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Deriving And Deducing The Equation Of The Curve Of Quickest Descent

Research Article

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Abstract: In Mathematics, we repeatedly come across the concept that a straight line is the shortest distance between two points. However hardly we find discussions on the curve that takes minimum time. My essay consider this issue not limited to 2-Dimensional but extended to spheres and other complex geometrical shapes and explores the research question How do we find the equation to the curve of quickest descent between two places using calculus and justify the same comparatively with different plane curves? The investigation aims to derive the equation of the curve of quickest descent. Since time is a crucial element in everyones life, achieving the aim can expunge some of the biggest problems.
The extended essay explores the research subject step wise with consideration of certain assumptions, however most of the paramount factors are included in the essay. It investigates different curves such as straight line, parabolic arc, elliptical arc and cycloid on 2D plane taking the example of a ball (negligible mass) allowed to travel along slides of different shapes. Then the result of part 1 is applied to a real life situation where it is proposed that if a road or an underground tunnel were built between the cities of Bhubaneswar, Orissa, India and New Delhi, India, which shape should it take to optimize travel time. Second part of essay examines a larger scope of problem. I employ important identities and formulae such as Beltrami Identity and Great Circle Distance Formula as well as calculus of Variations to derive the equations of the curves especially cycloid and hypocycloid.
The essay concludes in the first part that given a small-scale example, the curve of quickest descent is a cycloid. With this result, it further goes on to conclude that Hypocycloid is the Brachistochrone between Bhubaneswar and New Delhi.

Keywords: Brachistochrone, curve of quickest descent, hypocycloid.

1. Introduction

My extended essay investigates on the research question "**How do we find the equation to the curve of quickest descent between two places using calculus and justify the same comparatively with different plane curves?**" I aim to use calculus and different sub topics under Mathematics to derive the equation of the curve of quickest descent and calculate the total time of travel. Directly viewing this problem in real life scale complicates this problem. On account of which, I will first analyze a ball allowed to travel along slides that take on different shapes. This will give me a foundation to investigate further the application of the curve of quickest descent in real life.

With growing development of the world, cities are growing larger, small towns are expanding over large area. Contrarily at the same time rural areas are not able to receive sufficient support from the developed regions due to communication problems. Furthermore the lack of communication between regions is disrupting the growth of villagers who are not able to travel to cities and developed town because of long distances. Also those living in cities spend most of their time in cars, buses or trams on account of long transit time between different parts of the city.

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This topic is of personal interest to me. My curiosity in discovering this solution was the main motivation behind this research. Since a very young age I have been travelling around different parts of the world by train, car or flight. The journeys by car and train were very hectic. Once it took around 14 hours to travel from Zenica, Bosnia to Rome, Italy! On another case, one of my friend's relative died on the way to hospital because he could not reach on time. To save people's lives and make their journeys comfortable, I decided to investigate on optimization of time of travel.

I will take two real life examples; one is that of ball allowed to travel along slides of different shapes and the other of investigating the route between two cities that will minimize transit time. I have deliberately chosen to examine the two real life of different scope to conclude with a realistic solution. Hence when I studied calculus, I began modeling this problem in such a way so that I can bring about a solution mathematically that can solve one of the most crucial global issues in today's era.

Through this investigation, I will get dual benefit; one is expanding my knowledge on application of calculus in real life and the other is of providing a solution to one of the most pressing global issues.

1.1. Investigation

I shall begin my investigation with a small-scale problem by examining it on a 2D plane. First let us imagine that a small ball is allowed to travel along slides that take different shapes. For the benefit of my investigation, I will consider a straight-line, parabolic arc, elliptical arc and cycloid shaped slides. Let us assume that the starting and the ending points are coordinates on a Cartesian plane and carry out the mathematical calculation to conclude with a viable answer.

For each shape assumed by the slide, I will derive the equation and calculate the time. We must understand that a few assumptions have been considered for the investigation.

- The ball is assumed to be of negligible mass and
- The only force acting on it is gravity.

2. Part 1: Finding out the Curve which Takes Shortest Time Using Cartesian Plane

Mathematically it has been proven that the shortest distance between two points is a straight line. However, is it also the path that takes shortest time for a particle (acted only by gravity) to travel from one point to another? Or there exists a curve, which takes minimum time for the same particle to travel between the two points. In all cases g is assumed to be 10 m s^{-2}. With respect to the situation presented before, the two points on the Cartesian plane (the two end points of the slide) are A (0, 0) and B (6.283, 4).

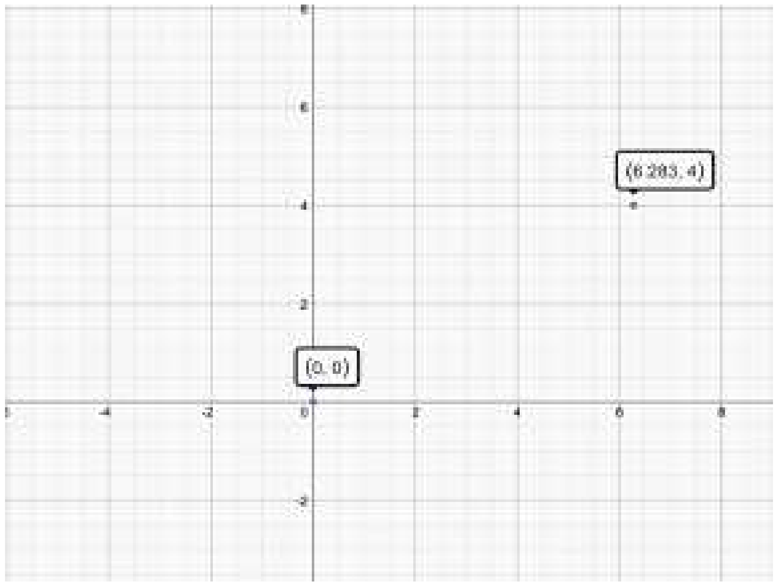

Figure 1. **Representing two end points on the Cartesian plane**

According to Mechanics in Physics, **Time taken = distance / speed**. Here distance refers to the length of the curve. The equation for the length of the curve is an integral of the small distances the curve is made up of. Each of the small infinitesimal lengths measures ds. ds can be measured in terms of dx and dy as illustrated in the following graph:

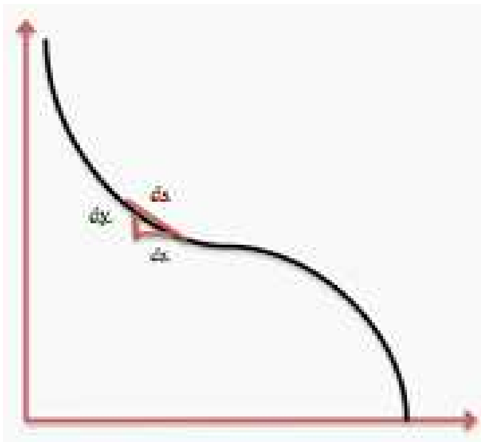

Figure 2.

Examining Figure 2, "ds" is also the hypotenuse of a right-angled triangle with the other two sides measuring dx and dy. Applying Pythagoras theorem:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$ds = \sqrt{(dx)^2 \times \left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\int ds = \int_{x1}^{x2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_{x1}^{x2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{1}$$

Not all functions can be represented in the form of $y = f(x)$. A few curves are defined using parametric equations. In that case, the parametric form of Equation (1) is used.

$$S = \int_{t1}^{t2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{2}$$

Equation (1) and (2) are integrals of a functional i.e. function of a function. The length of any curve with a function $f(x)$ or parametric equations of $x(t)$ and $y(t)$ can be deduced by applying Equation (1) and Equation (2) derived above.

