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Similarity Solution of Semilinear Parabolic Equations with Variable Coefficients

Research Article

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Abstract: In this paper we establish again that the nonclassical method accounts for more general results than those obtained by direct method and Lie's classical method with the help of a nonlinear parabolic equation with a variable coefficient $u_t = u_{xx} + V(t,x)u^p, p > 1$. A perturbation solution for the reduced equation $z^2 ff'' + l_5 f^2 + (1+p)/(1-p) z^2 {f'}^2 + \epsilon z^{n_1+2} = 0$ is obtained.

MSC: 34C14.

Keywords: Lie's classical method, Nonclassical method, Symmetries.

1. Introduction

The purpose of this work is to ascertain the superiosity of the non-classical method [1, 2] over the Lie's classical method [3] and CK-method [5–7]. Nucci and Clarkson [8] have already showed with the help of Fizugh-Nagumo equation that the nonclassical method is more general than the CK-method.

It may be recalled that the group theoretic explanation of CK-methodd is provided To achive our goal we consider the nonlinear parabolic equation with a variable coefficient

$$u_t = u_{xx} + V(t,x)u^p, \quad p > 1. \tag{1}$$

We show that the solution of (1) by Lie's classical method only solves its the elliptic counterpart

$$u_{xx} + V(x)u^p = 0. \tag{2}$$

Although the application of CK-method to (1) results in one solution it is the nonclassical method that yields two solutions.

This paper is divided into five sections. In section 2, 3 and 4 we apply the classical method, the nonclassical method and the CK-method respectively to (1). Section 5 is devoted to the summary of the present work.

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2. Classical Lie Group Method

We now seek Lie group of infinitesimal transformations

$$u^* = u + \epsilon U(t,x,u) + O(\epsilon^2), \quad t^* = t + \epsilon T(t,x,u) + O(\epsilon^2), \quad x^* = x + \epsilon X(t,x,u) + O(\epsilon^2), \tag{3}$$

under which (1) is invariant. Then

$$\begin{aligned} -pVu^{p-1}U - u^pV_xX - u^pV_tT - [U_{xx} + (2U_{xu} - X_{xx})u_x - T_{xx}u_t \\ +(U_{uu} - 2X_{xu})u_x^2 - 2T_{xu}u_xu_t - X_{uu}u_x^3 - T_{uu}u_x^2u_t + (U_u - 2X_x)(u_t - Vu^p) \\ -2T_xu_{xt} - 3X_uu_x(u_t - Vu^p) - T_uu_t(u_t - Vu^p) - 2T_uu_xu_{xt}] + U_t + (U_u - T_t)u_t \\ -X_tu_x - T_uu_t^2 - X_uu_xu_t &= 0, \end{aligned} \tag{4}$$

where we have replaced for u_{xx} using (1). Equating the coefficients of u_{xt}, u_xu_{xt},u_xu_t and u_x^2 in (4) to zero, we get $T_x = T_u = X_u = U_{uu} = 0$ resulting in $T = T(t)$, $X = X(x,t)$ and $U = f(x,t)u + g(x,t)$. Now (4) reduces to

$$-pVu^{p-1}[uf+g] - u^pV_xX - u^pV_tT - [uf_{xx}+g_{xx}] - u_x[2f_x - X_{xx} + X_t] + u_t[-T' + 2X_x] + Vu^p[f - 2X_x] + [uf_t + g_t] = 0. \tag{5}$$

Again equating the coefficients of u_x, u_t and u^0 in (5) to zero we have

$$2f_x - X_{xx} + X_t = 0, \tag{6}$$

$$T' - 2X_x = 0, \tag{7}$$

$$-pVu^{p-1}(uf+g) - u^pV_xX - u^pV_tT - uf_{xx} - g_{xx} + Vfu^p - 2u^pVX_x + uf_t + g_t = 0. \tag{8}$$

