International Journal of Mathematics And its Applications
Volume 4, Issue 3–A (2016), 93–97.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

A Fixed Point Theorem for Meir Keeler Type Contraction Using Result of Wong, Chi Song

Research Article

Sujata Goyal[1]*

1 Department of Mathematics, A.S.College, Khanna, India.

Abstract: In this paper, we establish a new fixed point theorem for Meir Keeler type contraction using result of Wong [2]. The presented theorem is an extension of the result of Wong [2].

Keywords: Complete metric space, Fixed point, Meir-Keeler type contraction.

© JS Publication.

1. Introduction

The Banach contraction principle [1] is the most celebrated fixed point theorem. It is very useful, simple and classical tool in nonlinear analysis. Moreover this principle has many generalizations (see for example [2–10]). Among the most relevant results in this direction, one can give that of Meir and Keeler [3] who proved the following fixed point result.

Theorem 1.1. *Let $(X,\ d)$ be a complete metric space and T be a self mapping from X into itself satisfying the following condition:*

$$\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0\ \text{ such that }\ \epsilon \leq d(x,\ y) < \epsilon + \delta(\epsilon) \Rightarrow d(Tx,\ Ty) < \epsilon$$

Then T has a unique fixed point $z \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to z.

Wong [2] proved the following fixed point result:

Theorem 1.2. *Let T be a self mapping on a complete metric space $(X,\ d)$. Suppose there exists symmetric functions α_i, i=1,2,..., 5, of $X \times X$ into [0, 1) such that*

(a). $r = \sup\{\sum_{i=1}^{5} \alpha_i(x,\ y) : x,\ y \in X\} < 1$

(b). for any x, y in X

$$d(Tx,\ Ty) \leq a_1 d(x,\ y) + a_2 d(x,\ Tx) + a_3 d(y,\ Ty) + a_4 d(x,\ Ty) + a_5 d(y,\ Tx),\ \ \text{where}\ \ a_i = \alpha_i(x,\ y)$$

then T has a unique fixed point.

In this paper, a new fixed point theorem of Meir Keeler type that generalizes Theorem 1.2 of Wong, Chi Song in the case when each a_i is assumed to be the constant in [0, 1/5] has been derived.

* *E-mail: sujatagoyal636@yahoo.com*

2. Main Result

Theorem 2.1. *Let $(X,\ d)$ be a complete metric space and T be a mapping of X into itself. We assume that the following hypothesis holds: given $\epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ such that*

$$5\epsilon \leq d(x,\ y) + d(x,\ Tx) + d(y,\ Ty) + d(x,\ Ty) + d(y,\ Tx) < 5\epsilon + \delta(\epsilon) \Rightarrow d(Tx,\ Ty) < \epsilon. \tag{1}$$

Then T has a unique fixed point $z \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to z.

To prove the Theorem we need the following lemma :

Lemma 2.2. *Let $(X,\ d)$ be a complete metric space and T be a mapping of X into itself. Then the conditions (A) & (B) are equivalent :*

(A). Given $\epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ such that $5\epsilon \leq d(x,\ y)+d(x,\ Tx)+d(y,\ Ty)+d(x,\ Ty)+d(y,\ Tx) < 5\epsilon+\delta(\epsilon) \Rightarrow d(Tx,\ Ty) < \epsilon$.

(B). Given $\epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ such that $d(x,\ y) + d(x,\ Tx) + d(y,\ Ty) + d(x,\ Ty) + d(y,\ Tx) < 5\epsilon + \delta(\epsilon) \Rightarrow d(Tx,\ Ty) < \epsilon$.

Proof. Clearly (B) $\Rightarrow$ (A). Now suppose condition (A) holds. For convenience, let $M(x,\ y) = \frac{1}{5}\{d(x,\ y) + d(x,\ Tx) + d(y,\ Ty) + d(x,\ Ty) + d(y,\ Tx)\}$. If $M(x,\ y) < \epsilon$, then by condition (A), we have $d(Tx,\ Ty) \leq M(x,\ y)$ and thus $d(Tx,\ Ty) < \epsilon$. If $M(x,\ y) \geq \epsilon$ then condition (B) immediately holds. □

Proof of Main Theorem. First note that trivially (1) implies

$$d(Tx,\ Ty) < \frac{1}{5}\{d(x,\ y) + d(x,\ Tx) + d(y,\ Ty) + d(x,\ Ty) + d(y,\ Tx) \quad \text{for} \quad x \neq y \tag{2}$$