Now let us apply Equation (1) in case of a straight line that passes through coordinates (0, 0) and (6.283, 4).

$$\text{Total Distance travelled} \;\; S = \int_{0}^{x(t2)} \sqrt{1 + (f'(u))^2} \tag{3}$$

Where $u = x(t)$, the function of the graph of straight line. In order to calculate time we must derive the equation $\mathbf{v}$ for velocity. Velocity is displacement per unit time or the rate of change of displacement that can be expressed as

$$v = \frac{ds}{dt} = s'(t)$$

To derive the equation for $\mathbf{v}$, I need to apply **The Fundamental Theorem of Calculus** to Equation (3). **The Fundamental Theorem of Calculus** is a theorem that links the concept of the derivative of a function with the concept of the function's integral. It states:

$$\frac{d}{dx}\left[\int_{a}^{x} f(y)\, dy\right] = f(x)$$

Applying the Fundamental Theorem of calculus and Chain Rule on Equation (3):

$$v(t) = \sqrt{1 + (f'(x(t)))^2} x'(t) \tag{4}$$

As it has been assumed that the particle is only being acted upon by gravity, it is also implied, no friction or external forces are acting on it. From the conditions of conservation of energy in Physics,

$$v = \sqrt{2gf(x)} \, (h = f(x)) \tag{5}$$

Now we can substitute Equation (5) in Equation (4) to solve the differential equation and derive the equation for time.

$$\sqrt{2gf(x)} = \sqrt{1 + (f'(x(t))^2} x'(t)$$

Solving the Differential equation by separating the variables:

$$dt = \frac{\sqrt{1 + (f'(x(t))^2}}{\sqrt{2gf(x)}} dx$$

dt is the time taken by the ball to travel the small distance of ds. The total time taken is the integral of the equation dt,

$$T = \int dt = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1 + (f'(x(t))^2}{f(x)}} dx$$

$$\text{Total time} = \int_{0}^{t1} dt = \frac{1}{\sqrt{2g}} \int_{0}^{y1} \sqrt{\frac{1 + (f'(x(t))^2}{f(x)}} dx \tag{6}$$

We can calculate the time taken along any curve by substituting $f(x)$ and $f'(x)$ in the equation for Time T.

2.1. Straight Line

Let us now find T in case of a straight line. The general equation of a straight-line graph is $y = mx + c$. The Gradient of the straight-line graph between Point A(0, 0) and Point B (6.283, 4) is 0.636 and it does not have any x-intercept or y-intercept (Evident from the graph below).

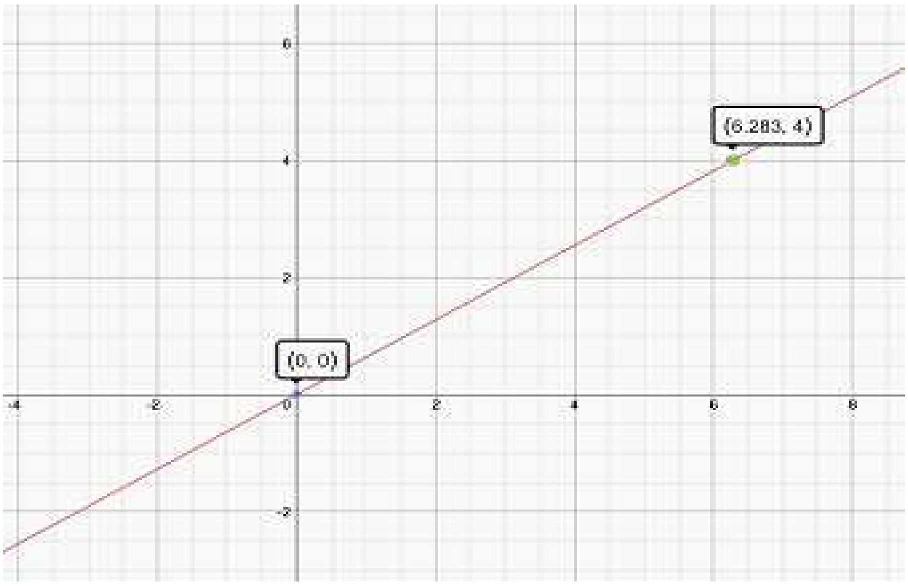

Figure 3. **Graph of the straight line between (0, 0) and (6.283, 4)**

Therefore the Equation of the straight line is:

$$y = 0.636x$$

Derivative of function y:

$$y' = 0.636$$

Solving the equation for Time:

$$T = \int_0^{6.283} \sqrt{\frac{1 + (f'(x(t))^2}{f(x)}} dx = \int_0^{6.283} \frac{1}{\sqrt{2g}} \sqrt{\frac{1 + (0.636)^2}{0.636x}} = 1.665 \text{ seconds}$$

Along a straight-line shaped slide, a ball will take 1.665 seconds to traverse the distance between the endpoints.

2.2. Parabolic Arc

Now let us consider a parabolic arc. To derive the equation of a parabolic arc, we need to take three points on a possible parabolic arc passing through both Points A and B. Let the third point be (4, 2) which is on one of the many possible parabolic arcs between the endpoints. The Three points on the parabolic arc are Point A (0, 0), Point B (6.283, 4) and Point C (4, 2). I shall consider the general equation of the parabolic arc, substitute the coordinates of the three points in three different equations and solve the three equations simultaneously for the 3 unknowns to derive the equation of the parabolic Arc. The general equation for a parabola is:

$$y = ax^2 + bx + c$$

Substituting the three points in the general equation, Point A(0, 0)

$$0 = 0 \times a + 0 \times b + c$$

Point B(6.283, 4)

$$4 = 39.476a + 6.283b + c$$

Point C(4, 2)

$$2 = \ 16a + 4b + c$$

Solving the three equations simultaneously to get:

$$a = 0.05985043735 \approx 0.06$$

$$b = 0.2605982506 \approx 0.26$$

$$c = 0$$

Substituting the values of a, b and c in the general equation to get:

$$y = 0.06x^2 + 0.260x \tag{7}$$

This is the Equation of the required parabolic arc.

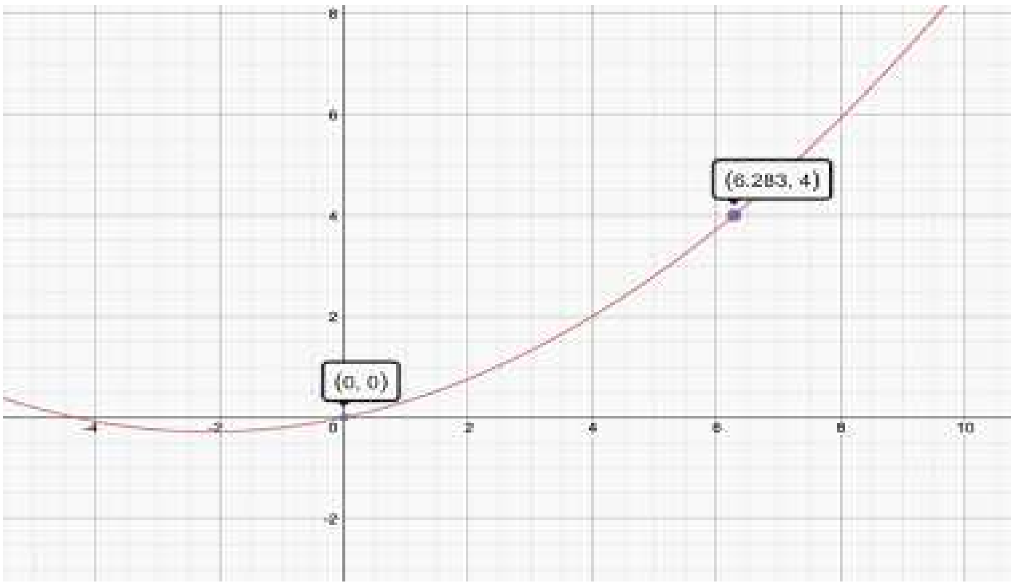

Figure 4. **Graph of Parabolic arc** $y = 0.06x^2 + 0.260x$

As noted before, the equation of time requires the derivative of the function too. Therefore the derivative of the parabolic function is:

$$y' = 0.12x + 0.260 \tag{8}$$

Substituting Equation (7) and (8) in Equation (6) (g taken as $10ms^{-2}$),

$$T = \frac{1}{\sqrt{2g}} \int_0^x \sqrt{\frac{1 + (f'(x))^2}{f(x)}} dx$$

$$T = \frac{1}{\sqrt{2g}} \int_0^{6.283} \sqrt{\frac{1 + (0.12x + 0.260)^2}{0.06x^2 + 0.260x}} dx$$

$$T = \frac{1}{\sqrt{2g}} \times 9.36$$

$$T = 2.09 \quad \text{seconds.}$$

Hence, along a parabolic arc shaped slide the ball will take 2.09 seconds to travel from Point B to Point A. With this calculation, we cannot conclude that the straight line is the curve of quickest descent. For accuracy of the result, in addition I will be investigating 2 different kinds of curves: an elliptical arc and a cycloid.

