International Journal of Mathematics And its Applications
Volume 4, Issue 3–A (2016), 63–72.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Time Dependent Solution of M/M/C Feedback Queue With Catastrophes

Research Article

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Abstract: In this paper we study the time dependent solution of M/M/C feedback queue under the catastrophic effect. To solve transient solution of this model, we use generating function technique. We derive stationary probability distribution of this model and we give numerical example for the feasibility of this model.

MSC: 60K25, 68M20, 90B22.

Keywords: Catastrophes, Departure Process, Feedback customer, Poisson process, Stationary probability distribution, Transient probability.

1. Introduction

In the year of 1909, queueing theory originated in telephony with the work of Erlang [2]. After his work many authors to develop different types of queueing models, incorporating different arrival patterns, different service time distributions and various service disciplines. In the year of 1963, Takacs [12] first introduced queues with feedback mechanism which includes the possibility for a customer return to the counter for additional service. In the year of 2000, Santhakumaran and Thangaraj [10] have studied a single server queue with impatient and feedback customers. In the year of 2008, Santhakumaran and Shanmugasundaram [11] have focused to study a Preparatory Work on Arrival Customers with a Single Server Feedback Queue.

In queueing theory, the time independent solutions only derived for a long time. According to the theory and applications of queueing theory time dependent solution is necessary. Parthasarathy [8] and Parthasarathy and Shafarali [7] have discussed single and multiple server poisson queues of transient state solution in easiest manner. Krishna Kumar and Arivudainambi [6] has proposed a transient state solution for the mean queue size of M/M/1 queueing model when catastrophes occurred at the service station. Parthasarathy and Sudesh [9] have introduced transient solution of a multiserver poisson queue with N-policy. Al-Seady, El-sherbiny, El-Sherhawy and Ammar [1] have discussed transient solution of the M/M/C queue with balking and reneging. Darmaraja and Rakesh Kumar [2] have studied transient solution of a Markovian queueing model with heterogeneous servers and catastrophes.

In this paper we analyze the time dependent solution of M/M/C feedback queue with catastrophes.

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2. Model Description and Analysis

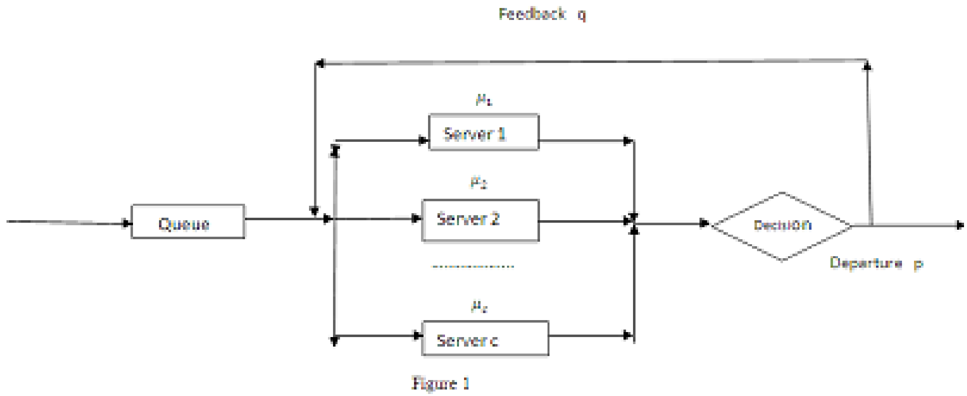


Figure 1.

Figure 1, illustrates the flow of customers through the queueing system. The queueing system is denoted by M/M/C heterogeneous queue with instantaneous Bernoulli feedback. Here the arrival rate of the customer enter into the queue is a poisson process with rate λ_t. If the server is idle upon an arrival, service of an arriving customer starts instantaneously. Service rate follows exponential distributions with rate $\mu_i(t)$. After receiving service the customer a decision is made whether or not feedback. If the customer does feedback, he joins the feedback stream with probability q. If the customer does not feedback, he joins the departure process with probability p. The queue discipline is FIFO and infinite in capacity, the service times are non-negative, independent and identically distributed random variables. Catastrophes occurs from the arrival and service process follows poisson process with rate ν_t. All the available customers are destroyed immediately when the catastrophes occurred in the system, the server gets inactivated. The server is ready for service when a new arrival happens. The motivation for this model comes from Hospitals, Production System, Banks, Restaurant, etc.

