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A Generalisation of Quasi-Convex Sequence Spaces

Research Article

Ajaya Kumar Singh[1*]

1 Department of Mathematics, P.N. College (Autonomous), Khordha, India.

Abstract: The object of the paper is to generalise the concept of quasi-convex sequence spaces by means of the generalised the fractional difference.

Keywords: Banach limit, Banach spaces, Binomial co-efficient, Cauchy sequence, fractional difference, Generalised almost convergence, Generalised binomial coefficient, Linear space, Quasi-convex sequence spaces, Sub-linear functional, Topological properties.

1. Introduction

Let $x = (x_n)$ be any real or complex sequence. Write $\Delta^0 x_n = x_n$, $\Delta x_n = x_n - x_{n+1}$, $\Delta^m x_n = \Delta(\delta^{m-1} x_n)$, where m is a positive integer. It may be easily verified that $\Delta^m x_n$ is represented by the sum

$$\Delta^m x_n = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{n+v} \ .$$

A sequence (x_n) is said to be the convex order m if $\Delta^m x_n = 0$ and quasi-convex of order m, if

$$\sum_{k=1}^{\infty} \begin{pmatrix} k+m-1 \\ k \end{pmatrix} |\Delta^m x_k| < \infty.$$

Let α denote a non-integral real number. Then $\Delta^\alpha x_n$ is defined by the infinite series

$$\Delta^\alpha x_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1} x_v,$$

provided by the series on the right converges, where the Binomial coefficients $A_n^\alpha = \begin{pmatrix} n+\alpha \\ \alpha \end{pmatrix}$ are defined by the power series

$$\frac{1}{(1-y)^\alpha} = \sum_{n=0}^{\infty} A_n^{\alpha-1} y^n, \ \ |y| < 1.$$

Let $\alpha > 0$, we note that $A_n^{\alpha-1}$ are all positive. A sequence (x_n) is said to be convex of order α, if $\Delta^\alpha x_n = 0$ and it is said to be quasi-convex of order α, if

$$\sum_{k=1}^{\infty} A_k^{\alpha-1} |\Delta^\alpha x_k| < \infty.$$

In this chapter we investigate some new matrix transformations on quasi-convex sequence spaces for non-integral positive values of α. Throughout the chapter we use K or K_1 as absolute constants, not necessarily same at each occurrence.

* *E-mail: ajayakumarsingh1966@gmail.com*

2. Some Notations

As in for any real α and β, $(\alpha > 0)$ we write

$$Q^{\alpha} = \left\{ x \in \omega : \sum_{k=1}^{\infty} A_k^{\alpha-1} \left| \Delta^{\alpha} x_k \right| < \infty \right\},$$
$$Q^{\alpha,\beta} = \left\{ x \in \omega : \sum_{k=1}^{\infty} A_k^{\alpha+\beta-1} \left| \Delta^{\alpha} x_k \right| < \infty \right\},$$
$$b^{\alpha,\beta} = l_{\infty} \cap Q^{\alpha,\beta},$$
$$q^{\alpha,\beta} = c \cap Q^{\alpha,\beta},$$
$$q^{\alpha} = c \cap Q^{\alpha,\beta}.$$

Note that

$$Q^{\alpha,0} = Q^{\alpha},\ Q^{0,1} = l_1,\ Q^{\alpha,0} = q^{\alpha},\ b^{\alpha,0} = b^{\alpha}$$

Das and Rao [2] observed that following inclusion relation hold.

$$Q^{\alpha,\beta} \subset Q^{\alpha} \left(\beta \leq 0,\ \alpha \leq 0\right),$$
$$l_1 \subset Q^{0,\beta} \left(\beta < 1\right),$$
$$Q^{0,\beta} \subset l_1 \left(\beta > 1\right).$$

The sequence space $X\left(\Delta\right)$ where $X = c_0$, c or l_{∞} is defined as follows :

$$X\left(\Delta\right) = \left\{x \in \omega : \left(\Delta x_k\right) \in X\right\}.$$

It is known that $X\left(\Delta\right)$ is a Banach space normed by $||x||_{\Delta} = |x_1| + ||\Delta x||_{\infty}$, where $||\Delta x||_{\infty} = \sup_k |x_k - x_{k+1}|$. Before we start characterizing the matrix A in the class $\left(l_{\infty}\left(\Delta\right),\ b^{\alpha,\beta,\delta}\right)$, we make the following preparations with a view to introduce new sequence spaces. We give a generalisation of binomial coefficient (A_n^{α}) in terms of the coefficient $\left\{A_n^{\alpha,\delta}\right\}$. Let the coefficient $\left\{A_n^{\alpha,\delta}\right\}$ be defined by the following power series

$$\sum A_n^{\alpha,\delta} x^n = (1-x)^{-\alpha-1} \left(log \frac{\alpha}{1-x} \right)^{\delta},\ \alpha > 2$$

Note that the following relations hold :

$$nA_n^{\alpha-1,\delta} = \alpha A_{n-1}^{\alpha,\delta} + sA_{n-1}^{\alpha,\delta-1}$$
$$A_n^{\alpha,\delta} = \sum_{v=0}^{n} A_v^{\alpha-1,\delta}$$

The following estimates are known :

$$A_n^{\alpha,\delta} \sim \frac{n^{\alpha}}{(\alpha+1)} (logn)^{\delta},\ \alpha \neq -1,\ -2, \ldots.$$
$$A_n^{\alpha,\delta} \sim \beta(-1)^{\alpha-1} \left(|\alpha| - 1\right)! n^{\alpha} (logn)^{\delta-1},\ \alpha = -1,\ -2, \ldots.$$

We now obtain the following new sequence spaces as generalisation of above quasi-convex sequence spaces :

$$\begin{aligned}
Q^{\alpha,\delta} &= \left\{x \in \omega : \sum A_k^{\alpha-1,\delta} \left|\Delta^{\alpha,\delta} x_k\right| < \infty\right\} \\
Q^{\alpha,\beta,\delta} &= \left\{x \in \omega : \sum A_k^{\alpha+\beta-1,\delta} \left|\Delta^{\alpha,\delta} x_k\right| < \infty\right\} \\
b^{\alpha,\beta,\delta} &= l_\infty \cap Q^{\alpha,\beta,\delta} \\
q^{\alpha,\beta,\delta} &= c \cap Q^{\alpha,\beta,\delta} \\
q^{\alpha,\delta} &= c \cap Q^{\alpha,\delta}
\end{aligned}$$

Note that

$$\begin{aligned}
A_r^{\alpha,0} &= A_r^{\alpha} \\
Q^{\alpha,0} &= Q^{\alpha} \\
Q^{\alpha,\beta,0} &= Q^{\alpha,\beta}
\end{aligned}$$

3. Topological and Other Properties

Before we study the sequence space $Q^{\alpha,\delta}$, it is necessary to develop some techniques to deal with new situations. It is easily verified that if m is a positive integer, then

$$\Delta^m x_n = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{n+v} \tag{1}$$

But if α is non-integral then $\Delta^\alpha x_n$ is defined by the infinite series

$$\Delta^\alpha x_n = \sum_{v=n}^{\infty} A_n^{-\alpha-1} x_v \tag{2}$$

provided the series on the right converges. Note that when α is a positive integer, then the series (2) reduces to (1). We first tabulate the known results that concern the fractional difference $\Delta^\alpha x_n$.

