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Snakes Related Sum Perfect Square Graphs

Research Article

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Abstract: Let $G = (V, E)$ be a simple (p, q) graph and $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ be a bijection. We define $f^* : E(G) \to \mathbb{N}$ by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(G)$. If f^* is injective, then f is called sum perfect square labeling. A graph which admits sum perfect square labeling is called sum perfect square graph. In this paper we prove that several snakes related graphs are sum perfect square.

MSC: 05C78.

Keywords: Sum perfect square graphs, triangular snakes, quadrilateral snakes.

1. Introduction

We consider simple, finite, undirected graph $G = (p, q)$ (with p vertices and q edges). The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. For all other terminology and notations we follow Harary[1]. Sonchhatra and Ghodasara[5] have initiated study of sum perfect square graphs in 2016. Due to [5] it becomes possible to construct a graph, whose all edges can be labeled by perfect square integers only. In [5] we have proved P_n, C_n, cycle C_n with one chord, cycle C_n with twin chords, tree, $K_{1,n}$, $T_{m,n}$ are sum perfect square graphs. In this paper we prove that several snakes related graphs are sum perfect square.

1.1. Definitions

Definition 1.1 ([5]). *Let $G = (p, q)$ be a graph. A bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ is called sum perfect square labeling of G, if the induced function $f^* : E(G) \to \mathbb{N}$ given by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$ is injective, for all $u, v \in V(G)$. A graph which admits sum perfect square labeling is called sum perfect square graph.*

Definition 1.2 ([4]). *The triangular snakes T_n, $n > 1$ are obtained by replacing each edge of path P_n by a triangle C_3.*

Definition 1.3 ([4]). *The double triangular snakes $D(T_n)$, $n > 1$ are obtained by replacing each edge of P_n by two triangles C_3.*

Definition 1.4 ([4]). *The alternating triangular snakes $A(T_n)$, $n > 1$ are obtained from a path $\{v_1, v_2, \ldots, v_n\}$ by joining v_i and v_{i+1} (alternatively) to new vertex u_i. i.e. every alternate edge of a path is replaced by C_3.*

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Definition 1.5 ([4]). *The double alternating triangular snakes $DA(T_n)$ are obtained from a path $\{v_1, v_2, \ldots, v_n\}$ by joining v_i and v_{i+1} (alternatively) to new vertices u_i and w_i. i.e. the double alternating triangular snakes $DA(T_n)$ consists of two alternative triangular snakes that have a common path.*

Definition 1.6 ([6]). *The alternating quadrilateral snakes $A(Q_n)$ are obtained from a path $\{v_1, v_2, \ldots, v_n\}$ by joining v_i and v_{i+1} (alternatively) to new vertices u_i and w_i, in which each u_i and w_i are also joined by an edge. i.e. every alternate edge of a path is replaced by C_4.*

Definition 1.7 ([6]). *The double alternating quadrilateral snakes $DA(Q_n)$ are obtained from a path $\{v_1, v_2, \ldots, v_n\}$ by joining v_i and v_{i+1} (alternatively) to new vertices u_i, w_i and u'_i, w'_i, in which each u_i is adjacent to w_i and each u'_i is adjacent to w'_i, for $1 \le i \le \lfloor \frac{n}{2} \rfloor$. i.e. double alternating quadrilateral snake $DA(Q_n)$ consists of two alternating quadrilateral snakes that have a common path.*

2. Main Results

Theorem 2.1. *T_n are sum perfect square graphs, $\forall n \in \mathbb{N}$.*

Proof. Let $V((T_n)) = \{v_i; 1 \le i \le n\} \cup \{v'_i; 1 \le i \le n-1\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and v'_i is adjacent with v_i and v_{i+1}, for $1 \le i \le n-1$, $E(T_n) = \{e_i = v_iv_{i+1}; 1 \le i \le n-1\} \cup \{e_i^{(1)} = v_iv'_i; 1 \le i \le n-1\} \cup \{e_i^{(2)} = v_{i+1}v'_i; 1 \le i \le n-1\}$. Here $|V(T_n)| = 2n-1$ and $|E(T_n)| = 3n-3$. We define a bijection $f : V(T_n) \to \{0, 1, 2, \ldots, 2n-2\}$ as $f(v_i) = 2i-2$, $1 \le i \le n$ and $f(v'_i) = 2i-1$, $1 \le i \le n-1$. Let $f^* : E(T_n) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(T_n)$.

Injectivity for edge labels:

For $1 \le i \le n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2})$, $1 \le i \le n-2$. Similarly $f^*(e_i^{(1)})$ and $f^*(e_i^{(2)})$ are also increasing, $1 \le i \le n-1$.

Claim: $f^*(e_i) \ne f^*(e_i^{(1)}) \ne f^*(e_i^{(2)})$, $1 \le i \le n-1$.