Let $Q_n(t) = Q[X(t) = n]$, $n = 0, 1, \ldots$ denote the transient state probabilities of n customers in the system at a time t. Let $Q(x,t)$ be the probability generating function. Generally we assume that there are no customers in the system at a time $t = 0$. The system of differential-difference equations for the probability $Q_n(t)$ is

$$Q_0'(t) = -(\lambda_t + \nu_t)Q_0(t) + [p\mu_1(t) + q\mu_1(t)]\, Q_1(t) + \nu_t$$

$$Q_0'(t) = -(\lambda_t + \nu_t)Q_0(t) + \mu_1(t)\,[p + q]\, Q_1(t) + \nu_t$$

$$Q_0'(t) = -(\lambda_t + \nu_t)Q_0(t) + \mu_1(t)Q_1(t) + \nu_t \tag{1}$$

$$Q_n'(t) = -\left(\lambda_t + \nu_t + p\sum_{i=1}^{n}\mu_i(t) + q\sum_{j=1}^{n}\mu_j(t)\right)Q_n(t) + \lambda_t Q_{n-1}(t) + \left[p\sum_{i=1}^{n+1}\mu_i(t) + q\sum_{j=1}^{n+1}\mu_j(t)\right]Q_{n+1}(t), \quad 1 \le n \le c-1 \tag{2}$$

$$Q'_c(t) = -\left(\lambda_t + \nu_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right)Q_c(t) + \lambda_t Q_{c-1}(t) + \left[p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right]Q_{c+1}(t), \tag{3}$$

$$Q_n'(t) = -\left(\lambda_t + \nu_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right)Q_n(t) + \lambda_t Q_{n-1}(t) + \left[p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right]Q_{n+1}(t), \quad n = c+1,\ c+2,\ \ldots \tag{4}$$

With $Q_n(0) = \delta_{0n}$, Kronecker delta symbol. The probability generating function $Q(x,t)$ for the transient state probabilities

$Q_n(t)$ is given by

$$Q(x,t) = p_c(t) + \sum_{n=1}^{\infty} Q_{n+c}(t)x^n; \quad Q(x,0) = 1 \tag{5}$$

$$p_c(t) = \sum_{n=0}^{\infty} Q_n(t)$$

$$p_c(t) = Q_0(t) + Q_c(t) + \sum_{n=1}^{c-1} Q_n(t)$$

$$p_c'(t) = Q_0'(t) + Q_c'(t) + \sum_{n=1}^{c-1} Q_n'(t) \tag{6}$$

Substituting equations (1), (2) & (3) we get

$$\frac{dp_c(t)}{dt} = -\lambda_t Q_c(t) + \left[p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right] Q_{c+1}(t) - \nu_t p_c(t) + \nu_t \tag{7}$$

Multiply equation (4) by x^n, we get,

$$\frac{d[\sum_{n=1}^{\infty} Q_{n+c}(t)\}x^n}{dt} = \left[-\left(\lambda_t + \nu_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) + \left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right)\right]\sum_{n=1}^{\infty} Q_{n+c}(t)x^n$$
$$+ \lambda_t x Q_c(t) - \left[p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right] Q_{n+c}(t) \tag{8}$$

Differentiate equation (5) with respect to t and substitute equations (7) & (8) we get,

$$\frac{\partial Q(x,t)}{\partial t} = \left[-\left(\lambda_t + \nu_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) + \left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right)\right] Q(x,t)$$
$$- \left[\left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right) - \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) p_c(t)\right] + \lambda_t(x-1)Q_c(t) + \nu_t \tag{9}$$

Using the integrating factor as

$$e^{-\left[\left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right) - \left(\lambda_t + \nu_t + p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right]t}$$

Solving the above first order differential equation (9) we get,

$$Q(x,t) = e^{-\left[\left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right) - \left(\lambda_t + \nu_t + p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right]t}$$
$$+ \int_0^t \left\{\lambda_t(x-1)Q_c(u) - \left[\left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right) - \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) p_c(u)\right]\right\}$$
$$e^{-\left[\left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right) - \left(\lambda_t + \nu_t + p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} du$$
$$+ \nu_t \int_0^t e^{-\left[\left(\lambda_t x + \frac{p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)}{x}\right) - \left(\lambda_t + \nu_t + p\sum\limits_{i=1}^{c}\mu_i(t) + q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} du \tag{10}$$

