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Interpretation of Planar Ternary Ring in Desargues Plane Like Usual Ring

Research Article

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Abstract: Use of “ring” in the determine of a planar ternary ring seems unjustified at first sight. In this paper we show that in Desargues affine plane, in certain condition, planar ternary ring (S, t) turns into usually associative ring $(S,\ +,\ \cdot)$. So, in an affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ coordinatized from a coordinative system (O, I, OX, OY, OI) and bijection $\sigma :\ (OI) \to S$, determine a ternary operation $t :\ S^3 \to S$, which we call planar ternary operation. We prove that every coordinatizing affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ determine a planar ternary ring (S, t), where S is the set of coordinates of affine plane and t its planar ternary operation, and vise versa. Also we introduce the binary operation of addition and multiplication in S and underline relations $a + b = t(a,\ 1,\ b)$, $a \cdot b = t(a,\ b,\ 0)$. Since affine plane is fulfilled with the first Desargues axiom D1, related to the structure $(S,\ +,\ \cdot)$ in that plane, considering some isomorphism imply that $(S,\ +)$ is abelian group. In the following, we show that when in an affine plane except first Desargues axiom D1 also hold the second Desargues axiom D2, then its planar ternary ring $(S,\ t)$ is the usual associative ring $(S,\ +,\ \cdot)$.

Keywords: Ternary operation, planar ternary operation, planar ternary ring, coordinatized affine plane, first D1 and the second D2 Desargues axiom.

1. Coordinatization of an Affine Plane

Let $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ be an affine plane [1]. The line $\ell \in \mathcal{L}$ considering like a set of points $P \in \mathcal{P}$, incident with that line we denote with (ℓ). The set of lines passing through the point P denote with (P). So, the set of lines from parallel with the line m denote with K_m. Let we fixed an arbitrary point $O \in \mathcal{P}$ and any three lines $d_1,\ d_2,\ d_3 \in \mathcal{L}$, passing through the point O. Let $X \neq O$, $X \in d_1$, $I \neq O$, $I \in d_2$ and $Y \neq O$, $Y \in d_3$. Line OX is abscissa, OY is ordinate, line OI denote with u we called unite line and a point I we called unite point. Besides the lines OX, OY, the line OI we called also unite line, since point I the unite point (Figure 1). Let S be the set of symbols such that

$$card\ S = card\ (OI) \tag{1}$$

and $\sigma :\ (OI) \to S$ a bijection between the set of points (OI) from the line OI and the set S. We denote with $0 = \sigma(O)$, $1 = \sigma(I)$. For every point $P \in (OI)$ there exist the symbol $p = \sigma(P) \in S$. The point $P(p, p) \in S \times S$.

Definition 1.1. *The coordinate of an ordered pair $(p,\ p) \in S \times S$ we called coordinate of the point P and we denote $P(p,\ p)$. Hence,*

$$\forall\ \ P \in (u),\ \ P(p,\ p) \Leftrightarrow p = \sigma(P) \tag{2}$$

we have $O(0,\ 0)$ and $I(1,\ 1)$.

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Let we denote ℓ_{OX}^{P} a line of $\mathcal{L}$, passing through the point P and parallel with the line OX. For every point $P \notin (u)$ there exists lines $\ell_{OX}^{P} \in K_{OX}$ and $\ell_{OY}^{P} \in K_{OY}$, that intersect the unit line u (Figure 1). Let $\ell_{OY}^{P} \cap OI = A$ and $\ell_{OX}^{P} \cap OI = B$.

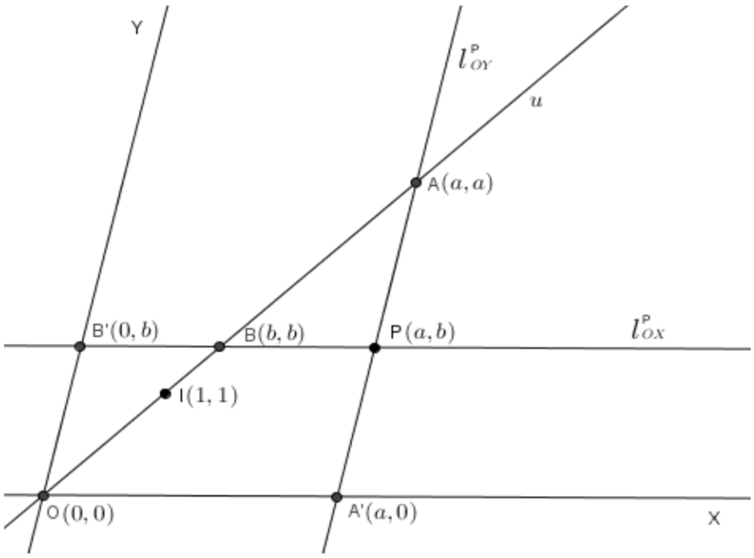

Figure 1.