We have $f^*(e_i) = (4i-2)^2$, $f^*(e_i^{(1)}) = (4i-3)^2$, $f^*(e_i^{(2)}) = (4i-1)^2$, $1 \le i \le n-1$. As $f^*(e_i^{(j)})$ are odd, $j = 1, 2$, $1 \le i \le n-1$ and $f^*(e_i)$ are even, it is enough to prove $f^*(e_i^{(1)}) \ne f^*(e_i^{(2)})$. Assume if possible $f^*(e_i^{(1)}) = f^*(e_i^{(2)})$, for some i, $1 \le i \le n-1$

$\Longrightarrow 4i-3 = 4i-1$ or $4i-3 = 1-4i$

$\Longrightarrow 3 = 1$ or $i = \frac{1}{2}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

So $f^* : E(T_n) \to \mathbb{N}$ is injective. Hence T_n are sum perfect square graphs, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea of defined labeling pattern in above theorem.

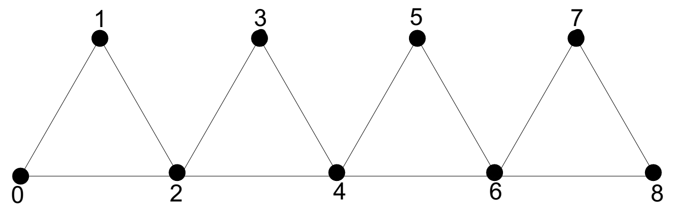

Figure 1. **Sum perfect square labeling of T_5.**

Theorem 2.2. *$D(T_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$.*

Proof. Let $V(D(T_n)) = \{v_i; 1 \leq i \leq n\} \cup \{v'_i; 1 \leq i \leq n-1\} \cup \{v''_i; 1 \leq i \leq n-1\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and v'_i, v''_i are adjacent with v_i, v_{i+1}, $1 \leq i \leq n-1$. $E(D(T_n)) = \{e_i = v_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i^{(1)} = v_iv'_i; 1 \leq i \leq n-1\} \cup \{e_i^{(2)} = v_{i+1}v'_i; 1 \leq i \leq n-1\} \cup \{e_i^{(3)} = v_{i+1}v''_i; 1 \leq i \leq n-1\} \cup \{e_i^{(4)} = v_iv''_i; 1 \leq i \leq n-1\}$. Here $|V(D(T_n))| = 3n-2$ and $|E(D(T_n))| = 5n-5$. We define a bijection $f : V(D(T_n)) \rightarrow \{0, 1, 2, \ldots, 3n-3\}$ as $f(v_i) = 3i-3$, $1 \leq i \leq n$, $f(v'_i) = 3i-2$, $1 \leq i \leq n-1$ and $f(v''_i) = 3i-1$, $1 \leq i \leq n-1$. Let $f^* : E(D(T_n)) \rightarrow \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(D(T_n))$.

Injectivity for edge labels:

For $1 \leq i \leq n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2})$, $1 \leq i \leq n-2$. Similarly $f^*(e_i^{(j)})$ are also increasing, $1 \leq j \leq 4$, $1 \leq i \leq n-1$.

Claim: $f^*(e_i) \neq f^*(e_i^{(1)}) \neq f^*(e_i^{(2)}) \neq f^*(e_i^{(3)}) \neq f^*(e_i^{(4)})$, $1 \leq i \leq n-1$.

We have $f^*(e_i) = (6i-3)^2$, $f^*(e_i^{(1)}) = (6i-5)^2$, $f^*(e_i^{(2)}) = (6i-2)^2$, $f^*(e_i^{(3)}) = (6i-1)^2$, $f^*(e_i^{(4)}) = (6i-4)^2$, $1 \leq i \leq n-1$. As $f^*(e_i)$ and $f^*(e_i^{(j)})$ are odd, $j = 1, 3$, $1 \leq i \leq n-1$, and $f^*(e_i^{(t)})$, $t = 2, 4$, $1 \leq i \leq n-1$ are even, it is enough to prove the following.

(1) $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(1)}), 1 \leq i \leq n-1\}$.

(2) $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$.

(3) $\{f^*(e_i^{(1)}), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$.

(4) $\{f^*(e_i^{(2)}), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(4)}), 1 \leq i \leq n-1\}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-1\} = \{f^*(e_i^{(1)}), 1 \leq i \leq n-1\}$, for some i.

$\Longrightarrow 6i-3 = 6i-5$ or $6i-3 = 5-6i$.

$\Longrightarrow 3 = 5$ or $i = \frac{2}{3}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-1\} = \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$, for some i.

$\Longrightarrow 6i-3 = 6i-1$ or $6i-3 = 1-6i$.

$\Longrightarrow 3 = 1$ or $i = \frac{1}{3}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i^{(1)}), 1 \leq i \leq n-1\} = \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$, for some i.

$\Longrightarrow 6i-5 = 6i-1$ or $6i-5 = 1-6i$.

$\Longrightarrow 5 = 1$ or $i = \frac{1}{2}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i^{(2)}), 1 \leq i \leq n-1\} = \{f^*(e_i^{(4)}), 1 \leq i \leq n-1\}$, for some i.

$\Longrightarrow 6i-2 = 6i-4$ or $6i-2 = 4-6i$.

$\Longrightarrow 2 = 4$ or $i = \frac{1}{2}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

So $f^* : E(D(T_n)) \rightarrow \mathbb{N}$ is injective. Hence $D(T_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea of defined labeling pattern in above theorem.