Let $\gamma = 2\sqrt{\lambda_t p \sum_{i=1}^{c} \mu_i(t) + q \sum_{j=1}^{c} \mu_j(t)}$ and $\delta = \sqrt{\frac{\lambda_t}{p \sum_{i=1}^{c} \mu_i(t) + q \sum_{j=1}^{c} \mu_j(t)}}$, then using the modified Bessel function of first kind $I_n(.)$ and the Bessel function properties, we get

$$e^{-\left[\left(\lambda_t x + \frac{p \sum_{i=1}^{c} \mu_i(t) + q \sum_{j=1}^{c} \mu_j(t)}{x}\right)\right] t} = \sum_{n=-\infty}^{\infty} (\delta s)^n I_n(\gamma t) \tag{11}$$

Substitute equation (11) in equation (10), we get,

$$\begin{aligned} Q(x,t) &= e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right]t} \sum_{n=-\infty}^{\infty} (\delta x)^n I_n(\gamma t) \\ &+ \lambda_t \int_0^t Q_c(u) e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \sum_{n=-\infty}^{\infty} (\delta x)^n (\delta)^{-1} \left[I_{n-1}(\gamma(t-u)) - I_n(\gamma(t-u))\right] du \\ &+ \int_0^t p_c(u) e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \sum_{n=-\infty}^{\infty} \left[-\lambda_t (\delta)^{-1} \left[I_{n-1}(\gamma(t-u))\right] du \right. \\ &+ \left. \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_n(\gamma(t-u)) - \left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_{n+1}(\gamma(t-u))\right] \\ &+ \nu_t \int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \sum_{n=-\infty}^{\infty} (\delta x)^n I_n(\gamma(t-u))\, du \end{aligned} \tag{12}$$

$$\begin{aligned} p_c(t) + \sum_{n=1}^{\infty} Q_{n+c}(t) x^n &= e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right]t} \sum_{n=-\infty}^{\infty} (\delta x)^n I_n(\gamma t) \\ &+ \lambda_t \int_0^t Q_c(u) e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \sum_{n=-\infty}^{\infty} (\delta x)^n (\delta)^{-1} \left[I_{n-1}(\gamma(t-u)) - I_n(\gamma(t-u))\right] du \\ &+ \int_0^t p_c(u) e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \sum_{n=-\infty}^{\infty} \left[-\lambda_t (\delta)^{-1} \left[I_{n-1}(\gamma(t-u))\right] du \right. \\ &+ \left. \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_n(\gamma(t-u)) - \left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_{n+1}(\gamma(t-u))\right] \\ &+ \nu_t \int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \sum_{n=-\infty}^{\infty} (\delta x)^n I_n(\gamma(t-u))\, du \end{aligned} \tag{13}$$

Equating the co efficient of x^n on both sides, and for $n = 1, 2, 3, \ldots$

$$\begin{aligned} Q_{n+c}(t) &= e^{-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)t} \delta^n I_n(\gamma t) \\ &+ \lambda_t \int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \left[I_{n-1}(\gamma(t-u))\,\delta^{n-1} - I_n(\gamma(t-u)\delta^n)\right] Q_c(u)\, du \\ &- \int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} p_c(u) \left[\lambda_t I_{n-1}(\gamma(t-u))\,\delta^{n-1} \right. \\ &- \left. \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_n(\gamma(t-u))\delta^n) + \left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_{n+1}(\gamma(t-u))\delta^{n+1})\right] du \\ &+ \nu_t \int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \delta^n I_n(\gamma(t-u))\, du \end{aligned} \tag{14}$$

Substituting $n = 0$ and comparing the constant terms of Equation (13) we get,

$$p_c(t) = e^{-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)t} I_0(\gamma t)$$
$$+\lambda_t\int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \left[I_{-1}(\gamma(t-u))\delta^{-1} - I_0(\gamma(t-u))\right] Q_c(u)\,du$$
$$-\int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} p_c(u)\left[-\lambda_t I_{-1}(\gamma(t-u)) - \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_0(\gamma(t-u))\right] du$$
$$+\nu_t\int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} I_0(\gamma(t-u))\,du \tag{15}$$