Differentiating (7) with respect to x gives $X_{xx} = 0$ so that (6) reduces to

$$2f_x + X_t = 0. \tag{9}$$

Integrating (7) with respect to x, we get

$$X(x,t) = \frac{T'(t)}{2}x + b(t), \tag{10}$$

where $b(t)$ is function of integration. Inserting (10) in (9) and integrating with respect to x, we find that

$$f = -\frac{1}{8}T''(t)x^2 - \frac{1}{2}b'(t)x + c(t), \tag{11}$$

where $c(t)$ is another function of integration. Now (8) assumes the form

$$\begin{aligned} Vu^p\left(-\frac{1}{8}T''(t)x^2 - \frac{1}{2}b'(t)x + c(t)\right)(1-p) - pVu^{p-1}g - Vu^pT'(t) - u^pV_tT \\ -u^pV_x[\frac{T'(t)}{2}x + b(t)] + \frac{uT''(t)}{4} - g_{xx} + u\left(-\frac{1}{8}T'''(t)x^2 - \frac{1}{2}b''(t)x + c'(t)\right) + g_t &= 0. \end{aligned} \tag{12}$$

Equating the coefficients of u^{p-1}, u and u^p in (12) to zero, we have

$$g = 0, \tag{13}$$

$$\frac{T''}{4} - \frac{T'''}{8}x^2 - \frac{b''}{2}x + c' = 0, \tag{14}$$

$$V\left[-\frac{T''}{8}x^2 - \frac{b'}{2}x + c\right](1-p) - VT' - V_xX - V_tT = 0. \tag{15}$$

The coefficients of x^2, x and x^0 in (14) and (15) when equated to zero give

$$T''' = b'' = T'' + 4c' = T'' = b' = 0, \tag{16}$$

$$[(1-p)c - A]\,V - (\frac{A}{2}x + b)V_x - (At+B)V_t = 0. \tag{17}$$

Equations in (16)-(17) are satisfied if b and c are constants,

$$T = At + B \quad \text{and} \quad V = \frac{1}{2}(At+B)^{\left[\frac{c(1-p)}{A} - \frac{3}{2}\right]}(\frac{A}{2}x + b), \tag{18}$$

where A, B and v_0 are constants. Substituting (18) into (10) and (11) (recall that $U = fu$), we have

$$X = \frac{A}{2}x + b, \quad T = At + B, \quad U = cu. \tag{19}$$

The invariant surface condition $\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}$ becomes

$$\frac{dx}{\frac{A}{2}x + b} = \frac{dt}{At+B} = \frac{du}{cu}. \tag{20}$$

Integration of equations (20) gives the similarity form of solutions of (2.1) as

$$u = (At+B)^{c/A}F(z), \quad z = \frac{(\frac{A}{2}x + b)^2}{A(At+B)}. \tag{21}$$

Putting (21) in (3) we get the following ordinary differential equation for the similarity function $F(z)$:

$$AzF'' + A\left(\frac{1}{2} + z\right)F' + 2z^{\frac{1}{2}}F^p - cF = 0. \tag{22}$$

Substituting $F = c_1 z^{c_2}$, we have

$$Ac_1c_2(c_2-1)z^{c_2-1} + \frac{A}{2}c_1c_2z^{c_2-1} + Ac_1c_2z^{c_2} + 2c_1^p z^{pc_2+\frac{1}{2}} - cc_1z^{c_2} = 0. \tag{23}$$

Case 1: $c_2 - 1 = pc_2 + \frac{1}{2}$

If we balance the first, second and third terms and the remaining terms equal to zero, we find that

$$c_1 = \left[\frac{1}{2}\left(Ac_2(1-c_2) - \frac{A}{2}c_2\right)\right]^{\frac{1}{p-1}}, \tag{24}$$

$$Ac_2 = c. \tag{25}$$

Thus

$$F = \left[\frac{1}{2}\left(Ac_2(1-c_2) - \frac{A}{2}c_2\right)\right]^{\frac{1}{p-1}} z^{\frac{3}{2(1-p)}}. \tag{26}$$

Corresponding solution of (1) is a solution of (1):

$$u = (At+B)^{\frac{c}{A}}\left[\frac{1}{2}\left(Ac_2(1-c_2) - \frac{A}{2}c_2\right)\right]^{\frac{1}{p-1}} z^{\frac{3}{2(1-p)}}, \tag{27}$$

where V(x, t) is given by

$$V(x,t) = \frac{1}{2}(At+B)^{\left[\frac{c(1-p)}{A} - \frac{3}{2}\right]}(\frac{A}{2}x + b). \tag{28}$$

Case 2: $c_2 = pc_2 + \frac{1}{2}$.