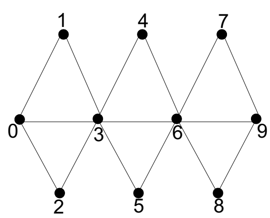

Figure 2. **Sum perfect square labeling of DT_4.**

Theorem 2.3. *$A(T_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$.*

Proof. **Case 1 :** Triangle starts from v_1.

$V(A(T_n)) = \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, where $\{v_1, v_2, \dots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively, such that u_i is adjacent with v_{2i-1} and v_{2i}, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. $E(A(T_n)) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i^{(1)} = u_i v_{2i-1}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{e_i^{(2)} = u_i v_{2i}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. Here $|V(A(T_n))| = n + \lfloor \frac{n}{2} \rfloor$ and $|E(A(T_n))| = n - 1 + 2\lfloor \frac{n}{2} \rfloor$. We define a bijection $f : V(A(T_n)) \to \{0, 1, 2, \dots, n + \lfloor \frac{n}{2} \rfloor - 1\}$ as

$$f(v_i) = \begin{cases} \frac{3i-4}{2}; & 1 \leq i \leq n, i \text{ } is \text{ } even. \\ \frac{3i-3}{2}; & 1 \leq i \leq n, i \text{ } is \text{ } odd. \end{cases}$$

$$f(u_i) = 3i - 1, \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor.$$

Let $f^* : E(A(T_n)) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(A(T_n))$.

Injectivity for edge labels for case 1:

For $1 \leq i \leq n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2})$, $1 \leq i \leq n-2$. Similarly $f^*(e_i^{(1)})$ and $f^*(e_i^{(2)})$ are also increasing, $1 \leq i \leq n-1$.

Claim: $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. We have $f^*(e_i) = (3i-2)^2$, $f^*(e_i^{(1)}) = (6i-4)^2$, $f^*(e_i^{(2)}) = (6i-3)^2$. As $f^*(e_i^{(1)})$ are even and $f^*(e_i^{(2)})$ are odd, it is enough to prove the following.

(1) $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

(2) $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-1\} = \{f^*(e_i^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\Longrightarrow 3i - 2 = 6i - 4$ or $3i - 2 = 4 - 6i$.

$\Longrightarrow i = \frac{2}{3}$ or $i = \frac{2}{9}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-1\} = \{f^*(e_i^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\Longrightarrow 3i - 2 = 6i - 3$ or $3i - 2 = 3 - 6i$.

$\Longrightarrow i = \frac{1}{3}$ or $i = \frac{5}{9}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Case 2 : Triangle starts from v_2.

Subcase 1 : n is odd.

For this subcase, the graph is isomorphic to the graph in case 1.

Subcase 2 : n is even.

$V(A(T_n)) = \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\}$, where $\{v_1, v_2, \dots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor - 1}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively, such that u_i is adjacent with v_{2i} and v_{2i+1}, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. $E(A(T_n)) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i^{(1)} = u_i v_{2i}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(2)} = u_i v_{2i+1}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\}$. Here $|V(A(T_n))| = n + \lfloor \frac{n}{2} \rfloor - 1$ and $|E(A(T_n))| = 2n - 3$. We define a bijection $f : V(A(T_n)) \to \{0, 1, 2, \dots, n + \lfloor \frac{n}{2} \rfloor - 2\}$ as

$$f(v_i) = \begin{cases} \lceil \frac{3i-5}{2} \rceil; & 2 \leq i \leq n, \ i \text{ is even.} \\ \lfloor \frac{3i-5}{2} \rfloor; & 2 \leq i \leq n, \ i \text{ is odd.} \end{cases}$$

$$f(u_i) = 3i, \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1 \ \text{ and } \ f(v_1) = 0.$$

Let $f^* : E(A(T_n)) \rightarrow \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(A(T_n))$.

Injectivity for edge labels for subcase 2:

For $1 \leq i \leq n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2})$, $1 \leq i \leq n-2$. Similarly $f^*(e_i^{(1)})$ and $f^*(e_i^{(2)})$ are also increasing, $1 \leq i \leq n-1$.

Claim: $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\} \neq \{f^*(e_i^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\}$.

We have $f^*(e_1) = 1$ $f^*(e_i) = (3i-3)^2$, $f^*(e_i^{(1)}) = (6i-2)^2$, $f^*(e_i^{(2)}) = (6i-1)^2$. For the arguments of rest of the edge labelings, we apply the similar arguments, which we provided in case 1. So $f^* : E(A(T_n)) \rightarrow \mathbb{N}$ is injective. Hence $A(T_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$, $n > 1$. □

The below illustration provides better idea of defined labeling pattern in above theorem.

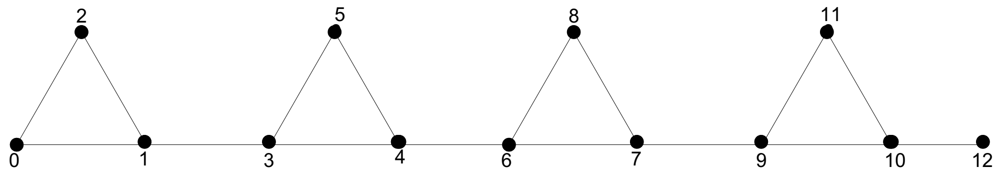

Figure 3. **Sum perfect square labeling of $A(T_9)$.**

Theorem 2.4. *$DA(T_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$.*

Proof. **Case 1 :** Triangles start from v_1.