Definition 1.2. *The coordinate of an ordered pair* $(a,\ b) \in S \times S$*, where* $A(a,\ a)$ *and* $B(b,\ b)$*, we called coordinate of the point P and we denote* $P(a,\ b)$*. Hence,*

$$\forall P \in \mathcal{P} \wedge P \notin (u),\ P(a,\ b) \Leftrightarrow l_{OY}^{P} \cap u = A(a,\ a) \wedge l_{OX}^{P} \cap u = B(b,\ b) \tag{3}$$

Hence we can obtain a function $\mathcal{P} \to S \times S$*, which is bijection.*

Definition 1.3. *(O, I, OX, OY, OI) is called coordinate system of an affine plane A, the point O called origin of the coordinate, the set of the symbols S called the set of coordinate of an affine plane* $\mathcal{A}$*, hence an affine plane* $\mathcal{A}$ *is coordinatized.*

The way of determine the coordinate of an arbitrary point $P \in \mathcal{P}$ (Figure 1) we can take in the form

$$\left.\begin{array}{l} P \in OI \\ p = \sigma(P) \end{array}\right] \Leftrightarrow P(p, p)$$

$$\left.\begin{array}{l} P \notin OI \\ \ell_{OY}^{P} \cap OI = A(a, a) \\ \ell_{OX}^{P} \cap OI = B(b, b) \end{array}\right] \Leftrightarrow P(a, b) \tag{4}$$

Hence, for a point $A' \in OX$ and $A' \neq O$ we have $A'(a,\ 0)$, for $B' \in OY$ and $B' \neq O$ we have $B'(0,\ b)$, while seems every point P of the plane $\mathcal{A}$ is the intersection of the lines ℓ_{OX}^{P} and ℓ_{OY}^{P}, we have

$$P(a,\ b) \Leftrightarrow \left[\begin{array}{l} \ell_{OY}^{P} \cap OX = A'(a, 0) \\ \ell_{OX}^{P} \cap OY = B'(0,\ b) \end{array}\right. \tag{5}$$

The line $\ell^I_{OY} \in K_{OY}$, passing through the unite point I and parallel with the line of ordinate OY, we called skew line (Figure 2). For every line $p \notin K_{OY}$, we denote with M the intersection $\ell^O_p \cap \ell^I_{OY}$.

Let $(m,\ m)$ be the coordinate of intersection $\ell^M_{OX} \cap OI$. Then, its obvious that the coordinate of the point M is $M(1,\ m)$ and of intersection $\ell^M_{OX} \cap OY$ is $(0,\ m)$. Let $(0,\ b)$ the coordinate of the intersection of lines $p \cap OY$.

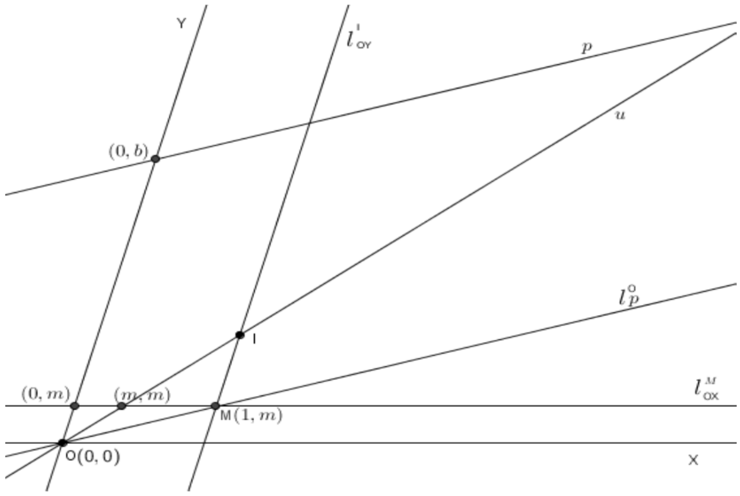

Figure 2.

