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Higher Order Backlund Transformation for $u_t + uu_x + \alpha u - u_{xx} + \gamma u_{xxx} + u_{yy} = 0$

Research Article

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Abstract: The homogeneous balance (HB) method has drawn lots of interests in seeking the solitary wave solution and other kinds of solutions. Wang showed the HB method is powerful for finding analytic solitary wave solutions of PDE. In this paper, the solitary wave solution of nonlinear PDE has been obtained by this method.

Keywords: Homogeneous Balance method, solitary wave solutions, auto-Backlund transformation.

1. Introduction

Homogeneous Balance method is easy to apply and always yields special exact solutions for nonlinear evolution equations (Biao Li, Yong Chen and Hongqing Zhang [1], Feng [3], Lei Yang.et.al [4], Wang Mingling [5–7], Yang Lei, Liu Jianbin, Yang Kongqing [8]. Recently Engui Fan [2] interpreted the transformation used in HBM as an auto-Backlund transformation as it involved first order derivatives. To be precide , any solution u of a pde $L[u] = 0$ is written in term of w and w_x, and a finite set of pdes are derived for w. In the present paper we relate u to w, w_x and w_{xx} and call the relation as a higher order BT.

2. Higher order Backlund Transformations

We consider the $(2+1)$-dimensional equation

$$u_t + uu_x + \alpha u - u_{xx} + \gamma u_{xxx} + u_{yy} = 0, \tag{1}$$

to find its exact solutions via HBM. In order that the highest nonlinear term uu_x partially balance with the third order derivative term u_{xxx}, we suppose that the solution of (1) is of the form

$$u(x,t,y) = f''w_x^2 + f'w_{xx} + af'w_x + b, \quad w = w(x,t,y), \tag{2}$$

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where the functions f and w as well as the constants a and b are to be determined. If follows from (2) that

$$
\begin{aligned}
u_t &= f'''w_t w_x^2 + 2f''w_{xt}w_x + f''w_t w_{xx} + f'w_{xxt} + af''w_t w_x + af'w_{xt}, &(3)\\
uu_x &= f''f'''w_x^5 + 3f''^2 w_x^3 w_{xx} + f'f''w_x^2 w_{xxx} + af''^2 w_x^4 + 5af'f''w_x^2 w_{xx}\\
&\quad + f'f'''w_x^3 w_{xx} + 3f'f''w_x w_{xx}^2 + f'^2 w_{xx}w_{xxx} + af'^2 w_{xx}^2\\
&\quad + af'f'''w_x^4 + af'^2 w_x w_{xxx} + a^2 f'^2 w_x w_{xx} + a^2 f'f''w_x^3 + bf'''w_x^3\\
&\quad + 3bf''w_x w_{xx} + bf'w_{xxx} + abf''w_x^2 + abf'w_{xx}, &(4)\\
-u_{xx} &= -\left[f^{IV}w_x^4 + 6f'''w_x^2 w_{xx} + 3f''w_{xx}^2 + 4f''w_x w_{xxx} + f'w_{xxxx} + af'w_{xxx} + af'''w_x^3 + 3af''w_x w_{xx}\right], &(5)\\
\gamma u_{xxx} &= \gamma\left[f^V w_x^5 + 10f^{IV}w_x^3 w_{xx} + 15f'''w_x w_{xx}^2 + 10f'''w_x^2 w_{xxx} + 10f''w_{xx}w_{xxx} + 5f''w_x w_{xxxx}\right.\\
&\quad \left. + f'w_{xxxxx} + af^{IV}w_x^4 + 6af'''w_x^2 w_{xx} + 3af''w_{xx}^2 + 4af''w_x w_{xxx} + af'w_{xxxx}\right] &(6)\\
u_{yy} &= f^{IV}e_y^2 w_x^2 + f'''w_{yyy}w_x^2 + 4f'''w_y w_x w_{xy} + 2f''w_{xy}^2 + 2f''w_{xyy}w_x + f'''w_y^2 w_{xx}\\
&\quad + f''w_{yy}w_{xx} + 2f''w_{xxy}w_y + f'w_{xxyy} + af'''w_y^2 w_x + af''w_{yy}w_x + 2af''w_{xy}w_y + af'w_{xyy}. &(7)
\end{aligned}
$$