$V(DA(T_n)) = \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{w_i; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor}\}$ and $\{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively, such that u_i and w_i are adjacent with v_{2i-1} and v_{2i}, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. $E(DA(T_n)) = \{e_i = v_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i^{(1)} = u_iv_{2i-1}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{e_i^{(2)} = u_iv_{2i}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{e_i^{(3)} = w_iv_{2i}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{e_i^{(4)} = w_iv_{2i-1}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. Here $|V(DA(T_n))| = n+2\lfloor \frac{n}{2} \rfloor$ and $|E(DA(T_n))| = n-1+4\lfloor \frac{n}{2} \rfloor$. We define a bijection $f : V(DA(T_n)) \rightarrow \{0, 1, 2, \ldots, n+2\lfloor \frac{n}{2} \rfloor - 1\}$ as $f(u_i) = 4i-3$ and $f(w_i) = 4i-2$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Subcase 1 : n is even.

$$f(v_i) = \begin{cases} 2i-1; 1 \leq i \leq n, \ i \text{ is even.} \\ 2i-2; 1 \leq i \leq n, \ i \text{ is odd.} \end{cases}$$

Subcase 2 : n is odd.

$$f(v_i) = \begin{cases} 2i-1; 1 \leq i \leq n-1, i \text{ is even.} \\ 2i-2; 1 \leq i \leq n-1, i \text{ is odd.} \end{cases}$$

Let $f^* : E(DA(T_n)) \rightarrow \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(DA(T_n))$.

Injectivity for edge labels for subcase 1:

For $1 \leq i \leq n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2})$, $1 \leq i \leq n-2$. Similarly $f^*(e_i^{(j)})$ are also increasing, $1 \leq j \leq 4$, $1 \leq i \leq n-1$.

Claim: $\{f^*(e_i), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(3)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}, \neq \{f^*(e_i^{(4)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

We have $f^*(e_i) = (4i-1)^2$, $f^*(e_i^{(1)}) = (8i-7)^2$, $f^*(e_i^{(2)}) = (8i-4)^2$, $f^*(e_i^{(3)}) = (8i-6)^2$, $f^*(e_i^{(4)}) = (8i-3)^2$. As $f^*(e_i)$ and $f^*(e_i^{(j)})$ are odd, $j = 1, 4$, and $f^*(e_i^{(t)})$, $t = 2, 3$, are even, it is enough to prove the following.

(1) $\{f^*(e_i), 1 \le i \le n-1\} \neq \{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

(2) $\{f^*(e_i), 1 \le i \le n-1\} \neq \{f^*(e_i^{(4)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

(3) $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(4)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

(4) $\{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

Assume if possible $\{f^*(e_i), 1 \le i \le n-1\} = \{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 4i-1 = 8i-7$ or $4i-1 = 7-8i$.

$\implies i = \frac{3}{2}$ or $i = \frac{2}{3}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i), 1 \le i \le n-1\} = \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 4i-3 = 8i-3$ or $4i-1 = 3-8i$.

$\implies i = 0$ or $i = \frac{1}{3}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(4)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 8i-7 = 8i-3$ or $8i-7 = 3-8i$.

$\implies 7 = 3$ or $i = \frac{5}{8}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 8i-4 = 8i-6$ or $8i-4 = 6-8i$.

$\implies 4 = 6$ or $i = \frac{5}{8}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Injectivity for edge labels for subcase 2:

The only difference in this case is due to $f^*(e_{n-1}) = (4n-6)^2$, which is the highest edge label among all the labelings in $DA(T_n)$. Rest of the arguments for edge labelings will be similar as we provided in subcase 1.

Case 2 : Triangles start from v_2.

Subcase 1 : n is even.

$V(DA(T_n)) = \{v_i; 1 \le i \le n\} \cup \{u_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor - 1}\}$ and $\{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor - 1}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively such that u_i and w_i are adjacent with v_{2i} and v_{2i+1}, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$. $E(DA(T_n)) = \{e_i = v_i v_{i+1}; 1 \le i \le n-1\} \cup \{e_i^{(1)} = u_i v_{2i}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(2)} = u_i v_{2i+1}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \cup \{e_i^{(3)} = w_i v_{2i}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(4)} = w_i v_{2i+1}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$. We note that $|V(DA(T_n))| = n + 2\lfloor \frac{n}{2} \rfloor - 2$ and $|E(DA(T_n))| = n - 5 + 4\lfloor \frac{n}{2} \rfloor$. We define a bijection $f : V(DA(T_n)) \to \{0, 1, 2, \ldots, n + 2\lfloor \frac{n}{2} \rfloor - 3\}$ as $f(v_1) = 0$, $f(v_n) = 2n-3$, $f(u_i) = 4i$, $f(w_i) = 4i-3$, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$.

$$f(v_i) = \begin{cases} \lfloor \frac{4i-3}{2} \rfloor, 2 \le i \le n-1, i \text{ is even.} \\ \lfloor \frac{4i-5}{2} \rfloor, 2 \le i \le n-1, i \text{ is odd.} \end{cases}$$

Injectivity for edge labels for subcase 1:

$f^*(e_1) = 4$ is the smallest edge label and $f^*(e_{n-1}) = (4n-8)^2$ is the highest edge label among all the labelings in $DA(T_n)$). For the injectivity of the remaining edge labels, we apply the similar arguments as per the subcase 1 of case 1.