(i) Let $\alpha > 0$ and let the series for $\Delta^{-\alpha} x_n$ converges; then $x_n = \Delta^\alpha \left(\Delta^{-\alpha} x_n\right)$.

(ii) Let $\alpha \geq 0,\ \beta > -1$. Let $\alpha + \beta > 0$ and $x \in l_\infty$. Then $\Delta^{\alpha+\beta} x_n = \Delta^\alpha \left(\Delta^\alpha x_n\right)$.

(iii) Let $\alpha \geq 0,\ \beta > -1,\ \alpha + \beta \geq 0$ and $x \in c_0$, then $\Delta^{\alpha+\beta} x_n = \Delta^\beta \left(\Delta^\alpha x_n\right)$

In particular, if $x_n = o\left(1\right)$, then for $0 \leq \alpha \leq 1$, $x_n = \Delta^{-\alpha} \left(\Delta^\alpha x_n\right)$.

3.1. A Generalisation of the Difference of Fractional Order

It may be noted that the fractional $\Delta^\alpha x_n$ has been defined by means of the binomial coefficients $\{A_n^\alpha\}$. We now generalise the fractional difference by means of the generalised binomial coefficients $\{A_n^\alpha\}$. We now write

$$\Delta^{\alpha,\delta} x_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1,-\delta} x_v \tag{3}$$

Whenever the series (3) converges. Note that $\Delta^{\alpha,0}x_n = \Delta^{\alpha}x_n$. We may remark that when α is an integer and $\delta \neq 0$, then the series (3) still continuous to be an infinite series. Therefore it will be interesting to study these new spaces. It follows the properties of $A_n^{\alpha,\beta}$ that

$$\sum_{k=0}^{n} A_{n-k}^{\alpha,\delta} A_k^{\alpha',\delta'} = A_n^{\alpha+\alpha',\delta+\delta'}$$

$$A_n^{\alpha+1,\beta} - A_{n-1}^{\alpha+1,\beta} = A_n^{\alpha,\beta}$$

It is familiar that if the series (2) exists for one n, then it exists for any other value of n. This naturally lead one to conjecture that a similar result holds for $\Delta^{\alpha,\delta}x_n$. In fact we prove

Theorem 3.1. *If $\Delta^{\alpha,\delta}x_n$ exists for one n, it exists for any other value of n.*

Proof. It is enough to prove that the series defining $\Delta^{\alpha,\delta}x_0$ converges if and only if $\Delta^{\alpha,\delta}x_1$ converges. Now

$$\Delta^{\alpha,\delta}x_0 = \sum_{v=n}^{\infty} A_v^{-\alpha-1,-\delta} x_v \tag{4}$$

$$\Delta^{\alpha,\delta}x_1 = \sum_{v=1}^{\infty} A_{v-1}^{-\alpha-1,-\delta} x_v \tag{5}$$

Thus converges of (4) imply the convergence of (5) If we prove that

$$\frac{A_{v-1}^{-\alpha-1,-\delta}}{A_v^{-\alpha-1,-\delta}} \tag{6}$$

Is of bounded variation. Similarly the converse implication apply if we prove that

$$\frac{A_v^{-\alpha-1,-\delta}}{A_{v-1}^{-\alpha-1,-\delta}} \tag{7}$$

is of bounded variation. Since (6) converges to 1 as $n \to \infty$ it is enough to prove that (6) is of bounded variation. It may be noted that in the case $\delta = 0$, there is a simple explicit expression for (6), since

$$\frac{A_{n-1}^{-\alpha-1}}{A_n^{-\alpha-1}} = \frac{n}{n-\alpha-1}$$

But when $\delta \neq 0$, there is so much simple expression for (6). Now

$$\Delta\left(\frac{A_{n-1}^{-\alpha-1,-\delta}}{A_n^{-\alpha-1,-\delta}}\right) = \frac{X_n}{A_n^{-\alpha-1,\delta} A_{n+1}^{-\alpha-1,-\delta}}$$

Where

$$\begin{aligned} X_n &= A_{n-1}^{-\alpha-1,-\delta} A_{n+1}^{-\alpha-1,-\delta} - \left(A_n^{-\alpha-1,-\delta}\right)^2 \\ &= \left(A_n^{-\alpha-1,-\delta} - A_n^{-\alpha-2,-\delta}\right)\left(A_n^{-\alpha-1,-\delta} + A_{n+1}^{-\alpha-2,-\delta}\right) - \left(A_n^{-\alpha-1,-\delta}\right)^2 \\ &\quad - A_n^{-\alpha-2,-\delta} A_{n+1}^{-\alpha-2,\delta} + A_n^{-\alpha-1,\delta}\left(A_{n+1}^{-\alpha-2,-\delta} - A_n^{-\alpha-2,\delta}\right) \\ &= A_n^{-\alpha-2,-\delta} A_{n+1}^{-\alpha-2,\delta} - A_n^{-\alpha-1,-\delta} A_{n+1}^{-\alpha-3,-\delta} \end{aligned}$$

Now it follows that

$$\Delta\left(\frac{A_{n-1}^{-\alpha-1,-\delta}}{A_n^{-\alpha-1,-\delta}}\right) = o\left(\frac{1}{n^2}\right)$$

Hence the result. □

Now we prove

Theorem 3.2. *Let α be non-negative integer, $\beta \neq 0$ and either $s > \alpha$, any δ or $s = \alpha$, $\delta > \beta$. Then the existence of $\Delta^{\alpha,\beta} x_n$ implies that of $\Delta^{s,\delta} x_n$.*

Proof. Because of Theorem 3.1, it is enough to consider the case $n = 0$. Thus we are given that

$$\Delta^{\alpha,\beta} x_0 = \sum_{v=0}^{\infty} A_v^{-\alpha-1,\beta} x_v$$

Converges and we have to show that

$$\sum_{v=0}^{\infty} A_v^{-s-1,-\delta} x_v$$

Converges. This will follow if we prove that $\frac{A_v^{-s-1,-\delta}}{A_v^{-\alpha-1,-\beta}}$ is of bounded variation. Now

$$\Delta\left(\frac{A_v^{-s-1,-\delta}}{A_v^{-\alpha-1,-\beta}}\right) = \frac{Y_v}{A_v^{-\alpha-1,-\beta} A_{v+1}^{-\alpha-1,-\beta}}$$

when

$$\begin{aligned} Y_v &= A_v^{-s-1,-\delta} A_{v+1}^{-\alpha-1,\beta} - A_{v+1}^{-s-1,-\delta} A_v^{-\alpha-1,\beta} \\ &= A_v^{-s-1,-\delta}\left(A_v^{-\alpha-1,\beta} + A_{v+1}^{-\alpha-2,\beta}\right) - A_v^{-\alpha-1,-\beta}\left(A_v^{-s-1,-\delta} + A_{v+1}^{-s-2,-\delta}\right) \\ &= A_v^{-s-1,-\delta} A_{v+1}^{-\delta-2,\beta} - A_v^{-\delta-1,-\beta} A_{v+1}^{-s-2,-\delta} \end{aligned}$$