Let $x_0 \in X$ and consider the sequence $\{x_n\} = \{T^n x_0\}$, $n \in N$, we will prove that $\{x_n\}$ is a Cauchy sequence in X. If there exists $t \in N$ such that $x_t = x_{t+1}$, the clearly x_t is a fixed point of T. So assume that $x_n \neq x_{n+1}$ for all n.
Define $s_n = d(x_n,\ x_{n+1})\ \forall\ n$. By (2), we have

$$\begin{aligned} s_n = d(x_n,\ x_{n+1}) = d(Tx_{n-1},\ Tx_n) &< \frac{1}{5}\{d(x_{n-1},\ x_n) + d(x_{n-1},\ x_n) + d(x_n,\ x_{n+1}) + d(x_{n-1},\ x_{n+1}) + d(x_n,\ x_n)\} \\ &\leq \frac{1}{5}\{d(x_{n-1},\ x_n) + d(x_{n-1},\ x_n) + d(x_n,\ x_{n+1}) + d(x_{n-1},\ x_n) + d(x_n,\ x_{n+1})\} \\ &= \frac{3}{5}s_{n-1} + \frac{2}{5}s_n \end{aligned}$$

$\Rightarrow s_n < s_{n-1}\ \forall\ n$. Thus the sequence $\{s_n\}$ is strictly decreasing. Therefore the sequence $\{s_n\}$ is convergent. Let it converge to s, where $s \geq 0$. If possible, let $s > 0$. Now $2s_n + s_{n-1} \to 5s$ as $n \to \infty$. According to condition (1), for $5s > 0$, there exists $\delta(5s) > 0$ with the given property. Now for $\delta(5s) > 0$, there exists M such that $5s \leq 2s_M + 3s_{M-1} < 5s + \delta(5s)$

$$\begin{aligned} &\Rightarrow 2d(x_M,\ x_{M+1}) + 3d(x_{M-1},\ x_M) < 5s + \delta(5s) \\ &\Rightarrow d(x_{M-1},\ x_M) + d(x_{M-1},\ Tx_{M-1}) + d(x_M,\ Tx_M) + d(x_{M-1},\ Tx_M) + d(x_M,\ Tx_{M-1}) \\ &= d(x_{M-1},\ x_M) + d(x_{M-1},\ x_M) + d(x_M,\ x_{M+1}) + d(x_{M-1},\ x_{M+1}) + d(x_M,\ x_M) \\ &\leq 2d(x_{M-1},\ x_M) + d(x_M,\ x_{M+1}) + d(x_{M-1},\ x_M) + d(x_M,\ x_{M+1}) \\ &= 2d(x_M,\ x_{M+1}) + 3d(x_{M-1},\ x_M) \\ &< 5s + \delta(5s) \end{aligned}$$

thus by Lemma 2.2, $d(Tx_{M-1},\ Tx_M) < s \Rightarrow d(x_M,\ x_{M+1}) < s \Rightarrow s_M < s$, which is a contradiction. Thus we deduce that $s_n \to 0$ as $n \to \infty$. Now, let $\delta^1(\epsilon) = \min\{\delta(\epsilon),\ \epsilon\}$. By the convergence of the sequence $\{s_n\}$ to 0, there exists $L \in N$ such that

$$d(x_n,\ x_{n+1}) < \frac{\delta^1(\epsilon)}{10}\ \ \forall\ \ n \geq L \tag{3}$$

Define $\Omega = \{x_p : p \geq L,\ d(x_p,\ x_L) < +\frac{\delta^1(\epsilon)}{5}\}$, we will show that

$$T(\Omega) \subset \Omega \tag{4}$$

let $u \in \Omega$, then there exists $p \geq L$ such that $u = x_p$ and $d(x_p,\ x_L) < \frac{5\epsilon}{3} + \frac{\delta^1(\epsilon)}{5}$. If $p = L$, then clearly $T(u) = x_{L+1} \in \Omega$ by (3). So assume that $p > L$. Now we have two cases:

Case I:

$$d(x_p,\ x_L) < \frac{5\epsilon}{3} + \frac{\delta^1(\epsilon)}{5} \tag{5}$$

We shall show that

$$\epsilon \leq \frac{1}{5}\{d(x_p,\ x_L) + d(x,\ x_{p+1}) + d(x_L,\ x_{L+1}) + d(x_p,\ x_{L+1}) + d(x_{p+1},\ x_L)\} < \epsilon + \frac{\delta^1(\epsilon)}{5} \tag{6}$$