Subcase 2 : n is odd.

For this subcase, the graph is isomorphic to the graph in subcase 2 of case 1. So $f^* : E(DA(T_n)) \to \mathbb{N}$ is injective. Hence $DA(T_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea of defined labeling pattern in above theorem.

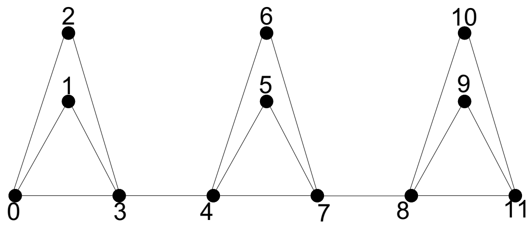

Figure 4. **Sum perfect square labeling of $DA(T_6)$.**

Theorem 2.5. *$A(Q_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$.*

Proof. **Case 1 :** Quadrilateral starts from v_1.

$V(A(Q_n)) = \{v_i; 1 \le i \le n\} \cup \{u_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor}\}$, $\{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively such that u_i is adjacent with w_i and v_{2i-1}, w_i is adjacent with u_i and v_{2i}, $1 \le i \le \lfloor \frac{n}{2} \rfloor$. $E(A(Q_n)) = \{e_i = v_i v_{i+1}; 1 \le i \le n-1\} \cup \{e_i^{(1)} = u_i v_{2i-1}; 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{e_i^{(2)} = u_i w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{e_i^{(3)} = w_i v_{2i}; 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$. Here $|V(A(Q_n))| = n + 2\lfloor \frac{n}{2} \rfloor$, $|E(A(Q_n))| = n - 1 + 3\lfloor \frac{n}{2} \rfloor$. We define a bijection $f : V(A(Q_n)) \to \{0, 1, 2, \ldots, n + 2\lfloor \frac{n}{2} \rfloor - 1\}$ as $f(u_i) = n + 2i - 2$ and $f(w_i) = n + 2i - 1$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$.

Subcase 1: n is even.

$f(v_i) = i - 1, 1 \le i \le n.$

Subcase 2: n is odd.

$f(v_i) = i, 1 \le i \le n - 1,\ f(v_n) = 0.$

Let $f^* : E(A(Q_n)) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(A(Q_n))$.

Injectivity for edge labels for subcase 1:

For $1 \le i \le n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2})$, $1 \le i \le n-2$. Similarly $f^*(e_i^{(j)})$ are also increasing, $1 \le j \le 3$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$.

Claim: $\{f^*(e_i), 1 \le i \le n-1\} \ne \{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \ne \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \ne \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

We have $f^*(e_i) = (2i-1)^2$, $f^*(e_i^{(1)}) = (n + 4i - 4)^2$, $f^*(e_i^{(2)}) = (2n + 4i - 3)^2$, $f^*(e_i^{(3)}) = (n + 4i - 2)^2$. As $f^*(e_i)$ and $f^*(e_i^{(2)})$ are odd and $f^*(e_i^{(j)})$, $j = 1, 3$ are even, it is enough to prove the following.

(1) $\{f^*(e_i), 1 \le i \le n-1\} \ne \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

(2) $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \ne \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

Assume if possible $\{f^*(e_i), 1 \le i \le n-1\} = \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\Longrightarrow 2i - 1 = 2n + 4i - 3$ or $2i - 1 = 3 - 2n - 4i$.

$\Longrightarrow 2i = 2 - 2n$ or $6i = 4 - 2n$.

$\Longrightarrow i = 1 - n$ or $i = \frac{4-2n}{6}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\Longrightarrow n + 4i - 4 = n + 4i - 2$ or $n + 4i - 4 = 2 - n - 4i$.

$\Longrightarrow -4 = -2$ or $8i = 6 - 2n$.

$\Longrightarrow 4 = 2$ or $i = \frac{6-2n}{8}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Injectivity for edge labels for subcase 2:

For $1 \le i \le n-2$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2})$, $1 \le i \le n-3$. Similarly $f^*(e_i^{(j)})$ are also increasing, $1 \le j \le 3$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$.

Claim: $\{f^*(e_i), 1 \le i \le n-2\} \neq f^*(e_{n-1}) \neq \{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

We have $f^*(e_i) = (2i+1)^2$, $f^*(e_{n-1}) = (n-1)^2$, $f^*(e_i^{(1)}) = (n+4i-3)^2$, $f^*(e_i^{(2)}) = (2n+4i-3)^2$,$f^*(e_i^{(3)}) = (n+4i-1)^2$. As $f^*(e_i)$ and $f^*(e_i^{(2)})$ are odd and $f^*(e_i^{(j)})$, $j = 1,3$ are even, it is enough to prove the following.

(1) $\{f^*(e_i), 1 \le i \le n-1\} \neq \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

(2) $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \neq f^*(e_{n-1})$.