The terms in left hand side of equation (14) have no negative powers of z, therefore left hand side replaced by zero and using $I_n(\gamma(t-u)) = I_{-n}(\gamma(t-u))$. We have

$$\int_0^t p_c(u) e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \left[\lambda_t I_{n+1}(\gamma(t-u))\delta^{n-1} - \left(\lambda_t + p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_n(\gamma(t-u)\delta^n)\right.$$
$$\left.+\left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) I_{n-1}\left(\gamma(t-u)\delta^{n+1}\right)\right] du = e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right]t}\delta^n I_n(\gamma t)$$
$$+\lambda_t\int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} Q_c(u)\left[I_{n+1}(\gamma(t-u))\delta^{n-1} - I_n(\gamma(t-u))\delta^n\right] du$$
$$+\nu_t\int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} I_n(\gamma(t-u))\delta^n du \tag{16}$$

Substituting equation (16) in equation (14) we get

$$Q_{n+c}(t) = n\delta^n\int_0^t e^{\left[-\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)\right](t-u)} \frac{I_n(\gamma(t-u))}{t-u} Q_c(u)\,du \tag{17}$$

The remaining probabilities $Q_n(t)$, $n = 0, 1, 2, \ldots c$ can be obtained by solving the equations (1) and (2). Equations (1) and (2) can be written in matrix form as

$$\frac{dQ(t)}{dt} = AQ(t) + \left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) Q_c(t) I_1 + \nu_t I_2 \tag{18}$$

Where $A = \begin{bmatrix} -(\lambda_t+\nu_t) & \mu_1(t) & \ldots & 0 \\ \lambda_t & -(\lambda_t+\nu_t+\mu_1(t)) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \left[p\sum\limits_{i=1}^{c-1}\mu_i(t)+q\sum\limits_{j=1}^{c-1}\mu_j(t)\right] \\ 0 & 0 & \ldots & -\left(\lambda_t+\nu_t+p\sum\limits_{i=1}^{c-1}\mu_i(t)+q\sum\limits_{j=1}^{c-1}\mu_j(t)\right) \end{bmatrix}$

And $Q(t) = (Q_0(t) \quad Q_1(t)\ldots Q_{c-1}(t))^T$, $I_1 = (0\ 0\ \ldots\ 1)^T$, $I_2 = (1\ 0\ \ldots\ 0)^T$ are column vector of order c. Let $Q^*(s) = ({Q_0}^*(s)\ {Q_1}^*(s)\ \ldots\ {Q_{c-1}}^*(s))^T$ denotes the Laplace transform of $Q(t)$. Taking Laplace transform of equation (18) we get,

$$sQ^*(s) - Q(0) = A\,Q^*(s) + \left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) {Q_c}^*(s) I_1 + \frac{\nu_t}{s} I_2$$

$$(sI - A)Q^*(s) = Q(0) + \left(p\sum_{i=1}^{c}\mu_i(t) + q\sum_{j=1}^{c}\mu_j(t)\right) {Q_c}^*(s) I_1 + \frac{\nu_t}{s} I_2$$

$$Q^*(s) = (sI - A)^{-1} \left\{ Q(0) + \left(p\sum_{i=1}^{c} \mu_i(t) + q\sum_{j=1}^{c} \mu_j(t) \right) Q_c{}^*(s)\, I_1 + \frac{\nu_t}{s} I_2 \right\} \tag{19}$$

With $Q(0) = (1 \;\; 0 \;\; \ldots \;\; 0)^T$. To find $Q_c{}^*(s)$, if $e = (1 \;\; 1 \;\; \ldots \;\; 1)^T{}_{c\times 1}$, then

$$e^T Q^*(s) + Q_c{}^*(s) = p_c{}^*(s) \tag{20}$$

Let $w = \left[\left(s + \lambda_t + \nu_t + p\sum\limits_{i=1}^{c} \mu_i(t) + q\sum\limits_{j=1}^{c} \mu_j(t) \right) - \sqrt{\left(s + \lambda_t + \nu_t + p\sum\limits_{i=1}^{c} \mu_i(t) + q\sum\limits_{j=1}^{c} \mu_j(t) \right)^2 - \alpha^2} \right]$