Definition 1.4. *The symbol $m \in S$ is called skew of a line p, while the symbol $b \in S$ is called ordinate intersection of the line p.*

The ways of determine the skew m and the ordinate intersection of a line $p \notin K_{OY}$ we can take in the form:

$$\left.\begin{array}{l} p \notin K_{OY} \\ M = \ell^O_p \cap \ell^I_{OY} \\ \ell^M_{OX} \cap OI = (m,\ m) \end{array}\right] \Rightarrow m \text{ is the skew of the line p;} \tag{6}$$

$$\left.\begin{array}{l} p \notin K_{OY} \\ p \cap OY = (0,\ b) \end{array}\right] \Rightarrow b \text{ is the ordinate intersection of the line p.} \tag{7}$$

So, to construct the line with the skew m and ordinate intersection b, we denote $p_{m,b}$:

$$\left.\begin{array}{l} (m,\ m) \in OI; \\ M(1,\ m) \in \ell^I_{OY}; \\ OM; \\ B'(0,\ b); \\ r \parallel OM \wedge B' \in r \end{array}\right] \Rightarrow p_{m,b} = r. \tag{8}$$

2. Planar ternary ring in Acoordinatizing Affine Plane

For every triple of symbols $(a,\ m,\ b) \in S^3$, in acoordinatizing affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ we construct the point $A(a,\ a)$, the line p with skew m and ordinate intersection b (Figure 3).

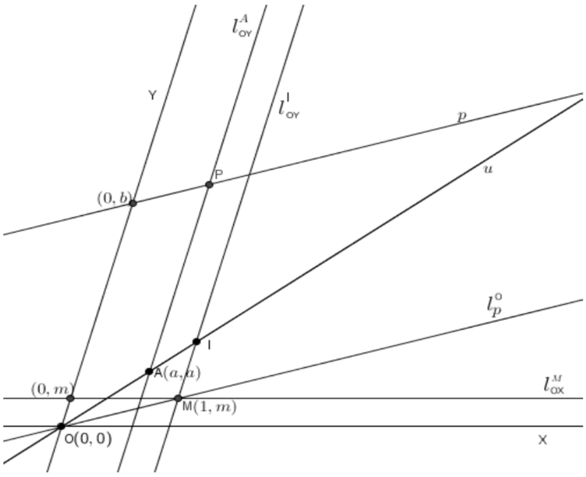

Figure 3.

Since the line $p \notin K_{OY}$, has the skew m, there exists the unique point $P = p \cap \ell^A_{OY}$. Let the symbol $c \in S$ be ordinate of the point P. Triples $(a,\ m,\ b) \in S^3$ we associate the symbol $c \in S$, which is unique like the ordinate of the point P. So, we obtain the ternary operation $t : S^3 \to S$, defined by $t(a,\ m,\ b) = c$, $\forall\ (a,\ m,\ b) \in S^3$.

Definition 2.1. *Ternary operationt : $S^3 \to S$, defined by $(a,\ m,\ b) \mapsto c$, $\forall\ (a,\ m,\ b) \in S^3$, we called planar ternary operation.*

By this definition, for every $(a,\ m,\ b) \in S^3$ we can write:

$$\left.\begin{array}{l} A(a,a); \\ p_{m,b}; \\ ordinate\, of\, p_{m,\ b} \bigcap \ell^A_{OY} = c \end{array}\right] \Leftrightarrow t(a,\ m,\ b) = c \tag{9}$$

By planar ternary operation t in S, for every line $p_{m,b}$ from coordinatizing affine plane $\mathcal{A}$, we can write the equation

$$y = t(x,\ m,\ b) \tag{10}$$

Definition 2.2 ([2])**.** *Let S be a set, that have at least two distinct elements 0 (called zero) and 1 (called unit) and t a ternary operation in S. Planar ternary ring is called ternary structure (S, t), that satisfy the following axioms:*