Writing $F = c_1 z^{c_2}$ in (22) and taking the coefficients of z^{c_2} and the remaining terms equal to zero separately, we have

$$c_2 = \frac{1}{2}, \quad p = 0, \quad c_1 = \frac{4}{2c - A}. \tag{29}$$

Substituting (29) in $F = c_1 z^{c_2}$ we finally arrive at

$$F = \frac{4}{2c - A} z^{\frac{1}{2}}. \tag{30}$$

Insertion of (30) into (21) and (18) lead to a solution of (1):

$$u = (At + B)^{\frac{c}{A}} \frac{4}{2c - A} z^{1/2}, \tag{31}$$

$$V(x,t) = \frac{1}{2}(At + B)^{\left[\frac{c(1-p)}{A} - \frac{3}{2}\right]}\left(\frac{A}{2}x + b\right). \tag{32}$$

3. Nonclassical Method

It follows from the invariant surface condition (where we have taken, without loss of generality, $T \equiv 1$)

$$u_t = U - X u_x. \tag{33}$$

In view of (33), equation (4) reduces to

$$\begin{aligned} -pVu^{p-1}U - u^p \frac{dV}{dt} T - u^p \frac{dV}{dx} X - \big[U_{xx} + (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})u_x^2 \\ -X_{uu}u_x^3 + (U_u - 2X_x)\left[(U - Xu_x) - Vu^p\right] - 3X_u u_x \left[(U - Xu_x) - Vu^p\right]\big] \\ +U_t + U_u(U - Xu_x) - X_t u_x - X_u u_x u_t = 0, \end{aligned} \tag{34}$$

Successively equating the coefficients of u^0, u_x, u_x^2 and $u_x u_t$ in (34) to zero we find that

$$-pu^{p-1}V(x)U - u^p V'(x)X - U_{xx} + 2UX_x + u^p V(x) U_u - 2u^p V(x) X_x + U_t = 0, \tag{35}$$

$$-X_t - 2U_{xu} + X_{xx} - 2XX_x + 3UX_u - 3X_u V(x) u^p = 0. \tag{36}$$

$$-U_{uu} + 2X_{xu} - 3XX_u = 0, \tag{37}$$

$$X_u = 0. \tag{38}$$

Again equating the coefficients of u^{p-1} and u^p in (35) to zero we have

$$U = 0. \tag{39}$$

$$V_t + XV'(x) + 2V(x)X_x = 0. \tag{40}$$

Equation (40) leads to

$$V(x) = v_0 X^{-2}. \tag{41}$$

Case 1: $X_t = 0$

Substituting (39) in (36), we get

$$X_{xx} - 2X_x = 0. \tag{42}$$

Now solving (42), we obtain the solution

$$X = -\frac{1}{x}. \tag{43}$$

On inserting (43), equation (41) leads to

$$V(x) = v_0 x^2. \tag{44}$$

Substituting (39), (43) and $T \equiv 1$, the invariant surface condition for z, namely $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ becomes

$$-xdx = dt = \frac{du}{0}. \tag{45}$$

Integration of equations (45) gives a similarity solution of (1) in the form

$$u = F(z), \tag{46}$$

$$z = t + \frac{x^2}{2}, \tag{47}$$

Substitution of (47) in (1) yields:

$$F'' + v_0 F^p = 0. \tag{48}$$

Equation (48) can be modified into

$$F'^2 + \frac{v_0}{1+p} F^{p+1} = 0. \tag{49}$$

Now solving (49), we obtain

$$F(z) = \left(\frac{1-p}{2} \left[\left(\frac{v_0}{-1-p} \right)^{1/2} z + C_2 \right] \right)^{2/(1-p)}, \tag{50}$$

where C_2 is an arbitrary constant. Thus the similarity solution of (1) in this case is

$$u(x,t) = \left(\frac{1-p}{2} \left[\left(\frac{v_0}{-1-p} \right)^{1/2} (t + \frac{x^2}{2}) + C_2 \right] \right)^{2/(1-p)}, \tag{51}$$

$$z(x,t) = t + \frac{x^2}{2}.$$

Case 2: $X_t \neq 0$

Substituting (39) in (36), we get

$$X_t - X_{xx} + 2X_x = 0. \tag{52}$$

Now solving (52), we obtain the solution

$$X = -\frac{x}{2t}. \tag{53}$$

On inserting (53), equation (40) leads to

$$V(x) = tx^{-4}. \tag{54}$$

Substituting (39), (53) and $T \equiv 1$, the invariant surface condition for z, namely $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ becomes