It now follows that

$$\Delta\left(\frac{A_v^{-s-1,-\delta}}{A_v^{-\alpha-1,\beta}}\right) = o\left(v^{-s-1+\alpha}(logv)^{\beta-\delta}\right)$$

This gives the conclusion when $s > \alpha$. We now need the case when $s = \alpha$. In that case

$$Y_v = A_v^{-\alpha-1,-\delta} A_{v+1}^{-\alpha-2,-\beta} - A_v^{-\alpha-1,-\beta} A_{v+1}^{-\alpha-3,-\delta}$$

So that

$$(v+1)\, Y_v = A_v^{\sigma-1,\delta}\left(-(\alpha+1)\, A_v^{-\alpha-1,-\beta} - \beta A_v^{-\alpha-1,\beta-1}\right) - A_v^{-\alpha-1,-\beta}\left(-(\alpha+1)\, A_v^{-\alpha-1,\delta} - \delta A_v^{-\alpha-1,-\delta} - \delta A_v^{-\alpha-1,-\delta-1}\right)$$

Hence in this case

$$\Delta\left(\frac{A_v^{-\alpha-1,-\delta}}{A_v^{-\alpha-1,-\beta}}\right) = o\left(\frac{1}{v}(logv)^{\beta-\delta-1}\right)$$

as the result follows. □

Theorem 3.3. *Let $\alpha > -1$. Let he series $\Delta^{\alpha,\beta} x_n$ converge. Then $\Delta^{r,\alpha}\left(\Delta^{s,\delta} x_n\right) = \Delta^{r+s,\alpha+\delta} x_n$. Provided that*

$$I_{mn} = \sum_{v=n}^{m} A_{v-n}^{-r-1,-\alpha} \sum_{k=m+1}^{\infty} A_{k-v}^{-s-1,-\delta} x_k \to 0 \text{ as } m \to \infty, \text{ for all } n.$$

Proof. Let the series $\Delta^{\alpha,\beta}x_n$ converge. Now

$$\begin{aligned}
\Delta^{r,\alpha}\left(\Delta^{s,\delta}x_n\right) &= \sum_{v=n}^{m} A_{v-n}^{-r-1,-\alpha}\left(\Delta^{s,\delta}x_v\right) \\
&= \sum_{v=n}^{m} A_{v-n}^{-r-1,-\alpha}\sum_{k=v}^{\infty} A_{k-v}^{-s-1,-\delta}x_k \\
&= \lim_{m\to\infty}\sum_{v=n}^{m} A_{v-n}^{-r-1,-\alpha}\left(\sum_{k=v}^{\infty}+\sum_{k=m+1}^{\infty}\right)A_{k-v}^{-s-1,-\delta}x_k \\
&= \lim_{m\to\infty}\sum_{k=v}^{\infty} x_k\sum_{v=n}^{k} A_{v-n}^{-r-1,-\alpha}A_{k-v}^{-s-1,-\delta} + \lim_{m\to\infty}\sum_{v=n}^{m} A_{v-n}^{-r-1,-\alpha}\sum_{k=m+1}^{\infty} A_{k-v}^{-s-1,-\delta}x_k \\
&= \lim_{m\to\infty}\sum_{k=n}^{\infty} A_{k-v}^{-r-s-1,-\alpha-\delta}x_k + \lim_{m\to\infty}\sum_{v=n}^{m} A_{v-n}^{r-1,-\alpha}\sum_{k=m+1}^{\infty} A_{k-v}^{-s-1,-\delta}x_k \\
&= \Delta^{r+s,\alpha+\delta}x_n + \lim_{m\to\infty} I_{mn} \\
&= \Delta^{r+s,\alpha+\delta}x_n.
\end{aligned}$$

This completes the proof. □

3.2. Topological Properties

Define

$$h : Q^{\alpha,\delta} \to \mathbb{R}^{+} \; as \; h(x) = \sum \left|A_n^{\alpha-1,\delta}\right|\left|\Delta^{\alpha,\delta}x_n\right|$$

assumed finite. Then it is easily verified that

$$x,\; y \,\epsilon\, Q^{\alpha,\delta} \;\Rightarrow x+y \,\epsilon\, Q^{\alpha,\delta}$$

For

$$\begin{aligned}
h(\lambda x+\mu y) &\le |\lambda|\sum\left|A_n^{\alpha-1,\delta}\Delta^{\alpha,\delta}x_n\right| + |\mu|\sum\left|A_n^{\alpha-1,\delta}\Delta^{\alpha,\delta}x_n\right| \\
&\le |\lambda|\, h(x) + \;|\mu|\, h(y)
\end{aligned}$$

Thus $Q^{\alpha,\delta}$ is a linear space. Since

$$h(x) = 0 \Rightarrow \Delta^{\alpha,\delta}x_n = 0 \Rightarrow x_n = \Delta^{-\alpha,-\sigma}\left(\Delta^{\alpha,\delta}x_n\right) = 0$$

It follows that h is a norm. Before we consider the space $Q^{\alpha,\beta,\delta}$, we need to note the following result:

Lemma 3.4. *Let* $x \in l_\infty$*. Let* $\alpha > 0,\ \delta$ *real or* $\alpha = 0,\ \delta > 0$*. Then* $\Delta^{\alpha,\ \delta}x_n$ *exists.*

Proof.

$$\begin{aligned}
\Delta^{\alpha,\delta}x_n &= \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1,-\delta}x_v \\
&= 0(1)\begin{cases} \sum_{v=n}^{\infty}(\alpha-n+1)^{-(\alpha+1)}(log\,(v-n+1))^{-\beta} & \alpha \ne 0,\ 1,\ 2,\ \ldots \\ \sum_{v=n}^{\infty}(\alpha-n+1)^{-(\alpha-1)}(log\,(v-n+2))^{-\delta-1} & \alpha = 0,\ 1,\ 2,\ \ldots \end{cases} \\
&= \begin{cases} 0\,(1) & if\ \alpha > 0, \quad any\ \delta \\ 0\,(1) & if\ \alpha = 0, \quad \delta > 0 \end{cases}
\end{aligned}$$

This completes the proof. □

Lemma 3.5. *Let $x_n = o(1)$. Then*

(i) $x_n = \Delta^{-\alpha,-\delta}(\Delta^{\alpha,\delta} x_n)$

(ii) $x_n = \Delta^{\alpha,\delta}(\Delta^{-\alpha,-\delta} x_n)$

Provided that $0 < \alpha < 1,\ \delta$ *real or* $\alpha = 0,\ \delta > 0$.