A1. $\forall\ a,\ m,\ b \in S,\ t(0,\ m,\ b) = t(a,\ 0,\ b) = b.$

A2. $\forall\ a \in S,\ t(a,\ 1,\ 0) = t(1,\ a,\ 0) = a.$

A3. $\forall\ m,\ m',\ b,\ b' \in S,$ *where* $m \neq m',\ \exists\ !x \in S,\ t(x,\ m,\ b) = t(x,\ m',\ b').$

A4. $\forall\ a,\ a',\ b,\ b' \in S,\ a \neq a',\ \exists\ !(x,\ y) \in S^2,\ t(a,\ x,\ y) = b \wedge t(a',\ x,\ y) = b'.$

A5. $\forall\ a,\ m,\ c \in S,\ \exists\ !x \in S,\ t(a,\ m,\ x) = c.$

Theorem 2.3. *If S is the set of coordinates of coordinatized affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ and t is planar ternary operation in S, then (S, t) is planar ternary ring.*

Theorem 2.4. *If (S, t) is a planar ternary ring, then S is the set of coordinates of a coordinatized affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ and t is planar ternary operation in S.*

3. Isomorphism Between Structures ((u), ⊕, *) and (S, +, ·)

(1) **Addition in (u):** Let A(a, a) and B(b, b) be any two point in (u) (Figure 4). Let we denote: $l_1 = \ell_{OX}^{B} \in K_{OX}$; $l_2 = \ell_u^{B'} \in K_u$, where $B'(0,\ b)$; $l_3 = \ell_{OY}^{A} \in K_{OY}$, intersect the line l_2 in a point P; $l_4 = \ell_{OX}^{P} \in K_{OX}$. The least line, intersect a unite line u in a unique point C (Figure 4). By associating a pair $(A,\ B) \in (u) \times (u)$ to the point $C \in (u)$ we obtain a function $\oplus : (u) \times (u) \to (u)$.

Definition 3.1. *The function $\oplus : (u) \times (u) \to (u)$ we called addition in (u), while the image C of the pair (A, B) we denote with $A \oplus B$ and called the sum of points A and B.*

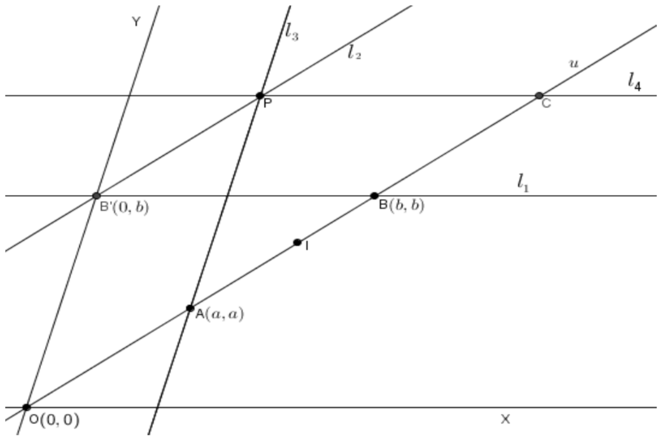

Figure 4.

(2) **Addition in S:** Let a and b be any two symbols in S. From (2), by the bijection $\sigma : (u) \to S$, there exists only one pair $(A,\ B) \in (u) \times (u)$ such that A(a, a) and B(b, b), where $a = \sigma(A)$ and $b = \sigma(B)$. Let (c, c) be the coordinates of the sum $A \oplus B = C$, i.e. $c = \sigma(A \oplus B) \in S$. If we associate in the pair $(a,\ b) \in S \times S$ the unique symbol $c = \sigma(A \oplus B) \in S$ we obtain a function $+ : S \times S \to S$.

Definition 3.2. *The function $+ : S \times S \to S$ we called addition in S, while the image c of the pair (a, b) we denote with $a + b$ and called sum of a and b.*

By this definition we have: $\forall\ (A,\ B) \in (u) \times (u),\ \sigma(A \oplus B) = \sigma(A) + \sigma(B)$.

