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Best Proximity Points for Generalized $\alpha - \psi$-Geraghty Contractive Mappings

Research Article

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Abstract: In this paper we consider a generalization of $\alpha - \psi$-Geraghty contractions and investigate the existence and uniqueness of best proximity point for the mappings satisfying this condition using the P-property.

MSC: AMS Classification

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1. Introduction and Preliminaries

Fixed point theory investigates whether a function f defined on abstract space, have at least one fixed point, that is a fixed point x such that $f(x) = x$ under some conditions on f and on the space. Results of this theory has been used in various fields. In Particular, the fixed point theory techniques plays a crucial role in the solutions of differential equations. Banach contraction principle [1] is one of the initial and also a fundamental results in the theory of fixed points. Which sates that every contraction on a complete metric space has a unique fixed point.

Many authors have generalized and extended the fixed point theory by defining new contractive conditions and replacing complete metric spaces with some convenient abstract space. Among them we would like to mention one of the interesting results given by Geraghty [2]. For the sake of completeness, we shall recall Geraghty's theorem. We first remind the class of F all functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the conditions: $\lim_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0$.

Theorem 1.1 ([2])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be an mapping. Suppose that there exists $\beta : [0, \infty) \to (0, 1)$ satisfying the following inequality:*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for any } x, y \in X,$$

then f has unique fixed point.

Recently Samet et al [5] has reported interesting fixed point results by introducing the notion of $\alpha - \psi$-contractive mappings. Let ψ denote the class functions.

(a) ψ is nondecreasing

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(b) ψ is sub additive that $\psi(s+t)\leq\psi(s)+\psi(t)$;

(c) ψ is continuous

(d) $\psi(t)=0$ if and only if $t=0$.

Definition 1.2 ([5]). *Let (X,d) be a metric space and $f:X\to X$ be a given mapping. We say that f is an $\alpha-\psi$ contractive mapping if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that*

$$\alpha(x,y)d(f(x),f(y))\leq\psi(d(x,y)),\ \textit{for all}\ x,y\in X.$$

Clearly any contractive mapping, that is, a mapping satisfying Banach contraction, is an $\alpha-\psi$ contractive mapping with $\alpha(x,y)=1$ for all $x,y\in X$ and $\psi(t)=kt$, where $k\in(0,1)$.

Definition 1.3 ([5]). *Let (X,d) be a metric space and $f:X\to X$ be a given mapping. We say that f is a generalized $\alpha-\psi$ contractive mapping if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that for all $x,y\in X$, and we have*

$$\alpha(x,y)d(f(x),f(y))\leq\psi(M(x,y)),$$

where $M(x,y)=\max\{d(x,y),\frac{(d(x,f(x))+d(y,f(y)))}{2},\frac{(d(x,f(y))+d(y,f(x)))}{2}\}$. Clearly, since ψ is nondecreasing, every $\alpha-\psi$ contractive mapping is a generalized $\alpha-\psi$ contractive mapping.

Definition 1.4 ([5]). *Let $f:X\to X$ be a map and $\alpha:X\times X\to R$ be a function. Then f is said to be α-admissible if*

$$\alpha(x,y)\geq1\Rightarrow\alpha(f(x),f(y))\geq1 \tag{1}$$

Definition 1.5 ([4]). *An α-admissible map $f$is said to be triangular α-admissible if*

$$\alpha(x,z)\geq1\quad \textit{and}\quad \alpha(z,y)\geq1\Rightarrow\alpha(x,y)\geq1 \tag{2}$$

In this paper the notion of generalized $\alpha-\psi$-Geraghty contraction type mapping is modified and the existence and uniqueness of best proximity point of mappings under the assumption of α-Geraghty contraction and P-property is researched in the setting of complete metric spaces.

Lemma 1.6 ([4]). *Let $f:X\to X$ be a triangular α-admissible map. Assume that there exists $x_1\in X$ such that $\alpha(x_1,f(x_1))\geq1$. Define a sequence $\{x_n\}$ by $x_{n+1}=f(x_n)$. Then we have $\alpha(x_n,x_m)\geq1$ for all $m,n\in N$ with $n<m$.*

Definition 1.7 ([3]). *Let (X,d) be a metric space, and let $\alpha:X\times X\to R$ be a function. A mapping $f:X\to X$ is said to be a $\alpha-\psi$-Geraghty contraction if there exists $\beta\in F$ such that*

$$\alpha(x,y)\psi(d(f(x),f(y)))\leq\beta(\psi(d(x,y)))\psi(d(x,y))\ \textit{for any}\ x,y\in X,$$