$$\frac{2tdx}{x} = \frac{dt}{1} = \frac{du}{0}. \tag{55}$$

Integration of equations (55) gives a similarity solution of (1) in the form

$$u = F(z), \tag{56}$$

$$z = tx^{-2}, \tag{57}$$

Substitution of (57) in (1) yields:

$$4z^2 F'' + (6z-1)F' + zF^p = 0. \tag{58}$$

4. Direct Similarity Method

We transform (1) through

$$u = [v(x,t)]^{2/(1-p)}, \tag{59}$$

to the following Clarkson and Kruskal [6]. We seek solutions of (59) in the form

$$v(x,t) = \alpha(x,t) + \beta(x,t)f(z), \quad z = z(x,t). \tag{60}$$

We substitute (59) in (1) and require the resulting equation in the following form of an ordinary differential equation governing the function $f(z)$:

$$\Lambda_1(z) + \Lambda_2(z)f' + \Lambda_3(z)f + \Lambda_4(z)ff' + \Lambda_5(z)f^2 + \Lambda_6(z)f'^2 + \Lambda_7(z)f'' + ff'' = 0. \tag{61}$$

The functions $\Lambda_n(z)$, $n = 1, 2, \cdots, 7$ are introduced according to

$$-\alpha\alpha_t + \frac{1+p}{1-p}\alpha_x^2 + \alpha\alpha_{xx} + \frac{(1-p)}{2}V = \beta^2 z_x^2 \Lambda_1(z), \tag{62}$$

$$-\alpha\beta z_t + \frac{2(1+p)}{(1-P)}\beta\alpha_x z_x + \alpha\beta z_{xx} + 2\alpha\beta_x z_x = \beta^2 z_x^2 \Lambda_2(z), \tag{63}$$

$$-\alpha\beta_t - \beta\alpha_t + \frac{2(1+p)}{1-p}\alpha_x\beta_x + \alpha\beta_{xx} + \beta\alpha_{xx} = \beta^2 z_x^2 \Lambda_3(z), \tag{64}$$

$$-\beta z_t + \frac{2(1+p)}{1-p}\beta_x z_x + \beta z_{xx} + 2\beta_x z_x = \beta z_x^2 \Lambda_4(z), \tag{65}$$

$$-\beta\beta_t + \frac{1+p}{1-p}\beta_x^2 + \beta\beta_{xx} = \beta^2 z_x^2 \Lambda_5(z), \tag{66}$$

$$\frac{1+p}{1-p}\beta^2 z_x^2 = \beta^2 z_x^2 \Lambda_6(z), \tag{67}$$

$$\alpha = \beta\Lambda_7(z). \tag{68}$$

Remark 4.1. *If $\alpha(x,t)$ is to be obtained from an equation of the form $\alpha(x,t) = \tilde{\alpha}(x,t) + \beta(x,t)\Lambda(z)$, then we may set $\Lambda(z) = 0$.*

Remark 4.2. *If $\beta(x,t)$ is given by an equation of the form $\beta(x,t) = \tilde{\beta}(x,t)\Lambda(z)$, then we may choose $\Lambda(z) = 1$.*

Remark 4.3. *If the equation $\Lambda(z) = \tilde{z}(x,t)$ is to be solved for z, then we may write $\Lambda(z) = z$.*

In view of Remark 4.1, we satisfy (68) by taking $\alpha = \Lambda_7 = 0$. And equation (67) simply gives $\Lambda_6 = \frac{1+p}{1-p}$. With $\alpha = 0$, equations (63), (64) and (62) become $\Lambda_2 = \Lambda_3 = 0$ and

$$\frac{1-p}{2}V(x) = \beta^2 z_x^2 \Lambda_1(z). \tag{69}$$

We assume that

$$\beta = \beta_0 e^{\sigma x + tht}, \quad z = z_0 e^{\phi x + \mu t}. \tag{70}$$