2.3. Elliptical Arc

Let us consider that Point A and Point B are points that lie on an elliptical arc. I will be using the Desmos Graphing software to fit an elliptical arc between both the points. The general equation of an elliptical arc is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{9}$$

Where (h, k) is the center of the ellipse and (x, y) is a point on the circumference of the ellipse. By fitting an ellipse curve using the Desmos graphing software I deduced the following values of the variables of the equation represented in the following figure:

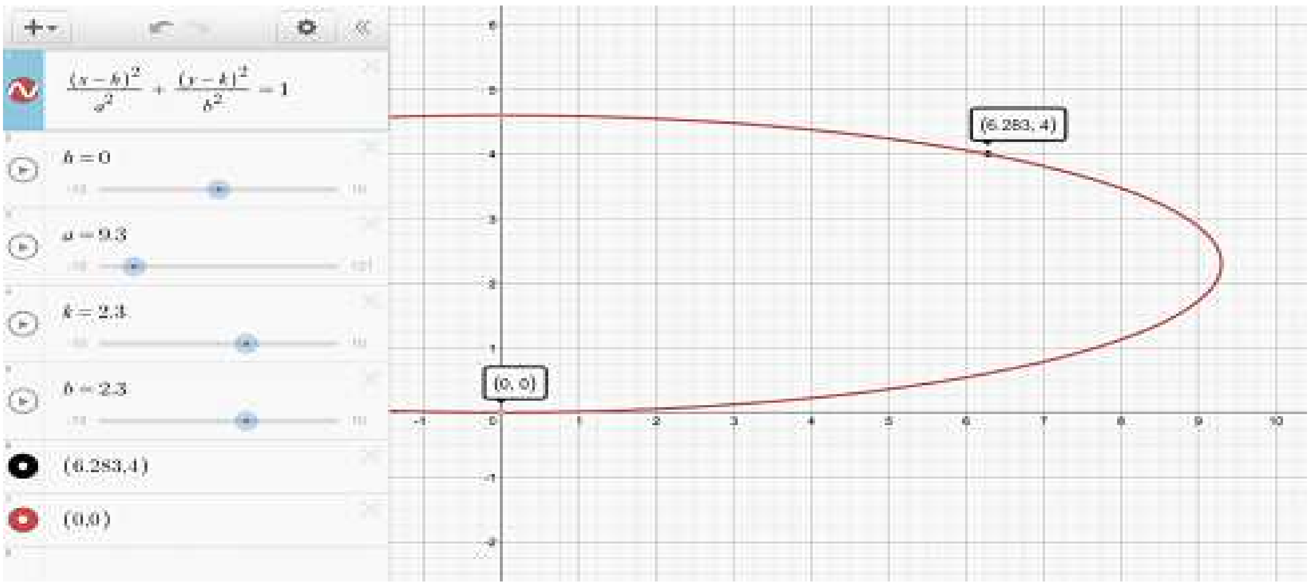

Figure 5. **Modeling an Elliptical Arc**

$$h = 0, \ k = 2.3, \ a = \ 9.3, \ b = 2.3$$

Substituting the values in Equation (9)

$$\frac{(x-0)^2}{9.3^2} + \frac{(y-2.3)^2}{2.3^2} = 1 \tag{10}$$

Rearranging the Equation (10) to express in terms of y

$$\begin{aligned}
\frac{(y-2.3)^2}{2.3^2} &= 1-\frac{(x-0)^2}{9.3^2} \\
\frac{(y-2.3)^2}{5.29} &= \frac{(1\times 86.49)-(x-0)^2}{86.49} \\
(y-2.3)^2 &= \frac{(1\times 86.49)-(x-0)^2}{86.49}\times 5.29 \\
(y-2.3)^2 &= ((86.49)-(x-0)^2)\times 0.061 \\
y-2.3 &= \sqrt{((86.49)-(x-0)^2)\times 0.06} \\
y &= \pm\sqrt{((86.49)-(x-0)^2)\times 0.06}+2.3 \qquad (11)
\end{aligned}$$

Since both the points lie above the x-axis, I will be considering:

$$y= \ +\sqrt{(86.49)-(x-0)^2)\times 0.06} \ +2.3$$

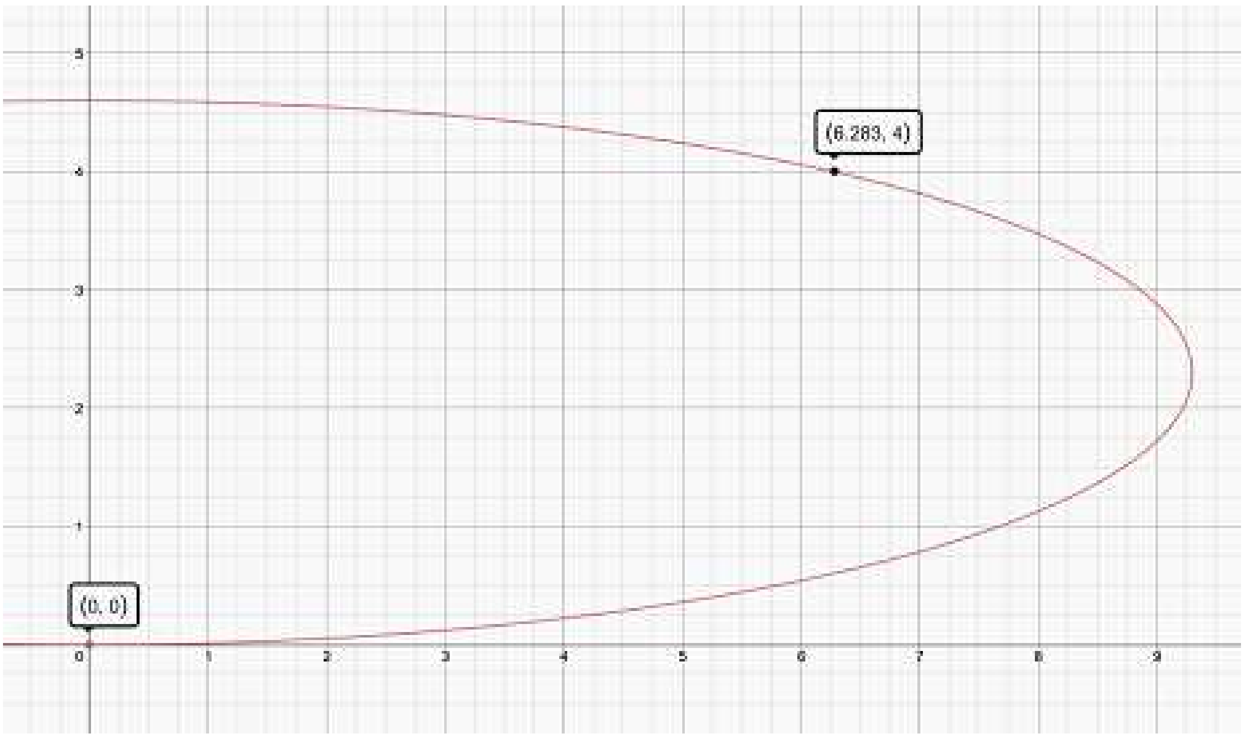

Figure 6. **Elliptical orbit between Point A and Point B**

Let us calculate the derivative of y

$$y' = \frac{2\times 0.06(-(x-0))}{2\left(\sqrt{0.061\times(86.49-\ (x-0)^2)}\right)} \qquad (12)$$