First collecting the terms with w_x^5 in (4), and (6), and setting its coefficient to zero, we obtain an ordinary differential equation for $f(w)$:

$$f''f''' + \gamma f^V = 0. \tag{8}$$

Two solutions of (8) are

$$f = \pm A \log w, \quad A = 12\gamma. \tag{9}$$

It follows from (9) that

$$f''^2 = -\frac{2\gamma}{\alpha}f^{IV}, \quad f'^2 - \frac{12\gamma}{\alpha}f'', \quad f''f''' = -\frac{\gamma}{\alpha}f^V, \quad f'f''' = \frac{4\gamma}{\alpha}f^{IV}, \quad f'f'' = -\frac{6\gamma}{\alpha}f'''. \tag{10}$$

Inserting (8)-(9) in (2) we relate u to w and its derivatives in the form

$$
\begin{aligned}
u_t + uu_x + \alpha u - u_{xx} + \gamma u_{xxx} + u_{yy} &= \left[w_{xxt} + aw_{xt} + bw_{xxx} + abw_{xx} + \alpha\left(w_{xx} + aw_x\right)\right.\\
&\quad \left. - w_{xxxx} - aw_{xxx} + \gamma\left(w_{xxxxx} + aw_{xxxx}\right) + w_{xxyy} + aw_{xyy}\right] f'\\
&\quad + \left[2w_x w_{xt} + w_t w_{xx} + aw_t w_x - Aw_{xx}w_{xxx} - aAw_{xx}^2 - aAw_x w_{xxx}\right.\\
&\quad - a^2 Aw_x w_{xx} + 3bw_x w_{xx} + abw_x^2 + \alpha w_x^2 - 3w_{xx}^2 - 4w_x w_{xxx}\\
&\quad - 3aw_x w_{xx} + \gamma\left(10w_{xx}w_{xxx} + 5w_x w_{xxxx} + 3aw_{xx}^2 + 4aw_x w_{xxx}\right)\\
&\quad + 2w_{xy}^2 + 2w_x w_{xyy} + w_{yy}w_{xx} + 2w_y w_{xxy} + aw_{yy}w_x\\
&\quad \left. + 2aw_y w_{xy}\right] f'' + \left[w_t w_x^2 - \frac{A}{2}w_x^2 w_{xxx} - \frac{5A}{2}w_x w_{xx}^2 - \frac{a^2 A}{2}w_x^3\right.\\
&\quad - bw_x^3 - 6w_x^2 w_{xx} - aw_x^3 + \gamma\left(10w_x^2 w_{xxx} + 15w_x w_{xx}^2 + 6aw_x^2 w_{xx}\right)\\
&\quad \left. + w_{yy}w_x^2 + 4w_x w_y w_{xy} + w_y^2 w_{xx} + aw_y^2 w_x\right] f''' + \left[-\frac{5A}{6}w_x^3 w_{xx}\right.\\
&\quad \left. - (1 + \frac{aA}{2})w_x^4 + \gamma\left(10w_x^3 w_{xx} + aw_x^4\right) + w_y^2 w_x^2\right] f^{IV}\\
&\quad + \left[-\frac{A}{12}w_x^5 + \gamma w_x^5\right] f^V + \alpha b, \qquad (11)
\end{aligned}
$$