Proof. Now

$$\Delta^{\alpha,\delta}\left(\Delta^{-\alpha,-\delta} x_n\right) = \lim_{N\to\infty} \sum_{v=n}^{N} A_{v-n}^{-\alpha-1,-\delta} \left(\sum_{r=v}^{N} + \sum_{r=N+1}^{\infty}\right) A_{r-v}^{\alpha-1,\delta} x_r$$
$$= x_n + \lim_{N\to\infty} \sum_{v=n}^{N} A_{v-n}^{-\alpha-1,-\beta} \sum_{r=N+1}^{\infty} A_{r-v}^{\alpha-1,\delta} x_r$$

Now since

$$\sum_{r=N+1}^{\infty} A_{r-v}^{\alpha-1} x_r = o(1)$$

as $N \to \infty$, it follows that

$$\sum_{v=n}^{N} A_{v-n}^{-\alpha-1,-\delta} \sum_{r=N+1}^{\infty} A_{r-v}^{\alpha-1,\delta} x_r = o\,(1) \sum_{v=n}^{N} \left|A_{v-n}^{-\alpha-1,-\delta}\right|$$
$$= \begin{cases} o(1) \sum_{v=n}^{N} (v-n+1)^{-(\alpha-1)} (log\,(v-n+2))^{-\delta}, & \alpha > 0,\ \beta\ \ real \\ o(1) \sum_{v=n}^{\infty} (v-n+1)^{-1} (log\,(v-n+2))^{-\beta-1}, & \alpha = 0,\ \beta > 0 \end{cases}$$
$$= o\,(1)$$

This completes the proof of the first part. The proof of the next part is similar. □

We now prove

Theorem 3.6. $Q^{\alpha,\delta} \cap c_0$ *is Banach space for* $\alpha > 0,\ \delta$ *real or* $\alpha = 0,\ \delta > 0$ *normed by* h *defined above.*

Proof. It is enough to show that it is complete. Let x^s be a Cauchy sequences in $Q^{\alpha,\delta} \cap c_0$. Then

$$h\left(x^s - x^t\right) = \sum_{n=0}^{\infty} \left|A_n^{\alpha-1,\delta} \Delta_n^{\alpha,\delta} \left(x_n^s - x_n^t\right)\right| \to 0 \quad as \quad s,t \to \infty$$

Now

$$x_n^s - x_n^t = \Delta^{-\alpha,-\delta} \Delta^{\alpha,\delta} \left(x_n^s - x_n^t\right)$$
$$= \sum_{v=n}^{\infty} A_{v-n}^{\alpha-1,\delta} \Delta^{\alpha,\delta} \left(x_v^s - x_v^t\right)$$

Hence

$$\left|x_n^s - x_n^t\right| \le \sum_{v=n}^{\infty} \left|A_{v-n}^{\alpha-1,\delta} \Delta^{\alpha,\delta} \left(x_v^s - x_v^t\right)\right|$$
$$= h\left(x^s - x^t\right) \to 0 \quad as \quad s,t \to \infty$$

Hence $(x_n^s)_{s=0}^{\infty}$ is a Cauchy sequence in c. Therefore there exists $x \in c$ such that

$$(x_n^s) \to (x_n) = x \quad (say) \quad as\ s \to \infty.$$

Now

$$\begin{aligned}\lim_{t\to\infty} \Delta^{\alpha,\delta}\left(x_n^s - x_n^t\right) &= \lim_{t\to\infty} \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1,\delta}\left(x_v^s - x_v^t\right) \\ &= \sum_{v=n}^{\infty} \lim_{t\to\infty} A_{v-n}^{-\alpha-1,\delta}\left(x_v^s - x_v^t\right) \\ &= \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1,\delta}\left(x_v^s - x_v\right) \\ &= \Delta^{\alpha,\beta}\left(x_n^s - x_n\right)\end{aligned}$$

It follows from this that given $\epsilon > 0$, there exists s_0 such that $h\left(x^s - x\right) \leq \in,\ s > s_0$. This proves that $(x^s - x) \in Q^{\alpha,\delta}$. This implies that $x = x - x^s + x^s \in Q^{\alpha,\delta}$. This proves the complete of $Q^{\alpha,\delta} \cap c_0$. □

4. Matrix Transformation

The following lemma is key to the development of this section.

Lemma 4.1. *Let $x\ \epsilon\ c$ and $0 < \alpha \leq 1,\ \delta$ real or $\alpha = 0,\ \delta > 0$. Then*

$$A_n\left(x\right) = \sum_{k=0}^{\infty} a_{nk}x_k = \left(\lim_k x_k\ \right)\sum_{k=0}^{\infty} a_{nk} + \sum_{v=0}^{\infty} \Delta^{\alpha,\delta}\left(x_v\right)\sum_{k=0}^{\infty} A_{v-k}^{\alpha-1,\delta}a_{nk}$$

In the sense that if the sum of two series exist then the sum of third will also exist and the equality holds.

Proof. Suppose that $x_n \to s$. Then we can write $x_n = s + \epsilon_n$, when $\epsilon_n \to 0$. If $A_n\left(x\right)$ exists for each $n \geq 0$, we can write

$$A_n\left(x\right) = s\sum_{k=0}^{\infty} a_{nk} + \sum_{k=0}^{\infty} a_{nk}\epsilon_k$$

Now using the result :

$$\epsilon_k = \sum_{v=k}^{\infty} A_{v-k}^{\alpha-1,\delta}\Delta^{\alpha,\delta}\left(\epsilon_v\right)$$

We obtain

$$\sum_{k=0}^{\infty} a_{nk}\epsilon_k = \sum_{v=0}^{\infty} \Delta^{\alpha,\delta}\left(\epsilon_v\right)\sum_{k=0}^{v} A_{v-k}^{\alpha-1,\delta}a_{nk}$$

But

$$\begin{aligned}\Delta^{\alpha,\delta}\left(x_v\right) &= \sum_{n=v}^{\infty} A_{n-v}^{-\alpha-1,-\delta}\left(x_v\right) \\ &= s\sum_{n=v}^{\infty} A_{n-v}^{-\alpha-1,-\delta} + \sum_{n=v}^{\infty} A_{n-v}^{-\alpha-1,-\beta}\epsilon_n \\ &= \Delta^{\alpha,\beta}\epsilon_v.\end{aligned}$$

This proves the result. □

Note: Since this equality is applicable in what follows, it is therefore always assumed that $0 < \alpha \leq 1,\ \delta$ real or $\alpha = 0,\ \delta > 0$.