Assume if possible $\{f^*(e_i), 1 \le i \le n-1\} = \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\Longrightarrow$ The highest edge label of f^*e_i is $(2n-3)^2$ and smallest edge label of $f^*(e_i^{(2)})$ is $(2n+1)^2$, which is larger than the largest edge label of f^*e_i. It is clear that the smallest edge label of $f^*(e_i^{(j)})$, $j = 1,2$ is larger than the value of $f^*(e_{n-1})$.

Assume if possible $\{f^*({e_i}^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(3)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\Longrightarrow n + 4i - 3 = n + 4i - 1$ or $n + 4i - 3 = 1 - n - 4i$.

$\Longrightarrow -3 = -1$ or $8i = 4 - 2n$.

$\Longrightarrow 4 = 2$ or $i = \frac{4-2n}{8}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Case 2 : Quadrilateral starts from v_2.

Subcase 1 : n is even.

$V(A(Q_n)) = \{v_i; 1 \le i \le n\} \cup \{u_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor - 1}\}$ and $\{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor - 1}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively such that u_i is adjacent with w_i and v_{2i}, w_i is adjacent with u_i and v_{2i+1}, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$. $E(A(Q_n)) = \{e_i = v_iv_{i+1}; 1 \le i \le n-1\} \cup \{e_i^{(1)} = u_iv_{2i}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(2)} = u_iw_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(3)} = w_iv_{2i+1}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$. Note that $|V(A(Q_n))| = n + 2\lfloor \frac{n}{2} \rfloor - 2$, $|E(A(Q_n))| = \frac{5n-8}{2}$. We define a bijection $f : V(A(Q_n)) \to \{0, 1, 2, \ldots, n + 2\lfloor \frac{n}{2} \rfloor - 3\}$ as $f(v_1) = 0$,$f(v_n) = 2n-3$, $f(v_i) = i-1, 2 \le i \le n-1$. $f(u_i) = n + 2i - 2$ and $f(w_i) = n + 2i - 1$, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$.

Injectivity for edge labels for subcase 1:

$f^*(e_1) = 1$ is the smallest edge label and $f^*(e_{n-1}) = (3n-5)^2$ is the highest edge label among all the labelings in $A(Q_n)$. For the injectivity of the remaining edge labels, we apply the similar arguments as per the subcase 1 of case 1.

Subcase 2 : n is odd.

Here the graph is isomorphic to the graph in subcase 2 of case 1. So $f^* : E(A(Q_n)) \to \mathbb{N}$ is injective. Hence $A(Q_n)$ are sum perfect square graphs, $\forall n, n \in \mathbb{N}$. □

The below illustration with figure provides the exact idea of defined labeling pattern in this theorem.

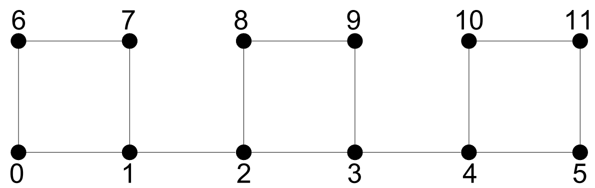

Figure 5. **Sum perfect square labeling of** $A(Q_6)$**.**

Theorem 2.6. $DA(Q_n)$ *are sum perfect square graphs,* $\forall n \in \mathbb{N}$.

Proof. **Case 1 :** Quadrilaterals start from v_1.

$V(DA(Q_n)) = \{v_i; 1 \le i \le n\} \cup \{u_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{u'_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{w'_i; 1 \le i \le$

$\lfloor\frac{n}{2}\rfloor\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \ldots, u_{\lfloor\frac{n}{2}\rfloor}\}$, $\{w_1, w_2, \ldots, w_{\lfloor\frac{n}{2}\rfloor}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively, such that u_i is adjacent with w_i and v_{2i-1}. Similarly $u'_1, u'_2, \ldots, u'_{\lfloor\frac{n}{2}\rfloor}, w'_1, w'_2, \ldots, w'_{\lfloor\frac{n}{2}\rfloor}$, are the vertices arranged between two consecutive vertices of P_n alternatively, such that u'_i is adjacent with w'_i and v_{2i}, $1 \le i \le \lfloor\frac{n}{2}\rfloor$. Moreover w'_i and $w_i$is adjacent with v_{2i}, $1 \le i \le \lfloor\frac{n}{2}\rfloor$. $E(DA(Q_n)) = \{e_i = v_iv_{i+1}; 1 \le i \le n-1\} \cup \{e_i^{(1)} = u_iv_{2i-1}; 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{e_i^{(2)} = u_iw_i; 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{e_i^{(3)} = w_iv_{2i}; 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{e_i^{(4)} = w'_iv_{2i}; 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{e_i^{(5)} = u'_iw'_i; 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{e_i^{(6)} = u'_iv_{2i-1}; 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$. Here $|V(DA(Q_n))| = n + 4\lfloor\frac{n}{2}\rfloor$, $|E(DA(T_n))| = n - 1 + 6\lfloor\frac{n}{2}\rfloor$. We define a bijection $f : V(DA(Q_n)) \to \{0, 1, 2, \ldots, n + 4\lfloor\frac{n}{2}\rfloor - 1\}$ as follows.