From (3) & (5), we have

$$\begin{aligned}
\epsilon \leq \frac{3}{5}d(x_p,\ x_L) &\leq \frac{3}{5}d(x_p,\ x_L) + \frac{2}{5}d(x_{p+1},\ x_p) + \frac{2}{5}d(x_{L+1},\ x_L)\\
&< \frac{3}{5}\left[\frac{5\epsilon}{3} + \frac{\delta^1(\epsilon)}{5}\right] + \frac{4}{5}\left[\frac{\delta^1(\epsilon)}{10}\right]\\
&= \epsilon + \frac{\delta^1(\epsilon)}{5}\\
&\Rightarrow \epsilon \leq \frac{3}{5}d(x_p,\ x_L)
\end{aligned} \tag{7}$$

and

$$\frac{3}{5}d(x_p,\ x_L) + \frac{2}{5}d(x_{p+1},\ x_p) + \frac{2}{5}d(x_{L+1},\ x_L) < \epsilon + \frac{\delta^1(\epsilon)}{5} \tag{8}$$

From (7), we have,

$$\epsilon + \frac{\delta^1(\epsilon)}{5} \leq \frac{1}{5}d(x_p,\ x_L) + \frac{1}{5}d(x_p,\ x_{p+1}) + \frac{1}{5}d(x_{p+1},\ x_L) + \frac{1}{5}d(x_p,\ x_{L+1}) + \frac{1}{5}d(x_{L+1},\ x_L) \tag{9}$$

From (8), we have,

$$\begin{aligned}
&\frac{1}{5}\{d(x_p,\ x_L) + d(x_p,\ x_{p+1}) + d(x_{p+1},\ x_L) + d(x_p,\ x_{L+1}) + d(x_{L+1},\ x_L)\}\\
&\leq \frac{1}{5}\{d(x_p,\ x_L) + d(x_p,\ x_{p+1}) + d(x_{p+1},\ x_p) + d(x_p,\ x_L) + d(x_p,\ x_L) + d(x_L,\ x_{L+1}) + d(x_{L+1},\ x_L)\}\\
&= \frac{3}{5}d(x_p,\ x_L) + \frac{2}{5}d(x_{p+1},\ x_p) + \frac{2}{5}d(x_{L+1},\ x_L)\\
&< \epsilon + \frac{\delta^1(\epsilon)}{5}
\end{aligned} \tag{10}$$

Combining (9) & (10), (6) is established. From (1), (3) & (6), we have

$$d(Tx_p,\ x_L) \leq d(Tx_p,\ Tx_L) + d(Tx_L,\ x_L) < \epsilon + \frac{\delta^1(\epsilon)}{10} < \frac{5\epsilon}{3} + \frac{\delta^1(\epsilon)}{5} \Rightarrow Tx_p \in \Omega \Rightarrow Tu \in \Omega$$

Case II: $d(x_p,\ x_L) < \frac{5\epsilon}{3}$

$$\begin{aligned} d(Tx_p,\ x_L) &\leq d(Tx_p,\ Tx_L) + d(Tx_L,\ x_L) \\ &< \frac{1}{5}d(x_p,\ x_L) + \frac{1}{5}d(x_{p+1},\ x_p) + \frac{1}{5}d(x_L,\ x_{L+1}) + \frac{1}{5}d(x_{L+1},\ x_p) + \frac{1}{5}d(x_L,\ x_{p+1}) + \frac{1}{5}d(x_{L+1},\ x_L) \\ &= \frac{3}{5}d(x_p,\ x_L) + \frac{2}{5}d(x_{p+1},\ x_p) + \frac{2}{5}d(x_{L+1},\ x_L) \end{aligned}$$

Thus $d(Tx_p,\ x_L) \leq \frac{3}{5}(\frac{5\epsilon}{3}) + \frac{2}{5}(\frac{\delta^1(\epsilon)}{10}) + \frac{3}{5}(\frac{\delta^1(\epsilon)}{10}) = \epsilon + \frac{\delta^1(\epsilon)}{10} \Rightarrow Tx_p \in \Omega \Rightarrow Tu \in \Omega$. Thus (4) is proved. Now from (4), we have,

$$\begin{aligned} &d(x_m,\ x_L) < \frac{5}{3}\epsilon + \frac{\delta^1(\epsilon)}{5} \quad \forall\, m \geq L \\ &\Rightarrow d(x_m,\ x_n) \leq d(x_m,\ x_L) + d(x_L,\ x_n) < \frac{10}{6}\epsilon + \frac{2}{5}\delta^1(\epsilon) \quad \forall\, m, n \geq L \\ &\Rightarrow d(x_m,\ x_n) < \frac{10}{6}\epsilon + \frac{2}{5}\delta^1(\epsilon) < 2\epsilon + \frac{2}{5}\epsilon < 3\epsilon \quad \forall\, m, n \geq L \end{aligned}$$