Laplace transform of $p_c(t)$ we get,

$$s(s + \nu_t)\, p_c{}^*(s) = (s + \nu_t) + sQ_c{}^*(s)\, \frac{(w - \gamma\delta)}{2} \tag{21}$$

$$p_c{}^*(s) = \frac{1}{s} + \frac{1}{(s+\nu_t)} Q_c{}^*(s)\, \frac{(w-\gamma\delta)}{2}$$

$$e^T Q^*(s) + Q_c{}^*(s) = \frac{1}{s} + \frac{1}{(s+\nu_t)} Q_c{}^*(s)\, \frac{(w-\gamma\delta)}{2}$$

$$Q_c{}^*(s) \left[1 - \frac{1}{(s+\nu_t)} \frac{(w-\gamma\delta)}{2} \right] = \frac{1}{s} + e^T Q^*(s)$$

Substitute equation (19) we get,

$$Q_c{}^*(s) \left[1 - \frac{1}{(s+\nu_t)} \frac{(w-\gamma\delta)}{2} \right] = \frac{1}{s} + e^T (sI - A)^{-1} \left\{ Q(0) + \left(p\sum_{i=1}^{c} \mu_i(t) + q\sum_{j=1}^{c} \mu_j(t) \right) Q_c{}^*(s)\, I_1 + \frac{\nu_t}{s} I_2 \right\}$$

$$Q_c{}^*(s) = \left(\frac{s+\nu_t}{s} \right) \times \frac{1 - se^T (sI - A)^{-1} \left(Q(0) + \frac{\nu_t}{s} I_2 \right)}{(s + \lambda_t + \nu_t) - \frac{w}{2} + (s + \nu_t)\, e^T (sI - A)^{-1} I_1 \left(p\sum\limits_{i=1}^{c} \mu_i(t) + q\sum\limits_{j=1}^{c} \mu_j(t) \right)} \tag{22}$$

By Raju and Bhat we get the value of the matrix $(sI - A)^{-1}$. Let us assume that $(sI - A)^{-1} = (a^*_{uv}(s))_{c\times c}$. For $u = 0,\ 1,\ 2,\ \ldots c-1$

$$a^*_{uv}(s) = \begin{cases} \frac{1}{p\sum\limits_{k=1}^{v+1} \mu_k(t) + q\sum\limits_{k=1}^{v+1} \mu_k(t)} \frac{g_{c,v+1}(s) g_{u,0}(s) - g_{u,v+1}(s) g_{c,0}(s)}{g_{c,0}(s)}, & v = 0,\ 1,\ 2,\ \ldots c-2 \\ \frac{g_{u,0}(s)}{g_{c,0}(s)} & v = c-1. \end{cases} \tag{23}$$

Where

$$g_{u,u} = 1, \qquad u = 0,\ 1,\ 2,\ \ldots c-1$$

$$g_{u+1,u} = \frac{s + \lambda_t + \nu_t + p\sum\limits_{k=1}^{u} \mu_k(t) + q\sum\limits_{k=1}^{u} \mu_k(t)}{p\sum\limits_{k=1}^{u+1} \mu_k(t) + q\sum\limits_{k=1}^{u+1} \mu_k(t)} \qquad u = 0,\ 1,\ 2,\ \ldots c-2$$

$$g_{u+1,u-v} = \frac{\left(s + \lambda_t + \nu_t + p\sum\limits_{k=1}^{u} \mu_k(t) + q\sum\limits_{k=1}^{u} \mu_k(t) \right) g_{u,u-v} - \lambda_t g_{u-1,u-v}}{p\sum\limits_{k=1}^{u+1} \mu_k(t) + q\sum\limits_{k=1}^{u+1} \mu_k(t)} \qquad v = u, \quad u = 1,\ 2,\ \ldots c-2$$

$$g_{c,v} = \begin{cases} \left(s + \lambda_t + \nu_t + p\sum\limits_{k=1}^{c-1} \mu_k(t) + q\sum\limits_{k=1}^{u} \mu_k(t) \right) g_{c-1,v} - \lambda_t g_{c-2,v} & v = 0,\ 1,\ 2,\ \ldots c-2 \\ \left(s + \lambda_t + \nu_t + p\sum\limits_{k=1}^{c-1} \mu_k(t) + q\sum\limits_{k=1}^{u} \mu_k(t) \right) & v = c-1 \end{cases} \tag{24}$$

And $g_{u,v} = 0$ for other u and v. Using the above equations in equation (21), we get

$$Q_c{}^*(s) = \left(\frac{s+\nu_t}{s}\right) \times \frac{1-(s+\nu_t)\sum_{u=0}^{c-1} a^*_{u,0}(s)}{(s+\lambda_t+\nu_t)-\frac{w}{2}+(s+\nu_t)\sum_{u=0}^{c-1} a^*_{u,c-1}(s)\left(p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)} \tag{25}$$