Lemma 4.2. *$\sum_{k=1}^{\infty} ka_k$ is convergent if and only if $\sum_{k=1}^{\infty} R_k$ is convergent with $nR_n \to 0$, where $R_n = \sum_{k=n+1}^{\infty} a_k$.*

Theorem 4.3. *Let $\alpha \geq 0$, $\beta \geq 0$. Then $A \in \left(l_\infty(\Delta),\ b^{\alpha,\beta,\delta}\right)$ if and only if A satisfies the following conditions:*

(i) $\sup_n \left[\sum_{k=1}^{\infty} k\,|a_{nk}|\right] < \infty$

(ii) $\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty}\delta_n^{\alpha,\delta}a_{nj}\right| < \infty$, *where* $\Delta_n^{\alpha,\delta}a_{nk} = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1,\delta}a_{vk}$.

Proof. Suppose condition (i) and (ii) hold. Let $x \in l_\infty(\Delta)$ with $x_1 = 0$, then there exists one and only one $y = (y_k) \in l_\infty$, such that

$$x_k = -\sum_{v=1}^{k} y_{v-1},\ y_0 = 0$$

Let $n \in N$ be fixed and let $m \in N$. Then by using condition (i) and Lemma 3.4 we may write

$$\sum_{k=1}^{m} a_{nk}x_k = -\sum_{k=1}^{m-1}\left(\sum_{j=k+1}^{\infty} a_{mj}\right)y_k + \sum_{j=n+1}^{\infty} a_{nj}\sum_{k=1}^{m-1} y_k$$

Further the second term in the right hand side of the above equation tends to zero as m tends to infinity and the first term converges. Therefore $\sum_{k=1}^{m} a_{nk}x_k$ converges and has the same sum as that of $-\sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} a_{nj}\right)\Delta x_k$ and therefore we have

$$\sum_{k=1}^{\infty} a_{nk}x_k = -\sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} a_{nj}\right)\Delta x_k$$

This gives

$$\begin{aligned}\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}A_n(x)\right| &= \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}\sum_{k=1}^{\infty} a_{nk}x_k\right| \\ &= \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}\left(-\sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} a_{nj}\right)\Delta x_k\right)\right| \\ &\leq K\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}\sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} a_{nj}\right)\right| < \infty, \text{ where } \quad K = \sup_k|\Delta x_k|\end{aligned}$$

Conversely, since $A \in \left(l_\infty(\Delta),\ b^{\alpha,\beta,\delta}\right)$ (i) follows by the fact that $\sup_n\left|\sum_{k=1}^{\infty} a_{nk}x_k\right| < \infty$ for each $x \in l_\infty(\Delta)$. In particular, by putting $(x_k) = (k)$ we get condition (i). Also by condition (i) and Lemma 3.4, we have

$$\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}A_n(x)\right| = \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}\left(-\sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} a_{n??}\right)\Delta x_k\right)\right|$$

Converges whenever $x \in l_\infty(\Delta)$. Now putting $x = (x_k) = (k)$ on the right hand side of the above equation we get

$$\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty}\Delta_n^{\alpha,\delta}a_{nj}\right)\right| < \infty.$$

□

Theorem 4.4. *Let $0 < \alpha \leq 1$, $\delta > 0$ and $\beta = 0$ and $p \in l_\infty$. Then $A \in \left(l_\infty(\Delta),\ b^{\alpha,\beta,\delta}\right)$ If and only if A satisfies the following conditions:*

(i) $\sum\limits_{k=1}^{\infty} a_{nk}$ *converges for each* n *and* $\sup_n \left|\sum\limits_{k=1}^{\infty} a_{nk}\right|^{p_n} < \infty$

(ii) $\sup_{n,v} \left|\left(A_v^{\alpha+\beta-1,\delta}\right)^{-1} \sum\limits_{k=1}^{v} a_{nk} A_{v-k}^{\alpha-1,\delta}\right|^{p_n} < \infty.$

Proof. Let $x \in q^{\alpha,\beta,\delta}$ with $\lim_n x_n = s$, we have to show that $\sum\limits_{k=1}^{\infty} a_{nk}x_k$ converges for all n and

$$\sup_n \left|\sum_{k=1}^{\infty} a_{nk}x_k\right|^{p_n} < \infty$$

Since $p \in l_\infty$, it is sufficient to prove the theorem for $p_n \leq 1$ for all n. Let $n \in N$, using Lemma 4.1,

$$A_n(x) = \left(\lim x\right) \sum_{k=0}^{\infty} a_{nk} + \sum_{v=0}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=0}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}$$

Since condition (i) holds, $A_n(x)$ converges if and only if $\sum\limits_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum\limits_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}$ converges. Now

$$\begin{aligned}
\sum_{v=0}^{\infty} \left|\Delta^{\alpha,\delta} x_v \sum_{k=0}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right| &\leq \sum_{v=0}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| A_v^{\alpha+\beta-1,\delta} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \left|\sum_{k=0}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right| \\
&\leq \left(\sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| A_v^{\alpha+\beta-1}\right) \left(\mathop{sup}_{x,v} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \left|\sum_{k=0}^{v} A_{v-k}^{\alpha-1} a_{nk}\right|\right) \\
&\leq K \sum_{v=0}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| A_v^{\alpha+\beta-1,\delta} < \infty.
\end{aligned}$$

Therefore the series converges absolutely and hence converges. This gives $\sum\limits_{k=1}^{\infty} a_{nk}x_k$ convergent for each $n \in N$. Now we have

$$\begin{aligned}
\sup_n |A_n(x)|^{p_n} &= \sup_n \left|\left(\lim x\right) \sum_{k=1}^{\infty} a_{nk} + \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right|^{p_n} \\
&\leq |s| \, |a_{nk}|^{p_k} + \sup_n \left|\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right|^{p_n} \\
&\leq \sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| A_v^{\alpha+\beta-1,\delta} \sup_{n,v} \left(\frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1} |a_{nk}|\right)^{p_n} + K
\end{aligned}$$

Since $x \in q^{\alpha,\beta,\delta}$ and condition (ii) holds,

$$\mathop{sup}_n |A_n(x)|^{p_n} < \infty$$

For necessity we are given that $sup_n |A_n(x)|^{p_n} < \infty$ whenever $x \in q^{\alpha,\beta,\delta}$. Since $x = e \in \left(q^{\alpha,\beta,\delta}\right)$ condition (i) follows. Using Lemma 4.1

$$\sup_n |A_n(x)|^{p_n} < \infty \Rightarrow \sup_n \left|\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right|^{p_n} < \infty$$

Or we can write

$$\sup_n \left|\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v A_v^{\alpha+\beta-1,\delta} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right|^{p_n} < \infty \tag{8}$$

put $y_v = A_v^{\alpha+\beta-1,\delta} \Delta^{\alpha,\delta} x_v$ and

$$b_{nv} = \begin{cases} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum\limits_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}, & v \leq n \\ 0, & v > n \end{cases}$$

Since by hypothesis $\sum |y_v| < \infty$ it follows from Maddox [4–6] that

$$\mathop{sup}_{n,v} |b_{nv}|^{p_n} \leq K$$

This completes the proof. □

Corollary 4.5. *Let $0<\alpha\leq 1,\ \beta\geq 0$ and $\delta>0$. Then $A\in\left(q^{\alpha,\beta,\delta},\ l_\infty(p)\right)$ if and only if*

$$\sum A\in\left(q^{\alpha,\beta,\delta},\ l_\infty(p)\right)\ \text{or}\ \sum A$$

satisfies (i) and (ii) of Theorem 4.4.