Evidently z lies in the interval $[l, 0)$ when $x \in [0, \infty)$ and $t \in [0, \infty)$. Equations (65), (66) and (69), requires that Λ_5 is proportional to z^{-2} , Λ_4 is proportional to z^{-1} and Λ_1 is proportional to z^{n_1} respectively we indeed find that

$$\Lambda_5 = l_5 z^{-2}, \quad l_5 = -\frac{\theta}{\phi^2} + \frac{2}{1-p}\frac{\sigma^2}{\phi^2}, \tag{71}$$

$$\Lambda_4 = l_4 z^{-1}, \quad l_4 = -\frac{\mu}{\phi^2} + \frac{4}{1-p}\frac{\sigma}{\phi} + 1, \tag{72}$$

$$\Lambda_1 = l_1 z^{n_1}, \quad V(x,t) = \frac{2}{1-p}\beta_0^2 z_0^{n_1+2}\phi^2 l_1 e^{2(\sigma x + \theta t)} e^{(2+n_1)(\phi x + \mu t)}. \tag{73}$$

Substituting for $\Lambda_n(z)$, $n = 1, 2, \ldots, 7$, equation (61) is

$$z^2 ff'' + \frac{1+p}{1-p} z^2 f'^2 + l_5 f^2 + l_4 z f f' + l_1 z^{n_1+2} = 0. \tag{74}$$

On inserting $f(z) = F(\theta)$, $\theta = \log z$, equation (74) transforms into

$$FF'' - \frac{p+1}{p-1} F'^2 + (l_4 - 1)FF' + l_5 F^2 + l_1 e^{(n_1+2)\theta} = 0. \tag{75}$$

The condition $f(0)$ is finite corresponds to $F(-\infty)$ is finite. We transform (75) through $F(\theta) = e^{\frac{n_1+2}{2}\theta} G(\theta)$, $l_1 = \epsilon$ to the autonomous equation

$$\left[\frac{-2(n_1+2)}{p-1} - 1 + l_4\right] GG' + GG'' + \left[\frac{-(n_1+2)^2}{2(p-1)} + (l_4-1)\frac{n_1+2}{2} + l_5\right] G^2 - \frac{p+1}{p-1} G'^2 + \epsilon = 0. \tag{76}$$

It is clear that $G(\theta) \to \infty$ as $\theta \to -\infty$ since $n_1 > 0$. Writing $G = G_0(z) + \epsilon G_1(z)$ in (76) and equating the coefficients of $\epsilon^i, i = 0, 1$ to 0, we have

$$\left[\frac{-2(n_1+2)}{p-1} - 1 + l_4\right] G_0 G_0' + G_0 G_0'' + \left[\frac{-(n_1+2)^2}{2(p-1)} + (l_4-1)\frac{n_1+2}{2} + l_5\right] G_0^2 - \frac{p+1}{p-1} G_0'^2 = 0, \tag{77}$$

$$\begin{aligned} &\left[\frac{-2(n_1+2)}{p-1} - 1 + l_4\right] (G_0 G_1' + G_1 G_0') + G_0 G_1'' + G_1 G_0'' \\ &+2\left[\frac{-(n_1+2)^2}{2(p-1)} + (l_4-1)\frac{n_1+2}{2} + l_5\right] G_0 G_1 + 2\frac{p+1}{p-1} G_0' G_1' + 1 = 0. \end{aligned} \tag{78}$$

It is easily verified that a solution of (77) is

$$G_0(z) = p_1 e^{k\theta}, \tag{79}$$

where k is a negetive root of

$$\left[1 + \frac{p+1}{p-1}\right] k^2 + \left[\frac{-2(n_1+2)}{p-1} - 1 + l_4\right] k + \left[\frac{-(n_1+2)^2}{2(p-1)} + (l_4-1)\frac{n_1+2}{2} + l_5\right] = 0, \tag{80}$$

and $p_1 > 0$. Solving (80) we find that

$$k = \frac{(1-p)}{4}\left[\left(\frac{2(n_1+2)}{p-1} + 1 - l_4\right) \pm \left((1-l_4)^2 - \frac{8l_5}{1-p}\right)^{1/2}\right]. \tag{81}$$

Then a solution of (78) is

$$G_1(z) = p_2 e^{-k\theta}, \tag{82}$$

where

$$p_2 = -\frac{1}{\frac{4p}{p-1} p_0 p_1^2 + 2\left[\frac{(n_1+2)^2}{2(1-p)} + \frac{(n_1+2)}{2}(l_4-1) + l_5\right] p_0}. \tag{83}$$

The corresponding solution of (76) is

$$G(\theta) = p_1 e^{k\theta} + \epsilon p_2 e^{-k\theta}, \tag{84}$$

where p_1 is an arbitrary constant. Then

$$F(\theta) = p_1 e^{k + \frac{(n_1+2)}{2}\theta} + \epsilon p_2 e^{-k + \frac{(n_1+2)}{2}\theta}. \tag{85}$$

