T^k_m(x) \neq x$ for all x ,if $k \geq 5$.

4. Conclusion

Theorem 2.15 and Theorem 3.9 provide some failure cases of the problem: $T^k_m(x) = 1$ for some k. This work has been carried out by having discussion on the possibilities: $T_m(x) = x, T^2_m(x) = x, T^3_m(x) = x, T^4_m(x) = x$. Further discussion may also be carried out for $T^k_m(x) = x$ with k=5,6,7.... They may provide a class of non-good numbers for the original collatz problem.

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