Theorem 4.6. *Let $0<\alpha\leq 1$ and δ real. Then $A\in\left(q^{\alpha,\beta,\delta},\ l_1\right)$ if and only if A satisfies the following conditions :*

(i) $\sum\limits_{k=1}^{\infty}\left|\sum_{k=1}^{\infty}a_{nk}\right|$ *is convergent,*

(ii) $\sum\limits_{n=1}^{\infty}\sup\limits_{v}\left|\left(A_v^{\alpha+\beta-1,\delta}\right)^{-1}\sum\limits_{k=1}^{v}a_{nk}A_{v-k}^{\alpha-1,\delta}\right|<\infty$

Theorem 4.7. *Let $0<\alpha\leq 1,\ \beta\geq 0$ and δ real. Then $A\in\left(q^{\alpha,\beta,\delta},\ q^{\alpha,\beta,\delta}\right)$ if and only if A satisfies the following conditions:*

(i) $\lim\limits_{n\to\infty}a_{nk}$ *exists, for each fixed k,*

(ii) $\lim\limits_{n\to\infty}\left(\sum\limits_{k=1}^{\infty}a_{nk}\right)$ *exists,*

(iii) $\left(\sum\limits_{k=1}^{\infty}a_{nk}\right)_{n=1}^{\infty}\in\left(q^{\alpha,\beta,\delta}\right)$,

(iv) $\sum\limits_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\sum\limits_{k=1}^{\infty}\left(\sum\limits_{j=1}^{\infty}\Delta_n^{\alpha,\delta}a_{nj}\right)\right|$ *is convergent.*

Proof. Let $A\in\left(q^{\alpha,\beta,\delta},\ q^{\alpha,\beta,\delta}\right)$. Conditions (i) and (ii) follow from the fact that $x=e\in\left(q^{\alpha,\beta,\delta}\right)$ and $z=e\in\left(q^{\alpha,\beta,\delta}\right)$ for each k. Now since

$$\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}A_n(x)\right|<\infty,\ \textit{whenever}\ x\in\left(q^{\alpha,\beta,\delta}\right),$$

Taking $x=e\in\left(q^{\alpha,\beta,\delta}\right)$, in the above inequality we get condition (ii). Further, summing by parts, we have for $n\in N$

$$\sum_{k=1}^{\infty}a_{nk}x_k=\sum_{k=1}^{n-1}\left(\sum_{j=1}^{k}a_{nj}\right)\Delta x_k+\left(\sum_{j=1}^{k}a_{nj}\right)x_n.$$

Now, if $x\in\left(q^{\alpha,\beta,\delta}\right)$, then $x\in c$, let $\lim\limits_{n\to\infty}x_n=s$. Then taking $n\to\infty$ in the above equation we get

$$\sum_{k=1}^{\infty}a_{nk}x_k=\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_{nj}\right)$$

Therefore, using (iii) there exists $K_1>0$ such that

$$\begin{aligned}\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha}\sum_{k=1}^{\infty}a_{nk}x_k\right| &= \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_{nj}\right)\Delta x_k+s\left(\sum_{j=1}^{\infty}a_{nj}\Delta_n^{\alpha,\delta}\right)\right|\\ &\geq \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_{nj}\right)\Delta x_k\right|-s\sum_{n=1}^{\infty}A_n^{\alpha+\beta-1,\delta}\left|\left(\sum_{j=1}^{\infty}\Delta_n^{\alpha}a_{nj}\right)\right|\\ &\geq \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_{nj}\right)\Delta x_k\right|-K_1.\end{aligned}$$

This gives

$$\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\Delta_n^{\alpha,\delta}A_n(x)\right|+K_1\geq\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\Delta_n^{\alpha,\delta}a_{nj}\right)\Delta x_k\right|$$

Since the left hand side of (**??**) is bounded, so is the right hand side. Write

$$D_{nk} = \begin{cases} A_v^{\alpha+\beta-1,\delta} \sum_{j=1}^{k}, & k \leq n \\ 0, & k > n \end{cases}$$

Since $\Delta x_k \in \left(q^{\alpha,\beta,\delta}\right)$, $D_{nk} \in \left(q^{\alpha,\beta,\delta},\ l_1\right)$ in view of (**??**). Now condition (iv) follows from Theorem 4.6. On the other hand, let the condition (i)-(iv) hold. Then, it is easy to see that $(A_n(x)) \in c$ whenever $x \in \left(q^{\alpha,\beta,\delta}\right)$ and $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} a_k.s$ where $a_k = \lim_{n\to\infty} a_{nk}$ and $s = \lim_{n\to\infty} x_k$. Summing by parts and using (1), we have

$$\sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta} a_{nk}x_k \sum_{k=1}^{\infty} \Delta_n^{\alpha,\delta} a_{nk}x_k\right| \leq \sum_{n=1}^{\infty}\left|A_n^{\alpha+\beta-1,\delta}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\Delta_n^{\alpha,\delta}a_{nj}\right)\right|.$$

Then by using conditions (iii) and (iv), it follows that $A \in \left(q^{\alpha,\beta,\delta},\ q^{\alpha,\beta,\delta}\right)$. This completes the proof. □

Theorem 4.8. *Let $0 < \alpha \leq 1$, $\beta \geq 0$ and $\delta > 0$ and $p \,\epsilon\, l_\infty$. Then $A \in \left(q^{\alpha,\beta,\delta},\ c_0(p)\right)$ if and only if A satisfies the following conditions :*

(i) $\lim_{n\to\infty} |a_{nk}|^{p_n} = 0$ *for all k,*

(ii) $\sum_{k=1}^{\infty} a_{nk}$ *converges for each n and* $\lim_{n\to\infty}\left|\sum_{k=1}^{\infty} a_{nk}\right|^{p_n} = 0,$

(iii) $\lim_{n\to\infty}\sup_{v}\left|\frac{1}{A_v^{\alpha+\beta-1,\delta}}\sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right|^{p_n} = 0.$

Proof. Let conditions (i)-(iii) hold. Let $x \in \left(q^{\alpha,\beta,\delta}\right)$ and $s = \lim_k x_k$. since $\sup_k P_k < \infty$ it is sufficient to prove that result for $p_k \leq 1$. For any $n,\ m \in N$ we have

$$\sum_{k=1}^{m} a_{nk}x_k = x_m\sum_{k=1}^{m} a_{nk} + \sum_{v=1}^{m}\Delta^{\alpha,\delta}x_v\sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta}a_{nk}.$$