Setting the coefficients of f^V, f^{IV}, f''', f'' and f' in (11) to zero respectively, we obtain the following four equations for the determination of $w(x,t)$:

$$w_{xxt} + aw_{xt} + bw_{xxx} + abw_{xx} + \alpha\left(w_{xx} + aw_x\right) - w_{xxxx} - aw_{xxx}$$
$$+\gamma\left(w_{xxxxx} + aw_{xxxx}\right) + w_{xxyy} + aw_{xyy} = 0, \quad (12)$$
$$2w_x w_{xt} + w_t w_{xx} + aw_t w_x - Aw_{xx}w_{xxx} - aAw_{xx}^2 - aAw_x w_{xxx}$$
$$-a^2 Aw_x w_{xx} + 3bw_x w_{xx} + abw_x^2 + \alpha w_x^2 - 3w_{xx}^2 - 4w_x w_{xxx}$$
$$-3aw_x w_{xx} + \gamma\left(10w_{xx}w_{xxx} + 5w_x w_{xxxx} + 3aw_{xx}^2 + 4aw_x w_{xxx}\right)$$
$$+2w_{xy}^2 + 2w_x w_{xyy} + w_{yy}w_{xx} + 2w_y w_{xxy} + aw_{yy}w_x + 2aw_y w_{xy} = 0, \quad (13)$$
$$w_t w_x^2 - \frac{A}{2}w_x^2 w_{xxx} - \frac{5A}{2}w_x w_{xx}^2 - \frac{a^2 A}{2}w_x^3 - bw_x^3 - 6w_x^2 w_{xx} - aw_x^3 + aw_y^2 w_x$$
$$+\gamma\left(10w_x^2 w_{xxx} + 15w_x w_{xx}^2 + 6aw_x^2 w_{xx}\right) + w_{yy}w_x^2 + 4w_x w_y w_{xy} + w_y^2 w_{xx} = 0, \quad (14)$$
$$-\frac{5A}{6}w_x^3 w_{xx} - (1 + \frac{aA}{2})w_x^4 + \gamma\left(10w_x^3 w_{xx} + aw_x^4\right) + w_y^2 w_x^2 = 0, \quad (15)$$

provided that $b = 0$. If we write

$$w(x,t) = w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}, \quad (16)$$

then (12)-(15) give

$$-\frac{5}{12}k_1^5 - (\frac{aA}{2} + 1)k_1^4 + \gamma(10k_1^5 + ak_1^4) + k_3^2 k_1^2 = 0, \quad (17)$$
$$-2Ak_1^5 - (\frac{5aA}{2} + 6)k_1^4 - \frac{a^2 A}{2}k_1^3 - ak_1^3 + \gamma(10k_1^5 + 15k_1^3 + 6ak_1^4) + 6k_3^2 k_1^2 + ak_1 k_3^2 = 0, \quad (18)$$
$$3k_1^2 k_2 + k_1 k_2 - Ak_1^5 - 2aAk_1^4 - 7k_1^4 - a^2 Ak_1^3 + \alpha k_1^2 - 3ak_1^3$$
$$+\gamma(15k_1^5 + 7ak_1^4) + 7k_1^2 k_3^2 + 3ak_1 k_3^2 = 0, \quad (19)$$
$$k_1^2 k_2 + ak_1 k_2 + \alpha(k_1^2 + ak_1) - k_1^4 - ak_1^3 + \gamma(k_1^5 + ak_1^4) + (1+a)k_1^2 k_3^2 = 0, \quad (20)$$

Substituting (16) and (9) in (2), we obtain a exact solution of (1) as

$$u(x,y,t) = 12\gamma e^{(k_1 x + k_2 t + k_3 y + k_4)}\left[-\frac{k_1^2}{[w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}]^2} + \frac{k_1^2 + ak_1}{[w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}]}\right]. \quad (21)$$

3. Results and Conclusions

In this paper, based on the idea of the HB method, we derive higher order Back-lund transformation for (1) is $u(x,t,y) = f'' w_x^2 + f' w_{xx} + af' w_x + b, \quad w = w(x,t,y)$. Based on these BTs, exact solution for (1) is

$$u(x,y,t) = 12\gamma e^{(k_1 x + k_2 t + k_3 y + k_4)}\left[-\frac{k_1^2}{[w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}]^2} + \frac{k_1^2}{[w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}]} + \frac{ak_1}{[w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}]}\right]. \quad (22)$$

The exact solution of (1) is in terms of exponential functions besides the usual solitary wave solutions.

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