Definition 1.8 ([3]). *Let (X,d) be a metric space, and let $\alpha:X\times X\to R$ be a function. A mapping $f:X\to X$ is said to be a generalized $\alpha-\psi$-geraghty contraction if there exists $\beta\in F$ such that*

$$\alpha(x,y)\psi(d(f(x),f(y)))\leq\beta(\psi(M(x,y)))\psi(M(x,y))\ \textit{for any}\ x,y\in X,$$

where $M(x,y)=\max\{d(x,y),d(x,f(x)),d(y,f(y))\}$, and $\psi\in\Psi$. Notice that if we take $\psi(t)=t$ in the definition, then f is called generalized α-Geraghty contraction mapping.

Definition 1.9 ([7]). *Let (X, d) be a metric space, and let $\alpha : X \times X \to R$ be a function. A map $f : X \to X$ is called a generalized α- Geraghty contraction type map if there exists $\beta \in F$ such that for all $x, y \in X$,*

$$\alpha(x,y)d(f(x),f(y)) \leq \beta(M(x,y))M(x,y).$$

Where $M(x,y) = \max\{d(x,y), d(x,f(x)), d(y,f(y))\}$.

Definition 1.10 ([7]). *Let (X, d) be a metric space, and let $\alpha : X \times X \to R$ be a function. A map $f : X \to X$ is called α-Geraghty contraction type map if there exists $\beta \in F$ such that for all $x, y \in X$,*

$$\alpha(x,y)d(f(x),f(y)) \leq \beta(d(x,y))d(x,y).$$

Definition 1.11. *Let (X, d) be a metric space. Let A and B be nonempty closed subsets of a complete metric space X and let $\alpha : X \times X \to R$ be a function. A mapping $f : A \to B$ is said to be a generalized $\alpha - \psi$-Geraghty contraction if there exists $\beta \in F$ such that*

$$\alpha(x,y)\psi(d(f(x),f(y))) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) \quad \text{for any} \quad x,y \in A, \tag{3}$$

where $M(x,y) = \max\{d(x,y), d(x,f(x)) - d(A,B), d(y,f(y)) - d(A,B)\}$ and $\psi \in \Psi$.

Define

$$A_0 = \{x \in A : d(x,y) = d(A,B), \quad \text{for some} \quad y \in B\}$$
$$B_0 = \{y \in B : d(x,y) = d(A,B), \quad \text{for some} \quad x \in A\} \text{ where } d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

Definition 1.12 ([6]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$*

$$d(x_1,y_1) = d(A,B) \text{ and } d(x_2,y_2) = d(A,B)$$

implies that

$$d(x_1,x_2) = d(y_1,y_2).$$

It is easy to see that, for any nonempty subset A of X, the pair (A, A) has the p-property. Also, it has been shown in [6], that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies the p-property. It is shown in [6] that strict convexity is equivalent to p-property.

2. Main Results

Theorem 2.1. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) satisfying the p-property, such that $A_0 \neq \emptyset$ and let $\alpha : X \times X \to R$ be a function, $f : A \to B$ be a map satisfying $f(A_0) \subseteq B_0$. Suppose that the following conditions are satisfied;*

(a) f is generalized $\alpha - \psi$-Geraghty contraction type map;

(b) f is triangular α-admissible;

(c) there exists $x_1 \in A$ such that $\alpha(x_1, f(x_1)) \geq 1$;

(d) f is continuous.

Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.

Proof. Choose $x_0 \in A_0$. Since $f(x_0) \in f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, f(x_0)) = d(A, B).$$

Since $f(x_1) \in f(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that

$$d(x_2, f(x_1)) = d(A, B).$$

We define a sequence $\{x_n\} \subset A_0$ satisfying $d(x_{n+1}, f(x_n)) = d(A, B)$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in N$. Then it is clear that x_{n_0} is a best proximity point of f. By the P-property we assume $x_n \neq x_{n+1}$ for each $n \in N$

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})).$$

where $x_n \neq x_{n+1}$ for all $n \in N$. Due to the Lemma 1.6 we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n \in N \tag{4}$$

From (3),

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(f(x_n), f(x_{n+1}))) \\ &\leq \alpha(x_n, x_{n+1}\psi(d(f(x_n), f(x_{n+1}))) \\ &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})) \end{aligned} \tag{5}$$

for all $n \in N$ where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, f(x_n)) - d(A, B), d(x_{n+1}, f(x_{n+1})) - d(A, B)\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}) + d(x_{n+1}, f(x_n)) - d(A, B), \\ &\quad d(x_{n+1}, x_{n+2}) + d(x_{n+2}, f(x_{n+1})) - d(A, B)\} \\ &= \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \end{aligned}$$