Let $k = \sup_n \sup_v = \left|\frac{1}{A_v^{\alpha+\beta-1,\delta}}\sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right|^{p_n}$, in view of (iii), $k < \infty$. For each fixed $n \in N$

$$\begin{aligned}\sum_{v=1}^{\infty}\left|\Delta^{\alpha,\delta}x_v\sum_{k=1}^{v}A_{v-k}^{\alpha-1,\delta}a_{nk}\right| &\leq \sum_{v=1}^{\infty}\left|\Delta^{\alpha,\delta}x_v A_v^{\alpha+\beta-1,\delta}\right|\left|\sum_{k=1}^{v}A_{v-k}^{\alpha-1,\delta}a_{nk}\right| \\ &\leq \sum_{v=1}^{\infty}\left|\Delta^{\alpha,\delta}x_v\right|\left|A_v^{\alpha+\beta-1,\delta}\right| \mathop{sup}_{n}\frac{1}{\left|A_v^{\alpha+\beta-1,\delta}\right|}\left|\sum_{k=1}^{v}A_{v-k}^{\alpha-1,\delta}a_{nk}\right| \\ &\leq (k)^{\frac{1}{P_n}}\sum_{v=1}^{\infty}\left|\Delta^{\alpha,\delta}x_v\right|\left|A_v^{\alpha+\beta-1,\delta}\right| < \infty.\end{aligned}$$

Since $x \in \left(q^{\alpha,\beta,\delta}\right)$, we have

$$\left(\sum_{v=1}^{\infty}\Delta^{\alpha,\delta}x_v\sum_{k=1}^{v}A_{v-k}^{\alpha-1,\delta}a_{nk}\right)$$

converges absolutely and hence converges. This gives

$$\sum_{k=1}^{\infty}a_{nk}x_k$$

converges for each n. Again from conditions (i)- (iii), Lemma 4.1 and the observations that $p_n \le 1$ for each n we have

$$\begin{aligned}
\lim_{n\to\infty} |A_n(x)|^{p_n} &= \lim_{n\to\infty} \left| s\sum_{k=1}^{\infty} a_{nk} + \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} \\
&\le \lim_{n\to\infty} \left| s\sum_{k=1}^{\infty} a_{nk} \right|^{p_n} + \lim_{n\to\infty} \left| \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} \\
&\le \lim_{n\to\infty} \sum_{k=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| A_v^{\alpha+\beta-1,\delta} \left(\left| \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \right| a_{nk} \right)^{p_n} \\
&\le \sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| A_v^{\alpha+\beta-1} \lim_{n\to\infty} \sup_v \left(\left| \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \right| a_{nk} \right)^{p_n} = 0.
\end{aligned}$$

Hence $\lim_{n\to\infty} |A_n(x)|^{p_n} = 0$. Conversely, since $A \in \left(q^{\alpha,\beta,\delta},\ c_0(p)\right)$ and $x = e \in \left(q^{\alpha,\beta,\delta}\right)$, we have condition (ii). Again $x = e \in \left(q^{\alpha,\beta,\delta}\right)$ for each k gives condition (i). Now to prove condition (iii), we have

$$\begin{aligned}
|A_n(x)|^{p_n} &= \left| \left(\lim_k x_k\right) \sum_{k=1}^{\infty} a_{nk} + \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} \\
&= \left| s\sum_{k=1}^{\infty} a_{nk} + \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} \\
&\ge \left| \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} - \left| s\sum_{k=1}^{\infty} a_{nk} \right|^{p_n}.
\end{aligned}$$

$$\Rightarrow \left| \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} \le |A_n(x)|^{p_n} + \left| s\sum_{k=1}^{\infty} a_{nk} \right|^{p_n}.$$

So

$$\lim_{n\to\infty} \left| \sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v A_v^{\alpha+\beta-1,\delta} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{p_n} = 0$$

Let

$$b_{nv} = \begin{cases} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}, & v \le n \\ 0, & v > n \end{cases}$$

and $y_v = A_v^{\alpha+\beta-1,\delta} \Delta^{\alpha,\delta} x_v$. Then

$$\lim_{n\to\infty} \left| \sum_{v=1}^{\infty} b_{nv}\, y_v \right|^{p_n} = 0.$$

since there is a one-to-one correspondence between l_1 and $c \cap Q^{\alpha,\beta,\delta}$, we have $B = (b_{nv}) \in (l_1, c_0(p))$. Therefore, condition (iii) holds. □

Corollary 4.9. *Let* $0 < \alpha \le 1,\ \beta \ge 0$ *and* $\delta \ge 0$. *Then* $A \in \left(q^{\alpha,\beta,\delta},\ \tau\right)$ *if and only if A satisfies the following conditions:*

(i) $\lim_{n\to\infty} |a_{nk}|^{\frac{1}{n}} = 0, \quad (k = 1,\ 2,\ \dots)$

(ii) $\sum_{k=1}^{\infty} a_{nk}$ *converges for each n and* $\lim_{n\to\infty} \left| \sum_{k=1}^{\infty} a_{nk} \right|^{\frac{1}{n}} = 0,$

(iii) $\sup_{n,v} \left| \frac{1}{A_v^{\alpha+\beta-1}} \sum_{k=1}^{v} a_{nk} A_{v-k}^{\alpha-1} \right|^{\frac{1}{n}} < \infty$ *and* $\lim_{n\to\infty} \sup_v \left| \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk} \right|^{\frac{1}{n}} = 0.$

Corollary 4.10. *Let* $0 < \alpha \le 1,\ \beta \le 0$ *and* $\delta \ge 0$. *Then* $A \in \left(q^{\alpha,\beta,\delta},\ \wedge\right)$ *if and only if A satisfies the following conditions:*

(i) $\sup_n |a_{nk}| < \infty, \quad (k = 1,\ 2,\ \dots),$

(ii) $\sup_n \left|\sum_{k=1}^{\infty} a_{nk}\right|^{\frac{1}{n}} < \infty$,

(iii) $\sup_{n,v} \left|\frac{1}{A_v^{\alpha+\beta-1}} \sum_{k=1}^{v} a_{nk} A_{v-k}^{\alpha-1}\right|^{\frac{1}{n}} < \infty$.