On inserting $f(z) = F(\theta)$, $\theta = \log z$ into (85), we have

$$f(z) = p_1 z^{k+\frac{(n_1+2)}{2}} + \epsilon p_2 z^{-k+\frac{(n_1+2)}{2}}. \tag{86}$$

Putting $\alpha = 0$, (86) and (70 in (59) we get a solution of (1):

$$u = \left[\beta_0 e^{\sigma(x+t)}\right]^{2/(1-p)} \left(p_1 z_0^{k+1+\frac{n_1}{2}} \exp\left[(\phi x + \mu t)(k+1+\frac{n_1}{2})\right] + \epsilon p_2 z_0^{-k+1+\frac{n_1}{2}} \exp\left[(\phi x + \mu t)(-k+1+\frac{n_1}{2})\right]\right)^{\frac{2}{1-p}}. \tag{87}$$

for

$$V(x,t) = \frac{2}{1-p}\beta_0^2 z_0^{n_1+2}\phi^2 l_1 e^{2(\sigma x+\theta t)} e^{(2+n_1)(\phi x+\mu t)}. \tag{88}$$

5. Results and Conclusions

The following solution

$$u(x) = \left[\frac{c^2}{2v_0}\right]^{1/(p-1)} (A/2x + b_0)^2, \tag{89}$$

of (2), when the variable coefficient

$$V(x) = v_0(\frac{Ax}{2} + b)^{-2p}, \tag{90}$$

is recovered from the classes of solutions of the nonlinear parabolic equation with the variable coefficient

$$u_t = \Delta u + v_0(\frac{Ax}{2} + b)^{-2p} u^p, \quad (x,t) \in \mathcal{R}^n \times (0,\infty). \tag{91}$$

By applying the Lie group of infinitesimal transformations to (91). In yet another study, we employed the direct method of Clarkson and Kruskal [6] to (91) to replace it by a nonlinear ordinary differential equation

$$z^2 f f'' + l_5 f^2 + \frac{1+p}{1-p} z^2 {f'}^2 + l_1 z^{n_1+2} = 0. \tag{92}$$

Perturbative solutions of (92) are derived and then the corresponding intermediate asymptotics of (91) is presented as

$$u(x,t) = \left[\beta_0 e^{\sigma(x+t)}\right]^{2/(1-p)} \left(p_1 z_0^{k+1+\frac{n_1}{2}} \exp\left[(\phi x + \mu t)(k+1+\frac{n_1}{2})\right] + \epsilon p_2 z_0^{-k+1+\frac{n_1}{2}} \exp\left[(\phi x + \mu t)(-k+1+\frac{n_1}{2})\right]\right)^{\frac{2}{1-p}}, \tag{93}$$

for

$$V(x,t) = \frac{2}{1-p}\beta_0^2 z_0^{n_1+2}\phi^2 l_1 e^{2(\sigma x+\theta t)} e^{(2+n_1)(\phi x+\mu t)}. \tag{94}$$

According to Barenblatt [2] similarity solutions of (91) play the role of intermediate asymptotics of the general solutions of classes of initial value problems.

References

[1] W.F.Ames, *Nonlinear Partial Differential Equations in Engineering*, Vol. 2, Academic Press, New York, (1972).

[2] G.I.Barenblatt, *Similarity, Self-similarity and Intermediate Asymptotics*, Consultant Bureau, New York, (1979).

[3] C.M.Bender and S.A.Orszag, *Advanced mathematical methods for Scientists and Engineers*, McGraw-Hill, New York, (1978).

[4] G.W.Bluman and J.D.Cole, *The general similarity solution of the heat equation*, J. Math. Mech., 18(1969), 1035-1047.

[5] G.W.Bluman and S.Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, (1989).

[6] P.A.Clarkson and M.D.Kruskal, *New similarity reductions of the Boussinesq equations*, J. Math. Phys., 36(1989), 2201-2213.

[7] George M.Murphy, *Ordinary Differential Equations and their Solutions*, D. Van Nostrand Company, Inc, New Jersey, (1960).

[8] M.C.Nucci and P.A.Clarkson, *The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzugh-Nagumo equation*, Phys. Lett., 164(1992), 49-56.