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})) \\ &\leq \beta(\psi(\max(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}))))\psi(\max(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}))) \end{aligned} \tag{6}$$

If

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\geq d(x_n, x_{n+1}) \\ \psi(d(x_{n+1}, x_{n+2}) &\leq \beta(\psi(d(x_{n+1}, x_{n+2})))\psi(d(x_{n+1}, x_{n+2})) \quad \text{from} \quad (7) \\ &< \psi(d(x_{n+1}, x_{n+2})) \quad \text{since} \quad \beta(x, y) = 1 \end{aligned}$$

which is a contradiction. Therefore

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \text{ for all } n \in N.$$

Hence sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and non-increasing. Consequently there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

We claim that $r = 0$.

Suppose on the contrary $r > 0$. Since

$$d(x_n, f(x_n)) - d(A,B) \leq d(x_n, x_{n+1}) \text{ and } d(x_{n+1}, f(x_{n+1})) - d(A,B) \leq d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}),$$

we have

$$M(x_n, x_{n+1}) = d(x_n, x_{n+1}).$$

Hence from (5) we have

$$\tfrac{\psi(d(x_{n+1},x_{n+2}))}{\psi(M(x_n,x_{n+1}))} \leq \beta(\psi(M(x_n, x_{n+1})) < 1.$$

Since ψ is non-decreasing and continuous,

$$\lim_{n\to\infty} \beta(\psi(M(x_n, x_{n+1}))) = 1.$$

Owing to the fact that $\beta \in F$, we have

$$\lim_{n\to\infty} \psi(M(x_n, x_{n+1})) = 0 \tag{7}$$

Hence we conclude that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{8}$$

we observe that

$$\begin{aligned} M(x_m, x_n) &= \max\{d(x_m, x_n), d(x_m, f(x_m)) - d(A,B), d(x_n, f(x_n)) - d(A,B)\} \\ &= \max\{d(x_m, x_n), d(x_m, x_{m+1}) + d(x_{m+1}, f(x_m)) - d(A,B), \\ &\quad d(x_n, x_{n+1}) + d(x_{n+1}, f(x_n)) - d(A,B)\} \\ &\leq \max\{d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1})\} \end{aligned}$$

Since $d(x_n, x_{n+1}) \to 0$. We have

$$\limsup_{m,n\to\infty} M(x_m, x_n) = \limsup_{m,n\to\infty} d(x_m, x_n). \tag{9}$$

We assert that $\{x_n\}$ is a cauchy sequence. By using the triangular inequality and since

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) \text{ and}$$
$$d(x_{n+1}, x_{m+1}) = d(f(x_n), f(x_m)),$$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= d(x_n, x_{n+1}) + d(f(x_n), f(x_m)) + d(x_{m+1}, x_m) \end{aligned} \tag{10}$$

combining (3) and (10) with the sub additive properties of ψ we have

$$\begin{aligned} \psi(d(x_n, x_m)) &\leq \psi(d(x_n, x_{n+1}) + d(f(x_n), f(x_m)) + d(x_{m+1}, x_m)) \\ &\leq \psi(d(x_n, x_{n+1})) + \psi(d(f(x_n), f(x_m))) + \psi(d(x_{m+1}, x_m)) \\ &\leq \psi(d(x_n, x_{n+1})) + \alpha(x_n, x_m)\psi(d(f(x_n), f(x_m))) + \psi(d(x_{m+1}, x_m)) \\ &\leq \psi(d(x_n, x_{n+1})) + \beta(\psi(M(x_n, x_m)))\psi(M(x_n, x_m)) + \psi(d(x_{m+1}, x_m)) \end{aligned} \tag{11}$$