From equation (18) for $= 0, 1, 2, \ldots, c-1$, we get

$$Q_k{}^*(s) = \left(\frac{s+\nu_t}{s}\right) a^*_{k,0}(s) + \left(p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right) a^*_{k,c-1}(s)\, Q_c{}^*(s) \tag{26}$$

We have seen that $a^*_{u,v}(s)$ are all rational algebraic functions in s. The cofactor of the $(u,v)^{th}$ element of $(sI-A)$ is a polynomial of degree $c-1-|u-v|$. The characteristic roots of the matrix A are all distinct. Let s_u , $u = 0,\ 1,\ \ldots c-1$ be the characteristic roots of the matrix A. Then

$$Q_c{}^*(s) = \frac{\left(1+\frac{\nu_t}{s}\right)B^*(s)}{\frac{w}{2}\left[1-\frac{2\left(p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)(1-C^*(s))}{\left(s+\lambda_t+\nu_t+p\sum_{k=1}^{c-1}\mu_k(t)+q\sum_{k=1}^{u}\mu_k(t)\right)-\sqrt{\left(s+\lambda_t+\nu_t+p\sum_{k=1}^{c-1}\mu_k(t)+q\sum_{k=1}^{u}\mu_k(t)\right)^2-\gamma^2}}\right]} \tag{27}$$

Where

$$B^*(s) = \sum_{u=0}^{c-1}\frac{\lim_{s\to s_u}(s-s_u)\left[1-\sum_{l=0}^{c-1}(s+\nu_t)\,a^*_{l,0}(s)\right]}{s-s_u} \tag{28}$$

$$C^*(s) = \sum_{u=0}^{c-1}\frac{\lim_{s\to s_u}(s-s_u)\left[\sum_{l=0}^{c-1}(s+\nu_t)\,a^*_{l,c-1}(s)\right]}{s-s_u} \tag{29}$$

$$Q_c{}^*(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\left(\frac{s+\nu_t}{s}\right)$$

$$Q_c{}^*(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\frac{(-1)^m}{\left(p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{s+\nu_t}{s}\right)B^*(s)\,(C^*(s))^m$$
$$\frac{\left[\left(s+\lambda_t+\nu_t+p\sum_{k=1}^{c-1}\mu_k(t)+q\sum_{k=1}^{u}\mu_k(t)\right)-\sqrt{\left(s+\lambda_t+\nu_t+p\sum_{k=1}^{c-1}\mu_k(t)+q\sum_{k=1}^{u}\mu_k(t)\right)^2-\gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}} \tag{30}$$

By taking inverse Laplace transform, we get

$$Q_c(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\left(p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}\begin{pmatrix} n \\ k \end{pmatrix}\int_0^t B(t-u)\int_0^u C^{c(m)}(u-v)e^{-\left(\lambda_t+\nu_t+p\sum_{k=1}^{c-1}\mu_k(t)+q\sum_{k=1}^{u}\mu_k(t)\right)v}$$
$$\frac{I_{n+1(v)}}{v}dudv+\nu_t\int_0^t G(t-u)\int_0^u C^{c(m)}(u-v)e^{-\left(\lambda_t+\nu_t+p\sum_{k=1}^{c-1}\mu_k(t)+q\sum_{k=1}^{u}\mu_k(t)\right)v}\frac{I_{n+1(\gamma v)}}{v}dudv \tag{31}$$

Where $G(t) = \int_0^t B(u)du$ and $C^{c(m)}(t)$ is m-fold convolution of C(t) with itself and $C^{c(0)} = \delta(t)$. By taking Laplace transform of equation (26), we get

$$Q_k(t) = a_{k,0}(t)+\nu_t\int_0^t a_{k,0}(u)du+\left(p\sum_{i=1}^{c}\mu_i(t)+q\sum_{j=1}^{c}\mu_j(t)\right)\int_0^t a_{k,c-1}(u)Q_c(t-u)\,du,\quad k=0,\ 1,\ \ldots c-1 \tag{32}$$

Hence equations (17), (31) & (32) completely determine all the state probabilities.