Cancelling 2 from both Numerator and Denominator to get

$$y' = \frac{0.502\times(-x+0)}{\left(\sqrt{0.061\times(86.64-\ (x-0)^2)}\right)}$$

Substituting Equation (11) and (12) in Equation (6) and calculating time around the belt. I will be calculating the time as sum of the time taken along the arc from (0, 0) to (9.29, 2.193) and from (9.29, 2.193) to (6.283, 4) to as shown in Figure 5.

$$T1 = \frac{1}{\sqrt{2g}} \int_0^{9.29} \sqrt{\frac{1+\left(\frac{0.06\times(-x+0)}{\left(\sqrt{0.06\times(86.49-\ (x-0)^2)}\right)}\right)^2}{(+\sqrt{0.06\times\left(86.49-\ (x-0)^2\right)}+\ 2.3)}}dx = 1.107\ seconds$$

Since the arc from(9.29, 2.193) to (6.283, 4) is in the upper half of the ellipse, I will

$$T2 = \left| \frac{1}{\sqrt{2g}} \int_{6.283}^{9.29} \sqrt{\frac{1+\left(\frac{0.06\times(-x+0)}{\left(\sqrt{0.06\times(86.49-\ (x-0)^2)}\right)}\right)^2}{(+\sqrt{0.06\times\left(86.49-\ (x-0)^2\right)}+\ 2.3)}}dx \right| = 0.4332\ seconds$$

$$T\,(total\ time) = T1 + T2 = 1.532\ seconds$$

The ball will take 1.532 seconds to travel from Point B to Point A along the elliptical arc shaped slide under the influence of gravity.

2.4. Cycloid

Till now, we examined parabolic arc, straight line, elliptical arc but now let's consider a cycloid. A Cycloid falls under a different category of curves. It is a curve traced out by a fixed point on the circumference of a rotating circle. However we should keep it in mind that a cycloid is not an elliptical arc or even a type of elliptical arc.

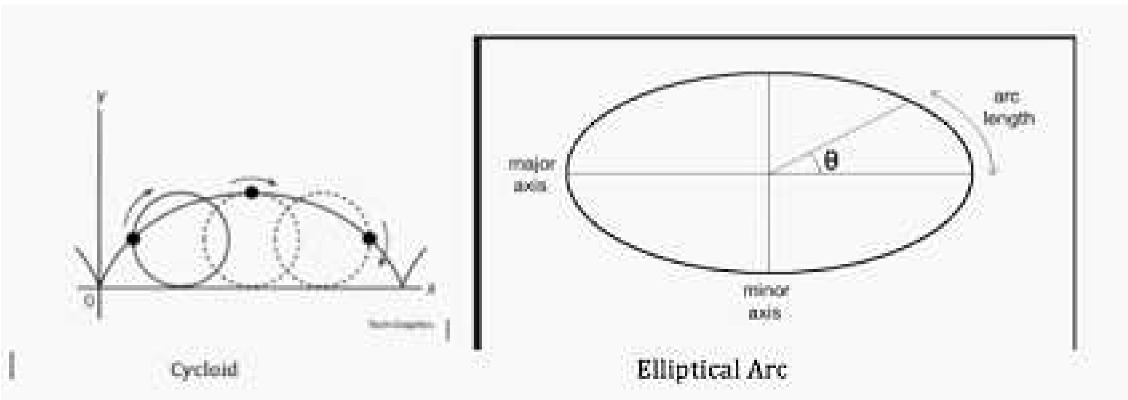

Figure 7. **Diagram of a Cycloid and an Elliptical Arc**

The general parametric equations of a cycloid are:

$$x\,(\theta) = \ a\,(\theta - sin\theta) \tag{13}$$

$$y\,(\theta) = \ a\,(1 - cos\theta) \tag{14}$$

In this equation there are two unknowns θ (angle swept by the rotating circle) and a (radius of the rotating circle). By considering the two unknown coordinates we shall deduce their values. Solving Equation (13) and (14) for unknowns by substituting the x and y coordinates of Point B.

$$6.283 = a\left(\theta - sin\theta\right)$$
$$4 = a\left(1 - cos\theta\right)$$
$$a = \frac{6.283}{\theta - sin\theta} \tag{15}$$
$$4 = \frac{6.283}{\theta - sin\theta} \times \left(1 - cos\theta\right)$$
$$4\left(\theta - sin\theta\right) = 6.283(1 - cos\theta)$$
$$\frac{4}{6.283}\left(\theta - sin\theta\right) = 1 - cos\theta \tag{16}$$

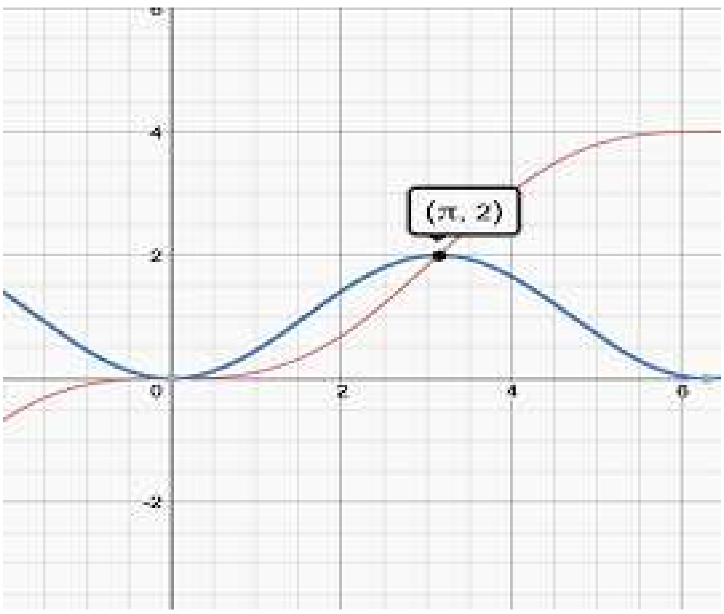

Figure 8. **Graph representing the unique solution of the Equation** $(6.283)/4(\theta - sin\theta) = 1 - cos\theta$

Therefore θ for Point B is π. Substituting π in Equation (15)

$$a = \frac{6.283}{\pi - sin\pi}$$
$$a = 2$$

Similarly $\theta = 0$ for Point A. Therefore $a = 2$. Substituting $a = 2$ and rewriting the Equation (13) and (14) again:

$$x\left(\theta\right) = 2\left(\theta - sin\theta\right) \tag{17}$$
$$y\left(\theta\right) = 2\left(1 - cos\theta\right) \tag{18}$$

To calculate time taken along the cycloid, we must find the derivative of Equation (17) and (18) and subsequently calculate time.

$$\frac{dy}{d\theta} = 2sin\theta \text{ and } \frac{dx}{d\theta} = 2 - 2cos\theta$$

Applying the derivations in Equation (6):

$$T = \frac{1}{\sqrt{2g}}\int_0^x \sqrt{\frac{1 + \left(f'\left(x\right)\right)^2}{f\left(x\right)}}dx$$