Theorem 3.3. *In an affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ coordinatized by (O, I, OX, OY, OI) and bijection $\sigma : (u) \to S$, the bijection σ is isomorphism of groupoid $((u),\ \oplus)$ intogroupoid $(S,\ +)$.*

Theorem 3.4. *Addition + in the coordinatizing set S is expressed with planar ternary operation t by equation*

$$a + b = t(a,\ 1,\ b),\ \forall\ (a,\ b) \in S \times S \tag{11}$$

Proof. From Definition 3.2 and 2.1 (Figure 4 and Figure 3), the sum $a + b$ is the ordinate of intersection of line $l_2 = \ell_u^{B'} \in K_u$, passing through $B'(0,\ b)$ and parallel with unit line u, with the line of class K_{OY}, passing through the point $A'(a,\ 0)$, which means that hold equation $a + b = t(a,\ 1,\ b)$. □

In a coordinatizing affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ the line with skew 1 and ordinate intersection b has the equation $y = x + b$.

(3) **Multiplication in (u):** Let be A(a, a) and B(b, b) two arbitrary points of (u) (Figure 5). Denote: $l = \ell^{B}_{OX} \in K_{OX}$, intersect the skew line s in point (1, b); line l_1, passing through the points (1, b) and O(0, 0), with skew line 1 and ordinate intersection 0; $l_2 = \ell^{A}_{OY} \in K_{OY}$ intersect the line l_1 in the point P; $l_3 = \ell^{P}_{OX} \in K_{OX}$. Line l_3 intersect the unite line in a unique point C (Figure 5). By associated the pair $(A,\ B) \in (u) \times (u)$ point $C \in (u)$ we obtain a function $* : (u) \times (u) \to (u)$.

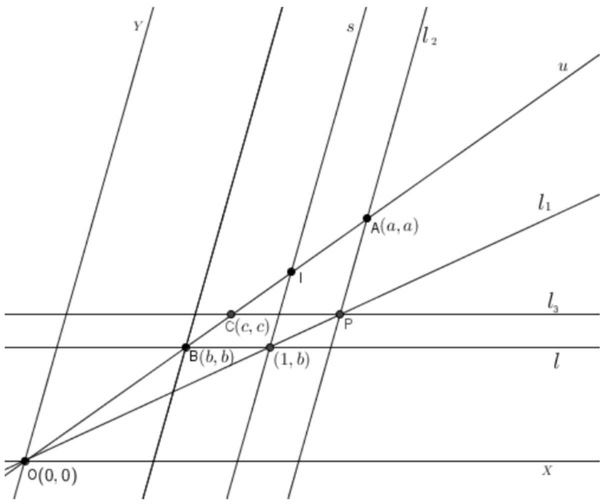

Figure 5.

Definition 3.5. *The function $* : (u) \times (u) \to (u)$ we called multiplication in (u), while the image C of the pair (A, B) we denote with $A * B$ and we called product of the points A and B.*

(4) **Multiplication in S:** For any a, b from S, there exists only one pair of points $(A,\ B) \in (u) \times (u)$ such that $A(a,\ a)$ and $B(b,\ b)$, where $a = \sigma(A)$ and $b = \sigma(B)$. Let (c, c) be the coordinate of the product $A * B = C$, i.e. $c = \sigma(A * B) \in S$. By associated in the pair $(a, b) \in S \times S$ the unique symbol $c = \sigma(A * B) \in S$ we obtain the function $\cdot : S \times S \to S$.

Definition 3.6. *The function $\cdot\ :\ S \times S \to S$ we called multiplication in S, while the image c of the pair (a, b) we denote with $a \cdot b$ and we called product of a and b.*

From this definition we have: $\forall\ (A,\ B) \in (u) \times (u),\ \sigma(A * B) = a \cdot b$, where $a = \sigma(A)$, $b = \sigma(B)$, which implies $\sigma(A * B) = \sigma(A) \cdot \sigma(B)$.

Theorem 3.7. *In an affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$, coordinatized from system (O, I, OX, OY, OI) and bijection $\sigma : (u) \to S$, the bijection σ is isomorphism between groupoids $((u),\ *)$ and $(S,\ \cdot)$.*

Theorem 3.8. *Multiplication $\cdot$ in coordinate set S is expressed with planar ternary operation t from equation:*

$$a \cdot b = t(a,\ b,\ 0),\ \forall\ (a,\ b) \in S \times S \tag{12}$$

From (12), for b=0 we have $a \cdot 0 = t(a,\ 0,\ 0)$, while, since $(S,\ t)$ is planar ternary ring, from axiom A_1 we have $t(a,\ 0,\ 0) = 0$. From(12) and A_1, for a=0 we have $0 \cdot b = t(0,\ b,\ 0) = 0$. Also from (12) and axiom A_2, for $b = 1$ we have $a \cdot 1 = t(a,\ 1,\ 0) = a$, while for $a = 1$ we have $1 \cdot b = t(1,\ b,\ 0) = b$.