Particular Case: The server is single server, i.e. when $c = 1$. From equations (1) & (6) we get

$$(s + \lambda_t + \nu_t)\, Q_0{}^*(s) = 1 + \mu_1 Q_1{}^*(s) + \frac{\nu_t}{s}$$

$$p_1{}^*(s) = Q_0{}^*(s) + Q_1{}^*(s)$$

Substituting the above equations in equation (22), we get

$$Q_1{}^*(s) = \left(\frac{s+\nu_t}{s}\right)\frac{1}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\left(\frac{\lambda_t}{(s+\lambda_t+\nu_t)}\right)^{n+1}\frac{\left[(s+\lambda_t+\nu_t+\mu_1)-\sqrt{(s+\lambda_t+\nu_t+\mu_1)^2-\gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}} \tag{33}$$

By taking inverse Laplace transform, we get

$$\begin{aligned} Q_1(t) = {} & \frac{1}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\int_0^t {\lambda_t}^{n+1}\frac{u^n e^{-(\lambda_t+\nu_t)u}}{n!}e^{-(\lambda_t+\nu_t+\mu_1)(t-u)}\frac{I_{n+1(\gamma(t-u))}}{(t-u)}du \\ & + \frac{\nu_t}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\int_0^t {\lambda_t}^{n+1}K_1(u)\, e^{-(\lambda_t+\nu_t+\mu_1)(t-u)}\frac{I_{n+1(\gamma(t-u))}}{(t-u)}du \end{aligned} \tag{34}$$

Where $K_1(t) = \int_0^t u^n \frac{e^{-(t+t)u}}{n!}du$. For $n = 0$ from equation (32), we get

$$Q_0(t) = e^{-(\lambda_t+\nu_t)t} + \mu_1\int_0^t e^{-(\lambda_t+\nu_t)(t-u)}Q_1(u)\,du + \nu_t\int_0^t e^{-(\lambda_t+\nu_t)u}du \tag{35}$$

For $n = 1,\ 2,\ \ldots$ from equation (17), we get

$$Q_{n+1}(t) = n\delta^n\int_0^t e^{-(\lambda_t+\nu_t+\mu_1)(t-u)}\frac{I_{n(\gamma(t-u))}}{(t-u)}Q_1(u)\,du \tag{36}$$

3. Stationary Probability Distribution

The Stationary Probability Distribution $_1$ for the M/M/C feedback queue with catastrophes $\nu_t > 0$ is

$$\pi_c = \lim_{t\to\infty} Q_c\ (t) = \lim_{s\to 0}\ sQ_c{}^*(s)\ = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\frac{(-1)^m}{\left(p\sum\limits_{i=1}^{c}\mu_i(t)+q\sum\limits_{j=1}^{c}\mu_j(t)\right)}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{s+\nu_t}{s}\right)B^*(s)\,(C^*(s))^m$$

$$\frac{\left[\left(\lambda_t+\nu_t+p\sum\limits_{k=1}^{c-1}\mu_k(t)+q\sum\limits_{k=1}^{u}\mu_k(t)\right)-\sqrt{\left(\lambda_t+\nu_t+p\sum\limits_{k=1}^{c-1}\mu_k(t)+q\sum\limits_{k=1}^{u}\mu_k(t)\right)^2-\gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}}$$

Let $c = 1$ we get,

$$\begin{aligned} \pi_1 &= \lim_{t\to\infty} Q_1(t) = \lim_{s\to 0}\ sQ_1{}^*(s) \\ &= \left(\frac{s+\nu_t}{s}\right)\frac{1}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\left(\frac{\lambda_t}{(s+\lambda_t+\nu_t)}\right)^{n+1}\frac{\left[(s+\lambda_t+\nu_t+\mu_1)-\sqrt{(s+\lambda_t+\nu_t+\mu_1)^2-\gamma^2}\right]^{n+1}}{(n+1)\,\gamma^{n+1}} \\ &= \frac{\nu_t}{\mu_1}\sum_{n=0}^{\infty}\left[\frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2\,(\lambda_t+\nu_t)}\right]^{n+1} \\ &= \frac{\nu_t}{\mu_1}\left[\frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2\,(\lambda_t+\nu_t)}\right]\left[1-\frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\nu_1)^2-\gamma^2}}{2\,(\lambda_t+\nu_t)}\right]^{-1} \end{aligned}$$