Subcase 1 : n is even.

$$f(v_i) = \begin{cases} 3i-3; 1 \le i \le n, \ i \text{ is even.} \\ 3i-1; 1 \le i \le n, \ i \text{ is odd.} \end{cases}$$

$$f(u_i) = 6i-6, \ f(w_i) = 6i-5, \ f(u'_i) = 6i-2, \ f(w'_i) = 6i-1, \ 1 \le i \le \lfloor\frac{n}{2}\rfloor.$$

Subcase 2 : n is odd.

$$f(v_i) = \begin{cases} 3i-3; 1 \le i \le n-1, \ i \text{ is even.} \\ 3i-1; 1 \le i \le n-1, \ i \text{ is odd.} \end{cases}$$

$f(v_n) = n + 4\lfloor\frac{n}{2}\rfloor - 2$, $f(u_i) = 6i-6$, $f(w_i) = 6i-5$, $f(u'_i) = 6i-2$, $1 \le i \le \lfloor\frac{n}{2}\rfloor$ and $f(w'_i) = 6i-1$, $1 \le i \le \lfloor\frac{n}{2}\rfloor - 1$, $f(w'_{\lfloor\frac{n}{2}\rfloor}) = n + 4\lfloor\frac{n}{2}\rfloor - 1$. Let $f^* : E(DA(Q_n)) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(DA(Q_n))$.

Injectivity for edge labels for subcase 1:

For $1 \le i \le n-1$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2})$, $1 \le i \le n-2$. Similarly $f^*(e_i^{(j)})$ are also increasing, $1 \le j \le 6$, $1 \le i \le \lfloor\frac{n}{2}\rfloor$.

Claim: $\{f^*(e_i), 1 \le i \le n-1\} \ne \{f^*(e_i^{(1)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(2)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(3)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(4)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(5)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(6)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

We have $f^*(e_i) = (6i-1)^2$, $f^*(e_i^{(1)}) = (12i-10)^2$, $f^*(e_i^{(2)}) = (12i-11)^2$, $f^*(e_i^{(3)}) = (12i-8)^2$, $f^*(e_i^{(4)}) = (12i-4)^2$, $f^*(e_i^{(5)}) = (12i-3)^2$, $f^*(e_i^{(6)}) = (12i-6)^2$. As $f^*(e_i)$, $f^*(e_i^{(2)})$, $f^*(e_i^{(5)})$ are odd and $f^*(e_i^{(j)})$, $j = 1, 3, 4, 6$ are even, it is enough to prove the following.

(1) $\{f^*(e_i), 1 \le i \le n-1\} \ne \{f^*(e_i^{(2)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(2) $\{f^*(e_i^{(2)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(5)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(3) $\{f^*(e_i), 1 \le i \le n-1\} \ne \{f^*(e_i^{(5)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(4) $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(3)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(5) $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(4)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(6) $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(6)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(7) $\{f^*(e_i^{(3)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(4)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(8) $\{f^*(e_i^{(3)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(6)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

(9) $\{f^*(e_i^{(4)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \ne \{f^*(e_i^{(6)}), 1 \le i \le \lfloor\frac{n}{2}\rfloor\}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-1\} = \{f^*(e_i^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 6i - 1 = 12i - 11$ or $6i - 1 = 11 - 12i$.

$\implies 6i = 10$ or $18i = 12$.

$\implies i = \frac{5}{3}$ or $i = \frac{2}{3}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(2)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(5)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 11 = 12i - 3$ or $12i - 11 = 3 - 12i$.

$\implies -11 = -3$ or $24i = 14$.

$\implies 11 = 3$ or $i = \frac{7}{12}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-1\} = \{f^*(e_i^{(5)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 6i - 1 = 12i - 3$ or $6i - 1 = 3 - 12i$.

$\implies 6i = 2$ or $18i = 4$.

$\implies i = \frac{1}{3}$ or $i = \frac{2}{9}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(3)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 10 = 12i - 8$ or $12i - 10 = 8 - 12i$.

$\implies -10 = -8$ or $24i = 18$.

$\implies 10 = 8$ or $i = \frac{3}{4}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(4)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 10 = 12i - 4$ or $12i - 10 = 4 - 12i$.

$\implies -10 = -4$ or $24i = 14$.

$\implies 10 = 4$ or $i = \frac{7}{12}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(6)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 10 = 12i - 6$ or $12i - 10 = 6 - 12i$.

$\implies -10 = -6$ or $24i = 16$.

$\implies 10 = 6$ or $i = \frac{2}{3}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(3)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(4)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 8 = 12i - 4$ or $12i - 8 = 4 - 12i$.

$\implies -8 = -4$ or $24i = 12$.

$\implies 8 = 4$ or $i = \frac{1}{2}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(3)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(6)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 8 = 12i - 6$ or $12i - 8 = 6 - 12i$.

$\implies -8 = -6$ or $24i = 14$.

$\implies 8 = 6$ or $i = \frac{7}{12}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i{}^{(4)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(6)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, for some i.

$\implies 12i - 4 = 12i - 6$ or $12i - 4 = 6 - 12i$.

$\implies -4 = -6$ or $24i = 10$.