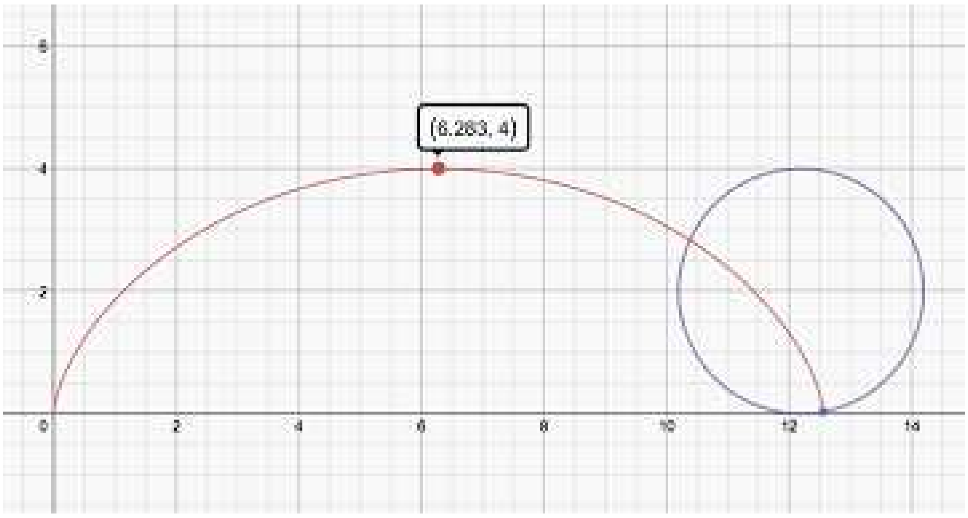

Figure 9. **Cycloid with parametric equation** $x(\theta) = 2(\theta - sin\theta)$**,** $y(\theta) = 2(1 - cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{d\theta} = asin\theta \ and \ \frac{dx}{d\theta} = a - acos\theta$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{asin\theta}{a(1 - cos\theta)}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2(sin\theta)^2}{a^2(1 - cos\theta)^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2(1 - (cos\theta)^2)}{a^2(1 - cos\theta)^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2(1 + cos\theta)(1 - cos\theta)}{a^2(1 - cos\theta)^2}$$

$$\frac{1 + cos\theta}{1 - cos\theta}$$

This is also equal to,

$$\frac{2a - y}{y}$$

$$\frac{2a - (a\,(1 - cos\theta))}{a(1 - cos\theta)}$$

$$\frac{1 + cos\theta}{1 - cos\theta}$$

$$\frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}$$

Separating the variables and rewriting equation in terms of dx

$$dx = dy \times \frac{1}{\sqrt{\frac{2a-y}{y}}}$$

Substituting $\left(\frac{dy}{dx}\right)^2$ and dx in Equation (6):

$$T = \frac{1}{\sqrt{2g}} \int_{\theta 1}^{\theta 2} \sqrt{\frac{1 + \left(\sqrt{\frac{2a-y}{y}}\right)^2}{y}} \times \ dy \times \frac{1}{\sqrt{\frac{2a-y}{y}}} \tag{19}$$

Solving for Equation (19)

$$T = \frac{1}{\sqrt{2g}}\int_{\theta 1}^{\theta 2}\sqrt{\frac{\frac{y+2a-y}{y}}{y\times\frac{2a-y}{y}}}\times dy$$

$$T = \frac{1}{\sqrt{2g}}\int_{\theta 1}^{\theta 2}\sqrt{\frac{\frac{2a}{y}}{2a-y}}\times\ dy$$

$$T = \frac{1}{\sqrt{2g}}\int_{\theta 1}^{\theta 2}\sqrt{\frac{2a}{y\times(2a-y)}}\times\ dy \tag{20}$$

Simplifying $y\times(2a-y)$ in terms of a and θ:

$$y(\theta) = \ a(1-cos\theta)$$

Substituting this in

$$y\times(2a-y) = a\times(1-cos\theta)\times(2a-a+acos\theta)$$

$$= (a(1-cos\theta))\times(a\times(1+cos\theta))$$

$$= a^2\times(1-(cos\theta)^2)$$

$$= a^2\ \times\ (sin\theta)^2$$

Substituting $a^2\times(sin\theta)^2$ in the Equation (20),

$$T = \frac{1}{\sqrt{2g}}\int_{\theta 1}^{\theta 2}\sqrt{\frac{2a}{a^2(sin\theta)^2}}dy$$

$$T = \frac{1}{\sqrt{2g}}\int_{\theta 1}^{\theta 2}\frac{\sqrt{2a}}{asin\theta}dy$$

But we know that,

$$dy = asin\theta\ d\theta$$

Substituting dy in the Equation gives:

$$T = \ \frac{1}{\sqrt{2g}}\int_{\theta 1}^{\theta 2}\frac{\sqrt{2a}}{asin\theta}\times a\ sin\theta\ d\theta$$

Cancelling $a\sin\theta$ from Numerator and Denominator

$$T = \ \frac{1}{\sqrt{2g}}\int_{0}^{\theta 2}\sqrt{2a}d\theta$$

The Integral of the Equation reduces to:

$$T = \frac{1}{\sqrt{2g}}\left(\sqrt{2a}\times\theta 2\right) \tag{21}$$

Where $\theta 2$ is the unique solution of $\frac{\theta - sin\theta}{1-cos\theta} = \frac{b1}{b2}$, a is the radius of the circle, which is rotated along its circumference to make the cycloid.

$$\frac{\theta-\sin\theta}{1-\cos\theta} = \frac{b1}{b2}$$

Where b1 and b2 are the coordinates of Point B therefore:

$$\frac{\theta-\sin\theta}{1-\cos\theta} = \frac{6.283}{4}$$

$$\frac{\theta-\sin\theta}{1-\cos\theta} = 1.57$$

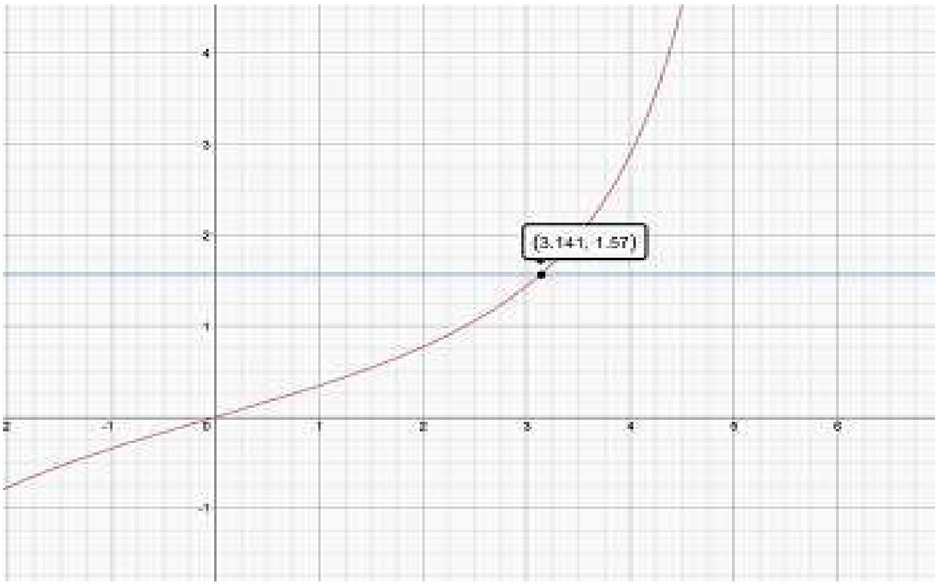

Figure 10. **Graph of** $y = (\theta - \sin\theta)/(1 - \cos\theta)$ **and** $y = 1.57$

Graphing the equation $\frac{\theta - \sin\theta}{1 - \cos\theta}$ and 1.57 on the calculator produces the following graph where the point of intersection is the unique solution to $\frac{\theta - \sin\theta}{1 - \cos\theta}$

Hence the unique solution $\theta 2 = 3.141 \approx 3.14$. Therefore Equation (21) becomes

$$T = \frac{1}{\sqrt{2g}} \times \sqrt{2a} \times 3.14$$

$$T = \frac{1}{\sqrt{2g}} \times \sqrt{2 \times 2} \times 3.14$$

$$T = \frac{1}{\sqrt{2 \times 10}} \times \sqrt{2 \times 2} \times 3.14$$

$$T = 1.404 \ seconds$$

The ball will take 1.404 seconds to traverse the distance between the endpoints. From the above result it can be inferred that the shortest distance between two points does not take the shortest time. From the investigation and the calculations made above, it can be concluded that compared to all the curves: Parabolic arc, elliptical arc, straight line, cycloid; cycloid is the curve of quickest descent.