In ancoordinatized affine plane $\mathcal{A}$, for any $a \in S$ we have

$$a \cdot 0 = 0 \cdot a = 0 \tag{12'}$$

$$a \cdot 1 = 1 \cdot a = a \tag{12''}$$

From Theorem 3.1, Theorem 3.3 and definition of isomorphism between two structures with binary operation [3], we obtain this:

Theorem 3.9. *There exists isomorphism between structures* $((u), \oplus, *)$ *and* $(S, +, \cdot)$.

4. Planar Ternary Ring Like Usual Ring

Proposition 4.1 (I Desargues axiom [4])**.** *If* AB, $A'B'$, $A''B''$ *are parallel line (Figure 11), then*

$$\left.\begin{array}{l} A\,A' || B\,B', \\ A'A'' || B'B'' \end{array}\right] \Rightarrow AA'' || BB''$$

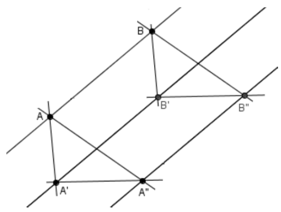

Figure 6.

An affine plane, in which hold the first Desargues axiom (denote with D_1) we called Desargues plane. A vector $\overrightarrow{AB}$ is an ordered pair (A, B) of any two points of $\mathcal{P}$. Hence, if we use axioms A_1, A_2, A_3, D_1, we can conclude that groupoid (V, +) is abelian group, where V is the set of vectors of Desargues plane and addition determined from triangle rule (Figure 7),

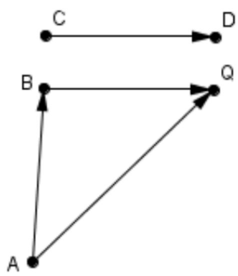

Figure 7.

$$i.e.\ \ \overrightarrow{AB} + \overrightarrow{CD} = \begin{cases} \overrightarrow{AD}, if\ \ B = C \\ \overrightarrow{AQ},\ where\ Q\ \ is\ \ such\ \ that\ \ \overrightarrow{CD} = \overrightarrow{BQ} \\ if\ B \neq C \end{cases}$$

$V_O = \{\overrightarrow{OZ} | Z \in (u)\}$ is the set with direction line of vectors the unite line u and with the same origin O (Figure 8), so $\forall\ \overrightarrow{OA},\ \overrightarrow{OB} \in V_0$ we have

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{BC} + \overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC} \in V_0. \tag{13}$$

Also, $\overrightarrow{OA} + (-\overrightarrow{OB}) \in V_0$. Substructure $(V_0,\ +)$ of the group $(V,\ +)$ is subgroup. In this way easy we can prove this:

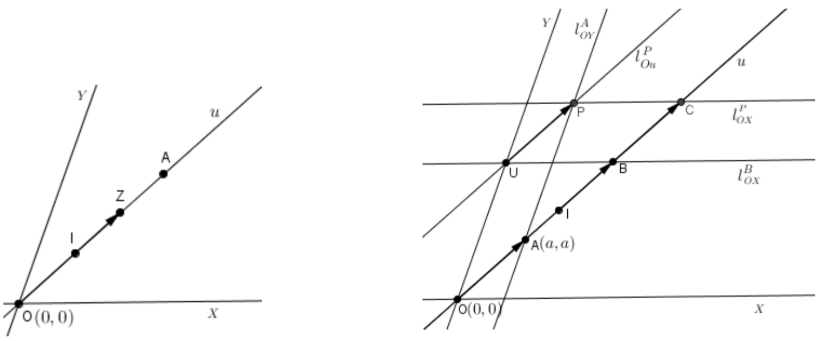

Figure 8.