Together with (9), (11) and (8),

$$\begin{aligned}\limsup_{m,n\to\infty}\psi(d(x_n,x_m)) &\leq \limsup_{m,n\to\infty}\beta(\psi(M(x_n,x_m)))\limsup_{m,n\to\infty}\psi(M(x_m,x_n))\\ &= \limsup_{m,n\to\infty}\beta(\psi(M(x_n,x_m)))\limsup_{m,n\to\infty}\psi(d(x_m,x_n))\end{aligned}$$

which implies that

$$\limsup_{m,n\to\infty} M(x_n,x_m)=0$$

and hence

$$\limsup_{m,n\to\infty} d(x_n,x_m)=0.$$

Therefore $\{x_n\}$ is a cauchy sequence. Since A is a closed subset of the complete metric space (X,d), $x_n\to x^*$ for some $x*\in A$. Since f is continuous we have $f(x_n)\to f(x^*)$. This implies that

$$d(x_{n+1},f(x_n))\to d(x^*,f(x^*)).$$

Taking into account that the sequence $\{d(x_{n+1},f(x_n))\}$ is a constant sequence with the value $d(A,B)$, we deduce that $d(x^*,f(x^*))=d(A,B)$. For the uniqueness, suppose that x_1 and x_2 are two best proximity points of f with $x_1\neq x_2$. This means that $d(x_i,f(x_i))=d(A,B)$ for $i=1,2$. Using the P-property we have $d(x_1,x_2)=d(f(x_1),f(x_2))$ and using the fact that f is a generalized $\alpha-\psi$-Geraghty contraction map we have

$$\begin{aligned}\psi(d(x_1,x_2)) &= \psi(d(f(x_1),f(x_2)))\\ &\leq \alpha(d(x_1,x_2))\psi(d(f(x_1),f(x_2)))\\ &\leq \beta(\psi(M(x_1,x_2)))\psi(M(x_1,x_2))\end{aligned}$$

we have

$$\begin{aligned}\psi(d(x_1,x_2)) &\leq \beta(\psi(M(x_1,x_2)))\psi(M(x_1,x_2))\\ &\leq \beta\psi(d(x_1,x_2))\psi(d(x_1,x_2))\\ &< \psi(d(x_1,x_2))\end{aligned}$$

which is a contradiction. This completes the proof. □

Definition 2.2 ([3]). *Let A and B be nonempty closed subsets of a complete metric space (X,d), and let $\alpha : X\times X\to R$ be a function, $f : A\to B$ be a map. We say that the sequence $\{x_n\}$ is a α-regular if the following condition is satisfied. If $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in A$ as $n\to\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x)\geq 1$ for all k.*

Theorem 2.3. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0\neq\emptyset$ and let $\alpha : X\times X\to R$ be a function, $f : A\to B$ be a map satisfying $f(A_0)\subseteq B_0$. Furthermore the pair (A,B) has the P-property. Suppose that the following conditions are satisfied;*

(a) f is generalized $\alpha-\psi$-Geraghty contraction type map;

(b) f is triangular α-admissible;

(c) there exists $x_1 \in A$ such that $\alpha(x_1, f(x_1)) \geq 1$;

(d) $\{x_n\}$ is α-regular.

Then there exists a best proximity point $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.

Proof. Following proof of Theorem 2.1, we know that the sequence $\{x_n\}$ satisfying $d(x_{n+1}, f(x_n)) = d(A, B)$ for all $n \geq 0$, converges to $x^* \in A$. From (4) and assumption (d) of the theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \alpha(x_{n_k}, x^*) \geq 1$. Applying (3) for all k, we get

$$\alpha(x_{n_k}, x^*)\psi(d(f(x_{n_k}), f(x^*))) \leq \beta(\psi(M(x_{n_k}, x^*)))\psi(M(x_{n_k}, x^*)) \tag{12}$$

On the other hand there exists $y \in A$ such that $d(y, f(x^*) = d(A, B)$ we have

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, f(x_{n_k})) - d(A, B), d(x^*, f(x^*)) - d(A, B)\} \\ &\leq \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, y)\} \end{aligned}$$

and hence

$$\lim_{k\to\infty} \psi(M(x_{n_k}, x^*)) = \psi(d(x^*, y)) \text{ From (12)}$$

and since $\alpha(x_{n_k}, x^*) \geq 1$ we have

$$\frac{\psi(d(x_{n_k+1}, y))}{\psi(d(x^*, y))} \leq \alpha(x_{n_k}, x^*)\frac{\psi(d(f(x_{n_k}), f(x^*)))}{\psi(M(x_{n_k}, x^*))} \leq \beta(\psi(M(x_{n_k}, x^*))) \leq 1.$$

Letting $k \to \infty$ in the above inequality we obtain

$$\lim_{k\to\infty} \beta(\psi(M(x_{n_k}, x^*))) = 1,$$

since ψ is non-decreasing and continuous,

$$\psi(d(x^*, y)) = \lim_{k\to\infty} \psi(M(x_{n_k}, x^*)) = 0.$$

Therefore $x^* = y$ and $d(x^*, f(x^*)) = d(A, B)$. □

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