3. Part 2: Modeling and Deriving the Equation of the Curve of Quickest Descent Between Bhubaneswar and New Delhi

Before examining the issue, it is pertinent to understand the minute details of the investigation to apply mathematics in solving the real life problem. Keeping in mind the result of the previous investigation, I will find out the path that takes the minimum time to travel from Bhubaneswar to New Delhi by Train. Assuming the top speed of the train to be 160 km/hr., I shall try out different curves that cross the two places on Earth. Using Google map, I found out the longitude and latitude coordinates of the two cities as:

Bhubaneswar-20.2700 degrees North, 85.8400 degrees East. New Delhi, Capital of India-28.6139 degrees North, 77.2090 degrees East. The above-mentioned coordinates have to be converted into Cartesian (x, y) coordinates for suitability to calculate total distance and time along a curve or path. The coordinates of the cities can also be expressed in the form of spherical coordinates: $(r,\ \theta,\ \varphi)$-Bhubaneswar-(6371, 20.2700, 85.8400) and New Delhi-(6371, 28.6139, 77.2090). This conversion makes it easy to convert it into Cartesian form by using the equation:

$$x = r \times \cos(\Phi) \times \sin(\theta)$$

$$y = r \times \sin(\Phi) \times \sin(\theta)$$

$$z = r \times \cos(\Phi)$$

Where Φ is longitude and θ is latitude

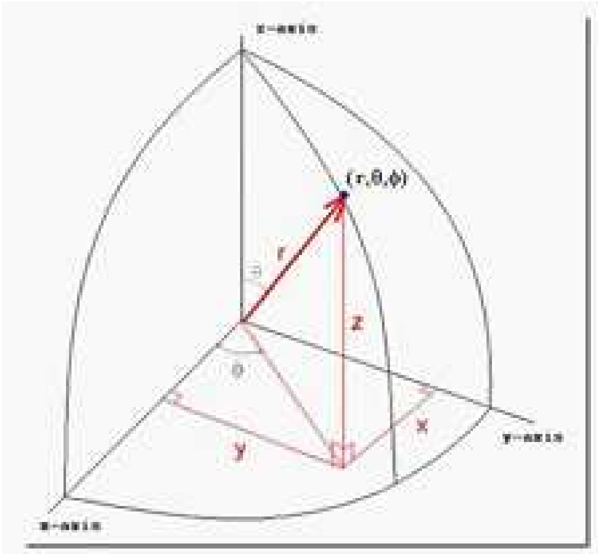

Figure 1 Spherical Coordinate System

Figure 11. **11 illustrates the above equations in terms of longitude θ and latitude φ**

1. **Bhubaneswar:**

$$x = 6371 \times \cos(85.8400) \times \sin(20.2700)$$

$$= 160.114$$

$$y = 6371 \times \sin(85.8400) \times \sin(20.2700)$$

$$= 2201.382$$

$$z = 6371 \times \cos(85.8400)$$

$$= 462.164$$

Bhubaneswar (160.114, 2201.382, 462.164)

2. **New Delhi:**

$$x = 6371 \times \cos(77.2090) \times \sin(28.6139)$$

$$= 675.499$$

$$y = 6371 \times \sin(77.2090) \times \sin(28.6139)$$

$$= 2975.387$$

$$z = 6371 \times \cos(77.2090)$$

$$= 1410.509$$

New Delhi (675.499, 2975.387, 1410.509576)

I will calculate the time and distance for two types of path: Straight line and a cycloid. In this process it has been assumed that there are no hurdles in the path and the train travels without a break with constant speed.

4. Great Circle Distance

Just like a straight line on a 2-D space, "The Great Circle or Orthodormic Distance" is the shortest distance between two points, measured along a sphere (Which represented by Figure 8). To calculate the Great Circle Distance, I will use The "Haversine" Formula that states:

$$\begin{aligned} a &= (\sin\frac{\triangle\theta}{2})^2 + \cos(\varphi 1)\times\cos(\varphi 2)\times(\sin\frac{\triangle\varphi}{2})^2 \\ c &= 2\times a\times\tan 2((\sqrt{a}),\ (\sqrt{1-a})) \\ d &= R\times c \end{aligned} \tag{22}$$

Where R is the Radius of the Earth = 6371 km, θ is latitude and φ is longitude. (The angles are in radians) and d is the distance in kilometers.

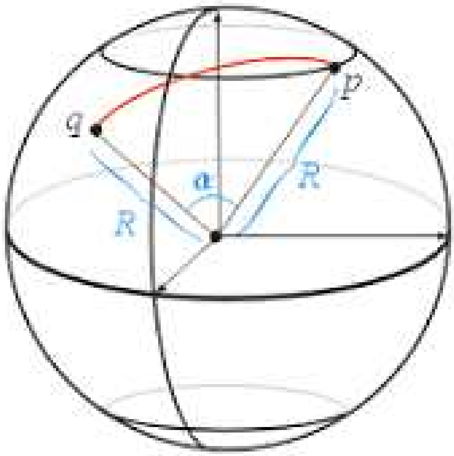

Figure 12.

Hence by substituting the values of longitude and latitude of the origin and the destination in Equation (22), I calculated **d** using software as 1273km and the midpoint of the path (in the form of spherical coordinates) to be (6371, 28.0034, 81.3305). According to a formula in Physics

$$Distance = Speed\ \times\ Time,$$

Time taken by train travelling at the speed of 160km/hr.:

$$T(Time) = \frac{Distance}{speed}$$

$$T = \frac{1273}{160} \quad \text{(160 km/hr- top speed of the train)}$$

$$T = 7.95\ hours$$

Therefore a train travelling at 160km/hr will take 7.95 hours to travel from Bhubaneswar to New Delhi.

4.1. Hypocycloid

There is a fair possibility that the shortest path may not take the shortest time of travel. Therefore, it's a safe approach to investigate further possibilities. For example, when we investigated on the 2D Cartesian plane, we found out that for a ball of negligible mass, the curve of quickest descent is a cycloid. Hence, Let us hypothetically assume that an underground tunnel in the shape of a hypocycloid (Which I will explain later on) is built between two cities. For this reason, we will have to find out the equation for total distance and velocity to calculate time taken by a gravity train. Before moving onto further calculations, we have to make few assumptions.

1. The only force acting upon the train is gravity.
2. Negligible frictional force resisting the motion of train
3. The train will travel without a break

To calculate time it is pertinent to derive the equation of **ds** (distance covered at an instant) and **v** (velocity of the train at a particular instant). According to the laws of physics, the velocity of the train will vary along the curve. The equation of velocity can be found out by considering conservation of energy.