Proof. It can be worked out on the lines of Theorem 4.4. □

Theorem 4.11. *Let $0 < \alpha \leq 1$, $\beta \geq 0$ and $\delta \geq 0$. Then $A \in \left(q^{\alpha,\beta,\delta},\ c_0(\Delta)\right)$ if and only if A satisfies the following conditions:*

(i) $A_1 = a_{1k}$ belongs to the β-dual of $q^{\alpha,\beta,\delta}$,

(ii) $\sum_{k=1}^{\infty} a_{nk}$ converges for all n, or equivalently

(iii) (a) $\sum_{k=1}^{\infty} a_{1k}$ converges and (b) $\sum_{k=1}^{\infty} \Delta a_{nk}$ converges for all n,

(iv) $\lim_{n\to\infty} \delta a_{nk} = 0$ for every k and $\lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \Delta a_{nk}\right) = 0$,

(v) $\sup_{n,v} \left|\frac{1}{A_v^{\alpha+\beta-1}} \sum_{k=1}^{v} \Delta a_{nk}\right| < \infty$, where $\Delta a_{nk} = (a_{nk} - a_{n+1,\ k})$

Proof. Suppose conditions (i)-(iv) hold and let $x \in q^{\alpha,\beta,\delta}$. We first prove that $B = (\Delta a_{nk}) \in \left(q^{\alpha,\beta},\ c_0\right)$. Since A_1 belongs to the β-dual of $q^{\alpha,\beta,\delta}$. We can say that $A \in \left(q^{\alpha,\beta,\delta},\ c_0(\Delta)\right)$. Now using Lemma 3.4 and condition (iii) it is sufficient to prove that for each n

$$\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk} \text{ converges and } \lim_{n\to\infty} \left|\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right| = 0.$$

Since condition (iv) holds therefore there exist a constant K such that

$$\begin{aligned}
\sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right| &\leq \sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| \left|\sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right| \\
&= \sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v\right| \left|A_v^{\alpha+\beta-1,\delta}\right| \frac{1}{A_v^{\alpha+\beta-1,\delta}} \left|\sum_{k=1}^{\infty} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right| \\
&\leq \sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v A_v^{\alpha+\beta-1,\delta}\right| \sup_v \frac{1}{\left|A_v^{\alpha+\beta-1,\delta}\right|} \left|\sum_{k=1}^{\infty} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right| \\
&\leq k \sum_{v=1}^{\infty} \left|\Delta^{\alpha,\delta} x_v A_v^{\alpha+\beta-1,\delta}\right| < \infty.
\end{aligned}$$

This gives $\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}$ converges absolutely an d hence converges for each n. Now using condition(iii) we have

$$\lim_{n\to\infty} \left|\sum_{v=1}^{\infty} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right| = 0.$$

Conversely, let $A \in \left(q^{\alpha,\beta,\delta},\ c_0(\Delta)\right)$. Since $x = e \in \left(q^{\alpha,\beta,\delta}\right)$. We have $\left(\sum_{k=1}^{\infty} a_{nk}\right) \in c_0(\Delta)$. So for all n, $\sum_{k=1}^{\infty} \Delta a_{nk}$ converges and $\lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \Delta a_{nk}\right) = 0$. Again since for each k, $x = e_k \in \left(q^{\alpha,\beta,\delta}\right)$, $(a_{nk})_{n=1}^{\infty} \in c_0(\Delta)$ for each k. This gives $\lim_{n\to\infty} \Delta a_{nk} = 0$ which are conditions (ii) and (iii). Condition (i) also holds. Now it is left to prove condition (iv). Using Lemma 4.1 and condition (iii) we get

$$\begin{aligned}
\lim_{n\to\infty} \left(A_n(x) - A_{n+1}(x)\right) &= \lim_{n\to\infty} \left(\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} \Delta a_{nk}\right) \\
&= \lim_{n\to\infty} \left(\sum_{v=1}^{\infty} \Delta^{\alpha,\delta} x_v A_v^{\alpha+\beta-1,\delta} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}\right).
\end{aligned}$$

Let

$$b_{nv} = \begin{cases} \frac{1}{A_v^{\alpha+\beta-1,\delta}} \sum_{k=1}^{v} A_{v-k}^{\alpha-1,\delta} a_{nk}, & v \leq n \\ 0, & v > n. \end{cases}$$

and $y_v = A_v^{\alpha+\beta-1,\delta} \Delta^{\alpha,\delta} x_v$. Then

$$\lim_{n\to\infty} \sum_{v=1}^{\infty} b_{nv} = \lim_{n\to\infty} \left(A_n(x) - A_{n+1}(x)\right) = 0 \text{ whenever } \sum_{v=1}^{\infty} |\, y_v| < \infty.$$

Since there is a one-to-one matrix correspondence between l_1 and $q^{\alpha,\beta,\delta}$, we have $B = (b_{nv}) \in (l_1, c_0(\Delta))$. Therefore condition (iv) also holds. □

Theorem 4.12. *Let $\alpha \geq 0,\ \beta \geq 0$ and $\delta > 0$. Then $A \in \left(q^{\alpha,\beta,\delta},\ c\right)$ if and only if A satisfies the following conditions:*

(i) $\sup_n \sum_{v=1}^{\infty} |a_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} a_{nk}$ *exists,*

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}$ *exists,*

(iv) $\sum_{k=1}^{\infty} \left| A_v^{\alpha+\beta-1,\delta} \sum_{k=1}^{v} \Delta_n^{\alpha,\delta} a_{nk} \right|$ *is convergent.*

Proof. The conditions (i)-(iii) are necessary since $A \in (c,\ c)$. It remains to show that (iv) is also necessary. Note that

$$\sum_{k=1}^{\infty} \left| A_v^{\alpha+\beta-1,\delta} \sum_{k=1}^{v} \Delta_n^{\alpha,\delta} a_{nk} \right| < \infty,$$

whenever $x \in c$. Therefore, by taking $x = e \in c$, we get (iv). On the other hand, let the conditions (i)-(iv) hold. Note that by conditions (i)-(iii) we have $A_n(x) \in c$ whenever $x \in c$. Further, since $x = e \in c$ there exists $k_1 > 0$ such that in view of condition (iv), we have

$$\sum_{n=1}^{\infty} \left| A_n^{\alpha+\beta-1,\delta} \Delta_n^{\alpha,\delta} \sum_{k=1}^{\infty} a_{nk} \right| = k_1 \sum_{n=1}^{\infty} \left| A_n^{\alpha+\beta-1,\delta} \Delta_n^{\alpha,\delta} \sum_{k=1}^{\infty} a_{nk} x_k \right| < \infty$$

□

Theorem 4.13. *Let $\alpha \geq 0,\ \beta \geq 0$ and $\delta > 0$. Then $A \in \left(q^{\alpha,\beta,\delta},\ c_0\right)$ if and only if A satisfies conditions (i), (ii) and (iv) of Theorem 4.12.*

Proof. It is similar to that of Theorem 4.12. □

Theorem 4.14. *Let $\alpha \geq 0,\ \beta \geq 0$ and $\delta > 0$. Then $A \in \left(q^{\alpha,\beta,\delta},\ \tau\right)$ if and only if A satisfies the condition (iv) of Theorem 4.12 together with the following conditions:*

(i) $|a_{nk}|^{\frac{1}{k}} \leq$ *for every n, k,*

(ii) $\lim_{n\to\infty} a_{nk} = a_k$ *for every n, k.*

Proof. Conditions (i) and (ii) are true since $A \in (\tau, c)$. Using the arguments of Theorem 4.12, we have condition (iv) of Theorem 4.12. Similarly we have the converse implication. □

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