$$\pi_1 = \frac{\nu_t}{\mu_1}\frac{\rho}{(1-\rho)}$$

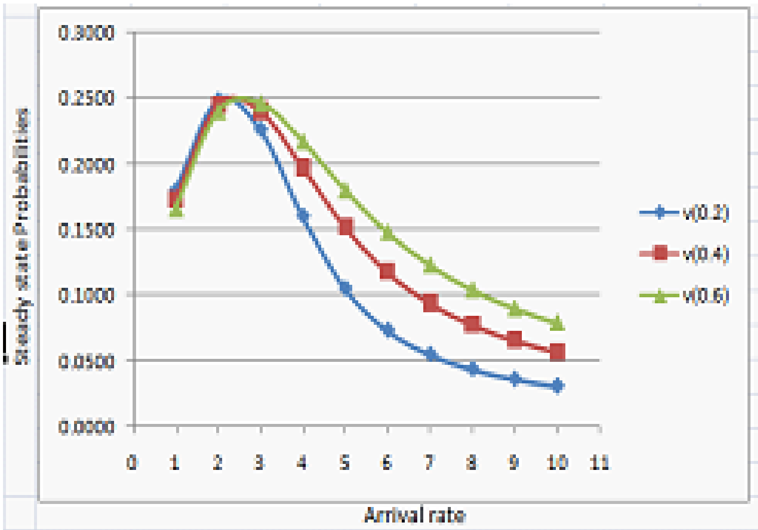
Where $\rho = \frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2(\lambda_t+\nu_t)}$.

4. Numerical Analysis

In this section some numerical analysis along the steady state probabilities with the related graph based on different arrival rate with fixed service rate $\mu_1 = 4$ and 8 and catastrophic effect $\nu_t = 0.2,\ 0.4,\ 0.6$.

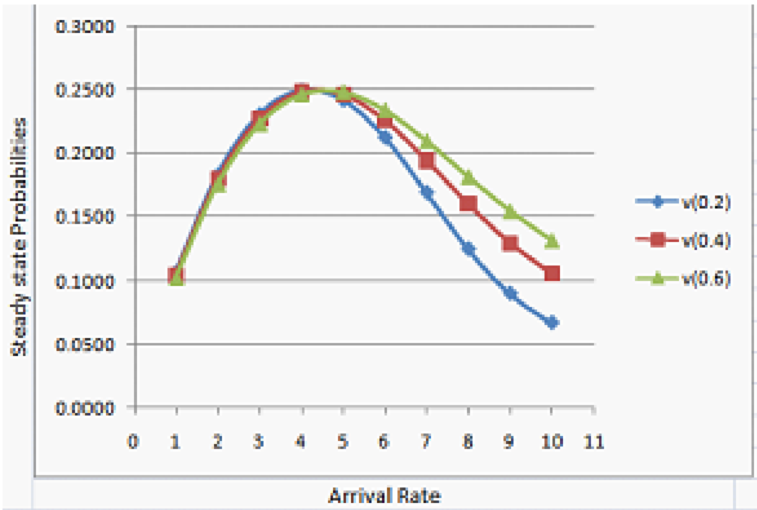
λ	$\nu(0.2)$	$\nu(0.4)$	$\nu(0.6)$
1	0.1796	0.1725	0.1660
2	0.2482	0.2445	0.2400
3	0.2260	0.2400	0.2465
4	0.1600	0.1972	0.2174
5	0.1050	0.1517	0.1802
6	0.0731	0.1174	0.1480
7	0.0549	0.0938	0.1231
8	0.0436	0.0773	0.1043
9	0.0360	0.0654	0.0900
10	0.0306	0.0566	0.0789

Table 1.



λ	$\nu(0.2)$	$\nu(0.4)$	$\nu(0.6)$
1	0.1068	0.1043	0.1020
2	0.1834	0.1796	0.1759
3	0.2306	0.2270	0.2234
4	0.2495	0.2482	0.2465
5	0.2421	0.2463	0.2486
6	0.2126	0.2260	0.2344
7	0.1692	0.1944	0.2100
8	0.1248	0.1600	0.1818
9	0.0900	0.1293	0.1549
10	0.0668	0.1050	0.1316

Table 2.



5. Conclusion

Time dependent solution of M/M/C feedback queue have derived using generating function method. The numerical example illustrate that the steady state probabilities decreased with arrival rate on different catastrophic effect and on different service rate.

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