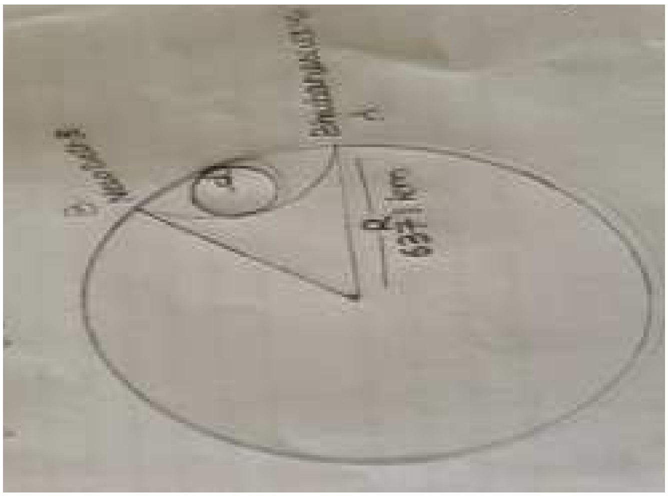

Figure 13. **Illustration of the proposed underground Tunnel**

Volume of the Earth is similar to that of sphere = $\frac{4}{3}\pi r^3$. Density ρ is Mass (M) per unit Volume ($\frac{4}{3}\pi r^3$). If we must find the total mass, we will have to find the integral of mass at a distance **r** from the center of earth.

$$
\begin{aligned}
M &= \int_0^r 4\pi r^2 \rho \, dr \\
&= \frac{4}{3}\pi r^3 \rho \qquad (23) \\
\rho &= \frac{M}{\frac{4\pi r^3}{3}} \qquad (24)
\end{aligned}
$$

From Equation (23) and (24),

$$
\begin{aligned}
M &= \frac{4}{3}\pi r^3 \rho \\
\rho &= \frac{M}{\frac{4\pi R^3}{3}} = \frac{\frac{4}{3}\pi r^3 \rho}{\frac{4\pi R^3}{3}}
\end{aligned}
$$

Cancelling $\frac{4}{3}\pi$ and ρ, we get,

$$r^2 = \frac{R^3}{r} \tag{25}$$

According to the Newton's Law of Gravitational Force of Attraction the force between two bodies is proportional to the product of the mass of the two bodies and inversely proportional to the distance between them. Acceleration due to gravity is

$$\frac{GMm}{r^2} = gm = ma$$

From Equation (25), $a = g =$ acceleration due to free fall

$$\frac{GM}{r^2} = \frac{GM}{\frac{R^3}{r}} = \frac{GM}{R^3}r$$

Total Potential Energy is the Integral of Force experienced by infinitesimal small lengths of the curve.

$$F\left(r\right) = \frac{GM}{R^3}\int_0^r r\ dr = \frac{GMr^2}{2R^3}$$

From the condition of conservation of energy in Physics,

$$v^2 = \frac{GM}{R^3}r^2$$

g is acceleration due to free fall

$$F = GMm/R^2 \text{ And } F = mg$$

$$g = \frac{GM}{R^2} \tag{26}$$

Substituting Equation (26) to get:

$$v^2 = \frac{gr^2}{R}$$

In my example **r** is the difference between **R**-Radius of Earth and Radius of the circle that forms the cycloid. Applying this concept modifies the equation to:

$$v^2 = \frac{g(R^2 - r^2)}{R}$$

$$v = \sqrt{\frac{g(R^2 - r^2)}{R}} \tag{27}$$

Where **r** is the distance from the center to the point on the hypocycloid where the train is at that instant of time. The total distance of a cycloid is an integral function of ds. Referring to Figure 2 We know that,

$$dt = \frac{ds}{v}$$

$$\int dt = \int \frac{ds}{v}$$

$$t = \int \frac{ds}{v}$$

Since we have taken the Cartesian coordinates of Bhubaneswar and New Delhi, let us explore the solution of Time in terms of it. Substituting Equation (1) and Equation (27)

$$t = \int \frac{\sqrt{(dx)^2 + (dy)^2}}{\sqrt{\frac{g(R^2 - r^2)}{R}}}$$

Rearranging the Equation to get:

$$t = \int \sqrt{\frac{R((dx)^2 + (dy)^2)}{g(R^2 - r^2)}}$$

$$r^2 = x^2 + y^2 \ \text{(in terms of Polar Coordinates)} \tag{28}$$

$$t = \int \sqrt{\frac{R((dx)^2 + (dy)^2)}{g\left(R^2 - (x^2 + y^2)\right)}} \tag{29}$$

Since the Equation derived above does not depend upon time, we shall use a derivation of the Euler-Lagrange equation to solve this problem. This derivation is known as the Beltrami Identity. The Beltrami identity named after Eugenio Beltrami is a simplified version of the Euler Lagrange equation that states

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\frac{\partial f}{\partial x'}$$

The Beltrami Identity States:

$$f - x'\left(\frac{\partial f}{\partial x'}\right) = C$$

I shall use this identity to aid me in deriving the Equation of the curve. Using the Beltrami to solve the Equation results in two-coupled Equation

$$f - x'\left(\frac{\partial f}{\partial x'}\right) = D1 = \frac{(y')^2\sqrt{\frac{R\left(x'^2 - y'^2\right)}{g\left(R^2 - x^2 - y^2\right)}}}{y'^2 - x'^2} \tag{30}$$

$$f - y'\left(\frac{\partial f}{\partial y'}\right) = D2 = \frac{(x')^2\sqrt{\frac{R\left(x'^2 - y'^2\right)}{g\left(R^2 - x^2 - y^2\right)}}}{x'^2 - y'^2} \tag{31}$$

Adding both Equations (30) and (31) as well as squaring them

$$(D1 + D2)^2 = \left(\left(\frac{(y')^2\sqrt{\frac{R\left(x'^2 - y'^2\right)}{g\left(R^2 - x^2 - y^2\right)}}}{y'^2 - x'^2}\right) + \left(\frac{(x')^2\sqrt{\frac{R\left(x'^2 - y'^2\right)}{g\left(R^2 - x^2 - y^2\right)}}}{x'^2 - y'^2}\right)\right)^2$$

$$= R\left(x'^2 + y'^2\right) = gD^2\left(R^2 - x^2 - y^2\right) \tag{32}$$

Where $D = D1 + D2$. The solution to the Equation (32) is a hypocycloid with the parametric equations:

$$x\left(t\right) = \left(R - d\right)cost + dcos\left(\frac{R - d}{d}t\right)$$

$$y\left(t\right) = \left(R - d\right)sint - dsin\left(\frac{R - d}{d}t\right)$$

with R being the Radius of the outside circle and "d" is the radius of the smaller circle. A hypocycloid is the curve traced by a point on the circumference of a circle, which is present within the interior of another circle. This is truly applicable to this problem, as the tunnel is present within the interior of Earth (Also in the shape of circle when observed on a 2D plane). Simplifying $x\left(t\right),\ x'\left(t\right),\ y\left(t\right)$ and $y'(t)$. Let $\frac{R-d}{d} = u$

$$(x(t))^2 = (R - d)^2(cost)^2 + d^2(cosut)^2 + 2(R - d)(d)(cost \times cosut)$$

$$\left(x'\left(t\right)\right)^2 = \left((R - d)^2 \times \left(sint^2 + sinut^2 + 2sint \times sinut\right)\right)$$

$$(y\,(t))^2 = (R-d)^2 (sint)^2 + d^2 (sinut)^2 - 2\,(R-d)\,(d)(sint \times sinut)$$

$$(y'(t))^2 = (R-d)^2 \times ((cost)^2 + (cosut)^2 - 2(cost \times cosut)$$

Now

$$\left(x'\,(t)\right)^2 + (y'(t))^2 = (R-d)^2 \left((sint)^2 + (cost)^2 + (sinut)^2 + (cosut)^2 + 2 \times ((sint \times sinut) - (cost \times cosut\;))\right)$$

According to the Trigonometric Identity $(sint)^2 + (cost)^2 = 1)$

$$= (R-d)^2 \times ((2 + 2\cos\,(ut+t)\;))$$

$$t + ut = \frac{(R-d)t}{d} + t = \frac{Rt}{d}$$

$$\left(x'\,(t)\right)^2 + (y'(t))^2 = (R-d)^2 \times 2 \times \left(1 + \cos\left(\frac{Rt}{d}\right)\right)$$

$$R^2 - ((x\,(t))^2 + \;(y\,(t))^2) = 2 \times \left(\left(-d^2 + dR\right) + \left((R-d)\,(d)\cos\left(\frac{Rt}{d}\right)\right)\right)$$