$\implies 4 = 6$ or $i = \frac{5}{12}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Injectivity for edge labels for subcase 2:

The change in this case is only due to edges e_{n-1}, $e^{(4)}_{\lfloor \frac{n}{2} \rfloor}$ and $e^{(5)}_{\lfloor \frac{n}{2} \rfloor}$. $f^*(e_{n-1}) = (6n-10)^2$, $f^*(e^{(4)}_{\lfloor \frac{n}{2} \rfloor}) = (6n-9)^2$, and $f^*(e^{(5)}_{\lfloor \frac{n}{2} \rfloor}) = (6n-8)^2$, which are the first three highest edge labels in their respective ascending order. So by applying the similar arguments, which we have applied in subcase 1, we get all the edge labels are distinct.

Case 2 : Quadrilaterals start from v_2.

Subcase 1 : n is even.

$V(DA(Q_n)) = \{v_i; 1 \le i \le n\} \cup \{u_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{u'_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{w'_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$, where $\{v_1, v_2, \ldots, v_n\}$ are successive vertices of P_n and $\{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor - 1}\}$, $\{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor - 1}\}$ are the vertices arranged between two consecutive vertices of P_n alternatively such that u_i is adjacent with w_i and v_{2i}. Similarly $u'_1, u'_2, \ldots, u'_{\lfloor \frac{n}{2} \rfloor - 1}, w'_1, w'_2, \ldots, w'_{\lfloor \frac{n}{2} \rfloor - 1}$ are the vertices arranged between two consecutive vertices of P_n alternatively such that u'_i is adjacent with w'_i and v_{2i}, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$. Moreover w'_i and w_i is adjacent with v_{2i+1}, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$. $E(DA(Q_n)) = \{e_i = v_i v_{i+1}; 1 \le i \le n-1\} \cup \{e_i^{(1)} = u_i v_{2i}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(2)} = u_i w_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(3)} = w_i v_{2i+1}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(4)} = w'_i v_{2i+1}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup$
$\{e_i^{(5)} = u'_i w'_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{e_i^{(6)} = u'_i v_{2i}; 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$. Note that $|V(DA(Q_n))| = n + 4\lfloor \frac{n}{2} \rfloor - 4$, $|E(DA(Q_n))| = n - 7 + 6\lfloor \frac{n}{2} \rfloor$. We define a bijection $f : V(DA(Q_n)) \to \{0, 1, 2, \ldots, n + 4\lfloor \frac{n}{2} \rfloor - 5\}$ as follows.

$$f(v_1) = 3n - 7,\ f(v_n) = 3n - 6,\ f(v_{n-1}) = 3n - 5.$$

$$f(v_i) = \begin{cases} 3i - 4; 2 \le i \le n-2, i \text{ is even.} \\ 3i - 6; 2 \le i \le n-2, i \text{ is odd.} \end{cases}$$

$$f(u_i) = 6i - 6 \text{ and } f(u'_i) = 6i - 2,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1.$$

$$f(w_i) = 6i - 5,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor - 2,\ f(w_{\lfloor \frac{n}{2} \rfloor - 1}) = 3n - 9.$$

$$f(w'_i) = 6i - 1,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor - 2,\ f(w'_{\lfloor \frac{n}{2} \rfloor - 1}) = 3n - 11.$$

Injectivity for edge labels for subcase 1:

The change in this case is only due to edges e_1, e_{n-1}, $e^{(2)}_{\lfloor \frac{n}{2} \rfloor - 1}$, $e^{(3)}_{\lfloor \frac{n}{2} \rfloor - 1}$, $e^{(4)}_{\lfloor \frac{n}{2} \rfloor - 1}$ and $e^{(5)}_{\lfloor \frac{n}{2} \rfloor - 1}$.
$f^*(e_1) = (6n-5)^2$, $f^*(e_{n-1}) = (6n-11)^2$, $f^*(e^{(2)}_{\lfloor \frac{n}{2} \rfloor}) = (6n-2)^2$, $f^*(e^{(3)}_{\lfloor \frac{n}{2} \rfloor}) = (6n-14)^2$, $f^*(e^{(4)}_{\lfloor \frac{n}{2} \rfloor}) = (6n-16)^2$ and $f^*(e^{(5)}_{\lfloor \frac{n}{2} \rfloor}) = (6n-19)^2$, which are unique edge labels in $DA(Q_n)$. For the rest of the edge labels, we apply the similar arguments as provided in subcase 1 of case 1.

Subcase 2 : n is odd.

Here we use the similar labeling pattern as defined in subcase 2 of case 1. So $f^* : E(DA(Q_n)) \to \mathbb{N}$ is injective. Hence $DA(Q_n)$ are sum perfect square graphs, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea of defined labeling pattern in above theorem.

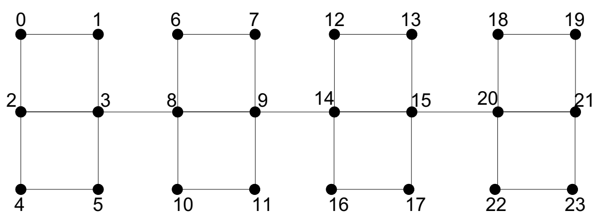

Figure 6. **Sum perfect square labeling of $DA(Q_8)$.**

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