Proposition 4.2. *Substructure $(V_0,\ +)$ of abelian group $(V,\ +)$, is abelian group.*

Theorem 4.3. *In Desargues plane $\mathcal{D} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$, coordinatized from system (O, I, OX, OY, u), abelian group $(V_O,\ +)$ is isomorph with groupoid $((u),\ \oplus)$.*

Proof. Let us consider $V_O = \{\overrightarrow{OZ} | Z \in (u)\}$. We associate the point $Z \in (u)$ vector $\overrightarrow{OZ} \in V_O$. Hence, is obtained a bijection $\varphi : V_O \to (u)$ (Figure 8) such that

$$\varphi(\overrightarrow{OZ}) = Z \tag{14}$$

By Definition 3.1, we have $A \oplus B = C$, where $C = \ell^P_{OX} \cap u$. Then, by (13) and (14), we have $\varphi(\overrightarrow{OA} + \overrightarrow{OB}) = \varphi(\overrightarrow{OC}) = C = A \oplus B$ and $\varphi(\overrightarrow{OA}) \oplus \varphi(\overrightarrow{OB}) = A \oplus B$ so, we get

$$\varphi(\overrightarrow{OA} + \overrightarrow{OB}) = \varphi(\overrightarrow{OA}) \oplus \varphi(\overrightarrow{OB}) \tag{15}$$

Equation (15) shows that abelian group $(V_O,\ +)$ is isomorph with groupoid $((u),\ \oplus)$ therefore, the second one is abelian group [3]. □

Theorem 4.4. *In Desargues plane $\mathcal{D} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$, coordinatized by system (O, I, OX, OY, u) and bijection $\sigma : (u) \to S$, grupoid (S, +) is abelian group.*

Proof. From Theorem 4.3 we have $(V_O,\ +) \cong ((u),\ \oplus)$, while from Theorem 3.3 we have $((u),\ \oplus) \cong (S,\ +)$. So, it result that $(V_O,\ +) \cong (S,\ +)$ [9]. From Proposition 4.2, $(V_O,\ +)$ is abelian group, therefore (S, +) is abelian group. □

Definition 4.5. *Planar ternary operation $t:\ S^3 \to S$, defined by $(a,\ m,\ b) \to a \cdot m + b$, for every $(a,\ m,\ b) \in S^3$, we called linear ternary operation, while planar ternaryring (S, t) we called linear ternary ring.*

Hence, if $t:\ S^3 \to S$ is linear ternary operation, then $\forall\ (a,\ m,\ b) \in S^3$,

$$t(a,\ m,\ b) = a \cdot m + b \tag{16}$$

Theorem 4.6. *If an coordinatized affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ is Desarguian,then its planar ternary ring $(S,\ t)$ is linear ternary ring.*

Theorem 4.7. *If a planar ternary ring (S, t) of a coordinatized affine plane is linear ternary ring, then that affine plane is Desarguian.*

Theorem 4.8. *In a linear ternary ring (S, t) multiplication is right distributive by multiplication. So, $\forall\ a,\ b,\ p \in S$,*

$$(a+b)p = ap + bp \tag{17}$$

In a linear ternary ring (S, t) in general multiplication is not left distributive by addition. It is such that if Desargues plane is fulfilled with second Desargues axiom which is:

Proposition 4.9 (II Desargues axiom [5]). *Let A, B, C, A', B', C', O be the point of an affine plane $\mathcal{A}$. If triples $(A,\ B,\ C)$, $(A',\ B',\ C')$ are collinear and the lines $AA',\ BB',\ CC'$ passing through the point O (Figure 9), then*

$$\left.\begin{aligned} AB||A'B', \\ BC||B'C' \end{aligned}\right] \Rightarrow AC||A'C'.$$

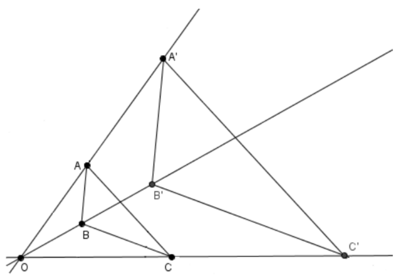

Figure 9.

Theorem 4.10. *If in an coordinatizing affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$ hold Proposition 4.9, then in its planar ternary ring (S, t) for every $a,\ b,\ c \in S$ hold the equation*

$$t(a,\ b,\ ac) = at(1,\ b,\ c). \tag{18}$$

Proof. By considering the axioms of an affine plane and equation $(12')$, we show that hold (18) when at least one of a, b, c are 0.