Substituting $(x\,(t))^2 + \;(y\,(t))^2$ and $(x'\,(t))^2 + (y'(t))^2$ in $R\left(x'^2 + \;y'^2\right) = gD^2\left(R^2 - x^2 - y^2\right)$

$$D^2 = \left(\frac{2R \times (R-d)^2 \times (1 + cos\left(\frac{Rt}{d}\right)}{2gd \times ((Rd - d^2) + \left(Rd - \;d^2 \times (cos\left(\frac{Rt}{d}\right))\;\;\right)}\right)$$

Cancelling 2 from Numerator and Denominator

$$= \frac{(R)\,(R-d)^2\left(1 + cos\left(\frac{Rt}{d}\right)\;\right)}{g((R-d)\,(d)\left(1 + cos\left(\frac{Rt}{d}\right)\;\right)}$$

$$= \frac{(R)\,(R-d)^2\left(1 + cos\left(\frac{Rt}{d}\right)\;\right)}{g((R-d)\,(d)\left(1 + cos\left(\frac{Rt}{d}\right)\;\right)}$$

Cancelling $\left(1 + cos\left(\frac{Rt}{d}\right)\;\right)$ and (R-d) from Numerator and Denominator we get:

$$D^2 = \frac{R(R-d)}{g(d)}$$

$$D = \sqrt{\frac{R(R-d)}{gd}}$$

From Equation (32) and (29), Time taken from A (Bhubaneswar) to B (New Delhi):

$$t = \int_A^B \frac{ds}{v} = \int_A^B \sqrt{\frac{R((dx)^2 + (dy)^2)}{g\,(R^2 - (x^2 + \;y^2))}} = \int_A^B D\;dt$$

$$\int_A^B \sqrt{\frac{R(R-d)}{gd}}\;dt \tag{33}$$

Time interval can be found in terms of **d** by squaring x (t), y (t) and adding them. We know that:

$$r^2 = \;x^2 + \;y^2$$

$$(x\,(t))^2 = \left((R-d)\,cost + dcos\left(\frac{R-d}{d}t\right)\right)^2$$

$$(x(t))^2 = (R-d)^2(cost)^2 + d^2\left(cos\left(\frac{R-d}{d}t\right)\right)^2 + 2\,(R-d)\,(d)\,(cost)\left(cos\left(\frac{R-d}{d}t\right)\right)$$

$$(y(t))^2 = \left((R-d)\ \ sint - dsin\left(\frac{R-d}{d}t\right)\right)^2$$

$$(y(t))^2 = ((R-d)^2(sint)^2 + d^2\left(sin\left(\frac{R-d}{d}t\right)\right)^2 - \left(2\,(R-d)\,(d)\,(sint)\left(sin\left(\frac{R-d}{d}t\right)\right)\right)$$

Adding $(y\ (t))^2$ and $(x\,(t))^2$ to get:

$$(R-d)^2 + d^2 + 2d\,(R-d) \times \left(\left(cost \times cos\left(\frac{R-d}{d}t\right)\right) - \left(sint \times sin\left(\frac{R-d}{d}t\right)\right)\right)$$

Applying the Sum-Difference formulas $\cos a \cos b - \sin a \sin b = \cos(a+b)$

$$= R^2 + \ d^2 - 2dr + d^2 + 2d(R-d) \times cos\left(\frac{td + (R-d)\,t}{d}\right)$$

$$= 2d^2 - 2dr + R^2 + 2d\,(R-d) \times cos\left(\frac{Rt}{d}\right)$$

$$1 = cos\left(\frac{Rt}{d}\right)$$

The periodicity of cosine angle repeats after 2π. Therefore, $\frac{Rt}{d} = 2\pi n$ (n is a positive integer). Solving for $n = 1$ to get the transit time along the hypocycloid:

$$t = \frac{2\pi d}{R}$$

Where $t = dt$

$$dt = \frac{2\pi d}{R} \tag{34}$$

Substituting Equation (34) in Equation (33)

$$t = \int_A^B \sqrt{\frac{R(R-d)}{dg}} \times \frac{2\pi d}{R}$$

$$t = \sqrt{\frac{R(R-d)}{dg}} \times \frac{2\pi d}{R}$$

Using the Equation $l = r\theta$ where l is the arc length $= 2\pi d$, R is the radius and θ is the angle formed by the sector in radians.

$$\theta = \frac{2\pi d}{R}$$

Since s is the arc-length between the two cities on Earth,

$$\theta = \frac{s}{R}$$

$$\theta R = s$$

Combining the previous equations

$$2\pi d = s$$

$$d = \frac{s}{2\pi}$$

Therefore, Time taken to travel between New Delhi and Bhubaneswar along the Hypocycloid: (Substituting d with s/2π)

$$t = \sqrt{\frac{R\left(R - \frac{s}{2\pi}\right)}{\frac{s}{2\pi}g}} \times \frac{2\pi \times \frac{s}{2\pi}}{R}$$

$$t = \sqrt{\frac{s}{R}\frac{(2\pi R - s)}{g}} \tag{35}$$

The Arc length distance between Bhubaneswar and New Delhi-(s) = 1273km, R (Radius of Earth) = 6371 km, g = $10ms^{-2}$. Finally Substituting s and R in Equation 34 to calculate the transit time (t):

$$t = 27.828\ minutes$$

It will take 27.828 minutes to travel from New Delhi to Bhubaneswar through the Hypocycloid by a gravity train. The present routes between New Delhi to Bhubaneswar are around 1626 km, which will take about 26 hours by car and about 25 hours by train. If a straight road (Great Circle Distance) were built between the two cities then a train constantly travelling at the rate of 160 km/hr would take 7.95 hours. On the contrary a gravity train would take 27 minutes travelling through the hypocycloid shaped underground tunnel, which is 5% of the time taken by the high-speed train. With this result we can successfully conclude that a hypocycloid is the Brachistochrone between New Delhi and Bhubaneswar. My investigation is applicable to intercontinental travels too. But there is limit to which this is applicable; this is further discussed in conclusion.

5. Conclusion

We can conclude from the investigation that albeit a straight line and arc length are the shortest distance between two points on a 2D plane and 3D plane respectively, they are not the curves of quickest descent. Under the influence of gravity, the curve of quickest descent or Brachistochrone is a cycloid on a 2D plane and Hypocycloid on a 3D plane. This investigation is a proof of the versatility of application of Mathematics to solve one of the most challenging problems and open up opportunity to discover the undiscovered knowledge. The amalgamation of the laws and theory of Physics with some of the most important concepts of Mathematics: especially calculus is itself the evidence of great power of Mathematics. I thoroughly enjoyed learning new concepts and exploring different possibilities through the course of my investigation. By using Mathematics, I have been able to come up with a solution to an impertinent global issue. However this result has some limitation that must be considered.

5.1. Limitation and Further Scope of Investigation

Throughout the investigation, several assumptions were made, which has the potential to affect the final solution and its application in real life. For instance the ball is considered to be of negligible mass, although it has considerable mass. It has also been assumed that the train travels consistently at top speed without any break. This can affect the outcome to a small extent. Nonetheless if all the variables would be included in the investigation, there would be too many unknown variables, which would make calculations very strenuous. For an initiative in exploring new possibilities, assumptions are necessary.There is a limit after which it is not possible to reduce the travel time regardless of the shape of the tunnel or the slide. If the distance between the cities is very large, there is no other solution apart from the Great Circle Distance. Many of the values used for calculations are estimations. For instance the value of g (acceleration due to free fall), radius of Earth, and the speed of train are estimated values. In addition, the results of several calculations were rounded to two significant figures. There is a possibility of systematic error in the calculator, used to calculate complex integrals.

My investigation could account for some of the variables not considered during calculations; such as the frictional force, or mass of the ball, then outcome would have been more accurate. Considering the possibilities to include the variables and carry out calculation accurately with the real value, this investigation could be strengthened and made more realistic. Notwithstanding the limitations, I have successfully deduced a feasible solution to my research question and pressing global issue while considering most of the crucial factors during calculation and deduction of outcome.

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