Thus the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, the sequence $x_n \to z$ (say). Now

$$\begin{aligned} d(Tz,\ z) \leq d(Tz,\ Tx_n) + d(Tx_n,\ z) &< \frac{1}{5}\{d(z,\ x_n) + d(z,\ Tz) + d(x_n,\ x_{n+1}) + d(z,\ Tx_n) + d(x_n,\ Tz)\} + d(x_{n+1},\ z) \\ \Rightarrow \frac{4}{5}d(Tz,\ z) &< \frac{1}{5}d(z,\ x_n) + \frac{1}{5}d(x_n,\ x_{n+1}) + \frac{1}{5}d(x_n,\ Tz) + \frac{6}{5}d(x_{n+1},\ z) \end{aligned}$$

Letting $n \to \infty$, we get:

$$\frac{4}{5}d(Tz,\ z) \leq \frac{1}{5}d(Tz,\ z) \Rightarrow d(Tz,\ z) \leq 0 \Rightarrow Tz = z.$$

Uniqueness : Let w be another fixed point of T, then $Tz = z$ & $Tw = w$, $z \neq w$. Now

$$\begin{aligned} d(z,\ w) = d(Tz,\ Tw) &< \frac{1}{5}\{d(z,\ w) + d(z,\ Tz) + d(w,\ Tw) + d(z,\ Tw) + d(w,\ Tz)\} \\ \Rightarrow d(z,\ w) &< \frac{3}{5}d(z,\ w) \Rightarrow 1 < \frac{3}{5}, \end{aligned}$$

a contradiction. Thus the uniqueness of fixed point is proved, and it completes the proof of the theorem. □

From the main theorem, we immediately get the following corollary which extends the known result of Wong [2] for the case when each a_i is assumed to be the constant in [0, 1/5]

Corollary 2.3 ([2])**.** *Let T be a self mapping on a complete metric space $(X,\ d)$. Suppose there exist a constant k such that for any x, y in X,*

$$d(Tx,\ Ty) \leq k\{d(x,\ y) + d(x,\ Tx) + d(y,\ Ty) + d(x,\ Ty) + d(y,\ Tx)\}$$

where $k \in [0,\ 1/5]$ then T has a unique fixed point $z \in X$. Also the sequence $\{T^n x\}$ converges to z for every $x \in X$.

References

[1] S.Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations integrals*, Fundamenta Mathematicae, 3(1)(1922), 133-181.

[2] C.S.Wong, *Fixed point theorems for generalized nonexpansive mappings*, Journal of the Australian Mathematical Society, 18(3)(1974), 265-276.

[3] A.Meir and E.Keeler, *A theorem on contraction mappings*, Journal of Mathematical Analysis and Applications, 28(2)(1969), 326-329.

[4] M.U.Ali, T.Kamran and E.Karapınar, *Fixed point of α-ψ-contractive type mappings in uniform spaces*, Fixed Point Theory and Applications, 2014(1)(2014), 1-12.

[5] V.Berinde, *Iterative approximation of fixed points*, Vol. 1912, Berlin, Germany: Springer, (2007).

[6] A.Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, International Journal of Mathematics and Mathematical Sciences, 29(9)(2002), 531-536.

[7] A.Latif, M.E.Gordji, E.Karapınar and W.Sintunavarat, *Fixed point results for generalized-meir-keeler contractive mappings and applications*, Journal of Inequalities and Applications, 2014(1)(2014), 1-11.

[8] L.B.Ćirić, *A generalization of Banach's contraction principle*, Proceedings of the American Mathematical Society, 45(2)(1974), 267-273.

[9] M.Edelstein, *An Extension of Banach's Contraction Principle*, Proceedings of the American Mathematical Society, 12(1)(1961), 7-10.

[10] R.Kannan, *Some Results on Fixed Points-II*, The American Mathematical Monthly, 76(4)(1969), 405-408.