Let $a,\ b,\ c \neq 0$. We denote with l the line with equation $y = xc$ and with B the point with coordinates $(1,\ c)$. Also, let we denote with C the point of an affine plane with coordinates $(1,\ t(1,\ b,\ c))$ (Figure 10). In these condition the line $l_1 = OC$ has the skew $t(1,\ b,\ c)$, therefore, its equation is $y = xt(1,\ b,\ c)$. Let we denote with C' the point with coordinates $(a,\ at(1,\ b,\ c))$. Let $A(0,\ c)$ and $y = t(x,\ b*,\ c)$ the equation of AC. In fact we have $b* = b$. If $b* \neq b$, from axiom A_3 of planar ternary rings, the equation

$$t(x,\ b*,\ c) = t(x,\ b,\ c) \tag{*}$$

has the one solution. From $t(0,\ b*,\ c) = c = t(0, b, c)$ implies that $x = 0$ is a solution of the equation (*). In the other side, $C(1,\ t(1,\ b,\ c)) \in AC$ with equation $y = t(x,\ b*,\ c)$. Since its coordinate prove this equation we take this equation

$t(1,\ b,\ c) = t(1,\ b*,\ c)$, which shows that also $x = 1$ is the solution of the equation (*). Hence we have a contradiction 1=0. So, the equation of AC is $y = t(x,\ b,\ c)$.

Now we denote with A' the point with coordinate $(0,\ t(1,\ 0,\ c))$ and B' the point of the line l with coordinate $(a,\ ac)$. We can see that the point $O,\ A,\ B,\ C,\ A',\ B',\ C'$ fulfilled the condition of D_2, so we have $AC \parallel A'C'$, so $A'C'$ has the same skew b with the line AC, so the equation of $A'C'$ is $y = t(x,\ b,\ ac)$. Thus, since $C'(a,\ at(1,\ b,\ c)) \in A'C'$ we have $t(a,\ b,\ ac) = at(1,\ b,\ c)$. □

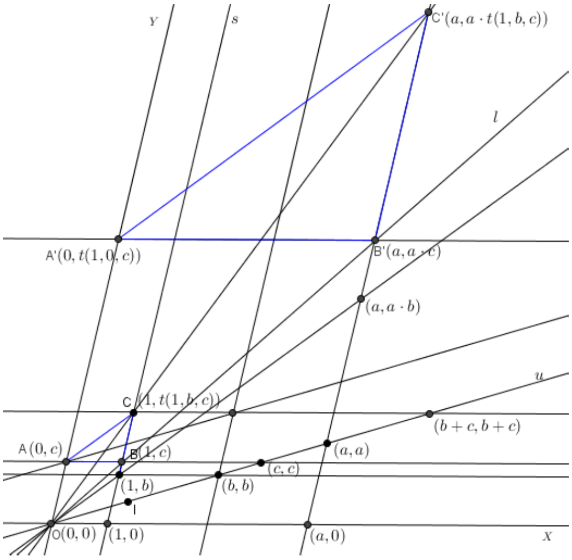

Figure 10.

Theorem 4.11. *In a linear ternary ring (S, t) of an affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$, where hold Proposition 4.9, multiplication is left distributive by addition.*

Proof. Since planar ternary ring (S, t) is linear, from (16) we have $t(1,\ b,\ c) = 1 \cdot b + c$ and $t(a,\ b,\ ac) = ab + ac$. Since also hold D_2, from (18) we have $t(a,\ b,\ ac) = at(1,\ b,\ c)$. This imply $a(b + c) = ab + ac$, because from $(12'')$, $1 \cdot b = b$. □

From Theorem 4.5 and Theorem 4.7 results that multiplication is distributive in linear ternary ring of an affine plane, where both Desargues axioms hold.

Theorem 4.12. *In a planar ternary ring (S, t) of an affine plane $\mathcal{A} = (\mathcal{P},\ \mathcal{L},\ \mathcal{I})$, where hold Proposition D_2, multiplication is associative.*

By Theorem 4.4, which shows that (S, +) is abelian group, Theorem 4.12, which shows that multiplication is associative in S, Theorems 4.8 and 4.11 which shows that multiplication is distributive by addition in S, hence we obtain this:

Theorem 4.13. *Planar ternary ring (S, t) of ancoordinatized affine plane,where hold Proposition D_1 and Proposition D_2, is an usual ring in relation with addition + and multiplication · determined in S like its coordinate set.*

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