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Edge Geodetic Domination Number of a Graph

Research Article

P.Arul Paul Sudhahar[1]*, A.Ajitha[2] and A.Subramanian[3]

1 Department of Mathematics, Rani Anna Government College (W), Tirunelveli, Tamilnadu, India.

2 Department of Mathematics, Nanjil Catholic College of Arts and Science, Kaliakkavilai, Tamilnadu, India.

3 Research Department of Mathematics, MDT Hindu College, Tirunelveli, Tamilnadu, India.

Abstract: In this paper the concept of edge geodetic domination number of a graph is introduced. A set of vertices S of a graph is an edge geodetic domination set (EGD) if it is both edge geodetic set and a domination set of G. The edge geodetic domination number (EGD number) of G, γg_e(G) is the cardinality of a minimum EGD set. EGD numbers of some connected graphs are realized. Connected graphs of order p with EGD number p are characterized. It is shown that for any two integers p and q such that $2 \leq p \leq q$, there exist a connected graph G with γg(G) $= p$ and γg_e(G) $= q$. Also it is shown that there is a connected graph G such that γ(G) $= p$, g_e(G) $= q$ and γg_e (G) $= p + q$.

MSC: 05C12.

Keywords: Geodetic number, Edge geodetic number, Geodetic domination number, Edge geodetic domination number.

1. Introduction

By a graph $G = (V, E)$ we consider a finite undirected graph without loops or multiple edges. The order and size of a graph are denoted by p and q respectively. For the basic graph theoretic notations and terminology we refer to Buckley and Harary [2]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A geodetic set of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets.

The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an extreme vertex if the subgraph induced by its neighborhood is complete. A vertex v in a connected graph G is a cut vertex of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if $\Delta(< N(v) >) = |N(v)| - 1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. An acyclic connected graph is called a tree [2]. A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbor in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. A geodetic domination set of G is a subset of $V(G)$ which is both geodetic and dominating set of G. The minimum cardinality of a geodetic domination set is denoted by $\gamma g_e(G)$. A detailed study of geodetic domination set is available in [6]. A vertex v is a universal vertex of a graph G if $deg(v) = p - 1$. Edge geodetic set of a connected graph is studied in [1, 10].

* E-mail: arulpaulsudhar@gmail.com

2. Basic Concepts and Definitions

Definition 2.1. *A set of vertices S of a graph G is an edge geodetic domination set (EGD set) if it is both edge geodetic set and a domination set of G. The minimum cardinality among all the EGD sets of G is called edge geodetic domination number (EGD number) and is denoted by* $\gamma g_e(G)$.

Example 2.2. *Consider the graph G given in Figure 1. Here* $M = \{v_4, v_6, v_7\}$ *is an edge geodetic set.* $N = \{v_4,\ v_5\}$ *is a dominating set and* $S = \{v_4, v_5, v_6, v_7\}$ *is a minimum EGD set. Hence* $\gamma g_e(G) = 4$.

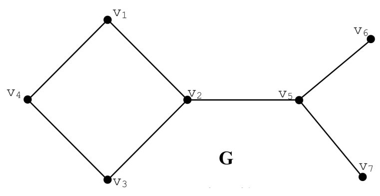

Figure 1:

Theorem 2.3. *Let G be a connected graph. Then* $2 \leq g_e(G) \leq \gamma g_e(G) \leq\ p$.

Proof. Any edge geodetic set has at least two vertices. Therefore $2 \leq g_e(G)$. Since every EGD set is an edge geodetic set $g_e(G) \leq \gamma g_e(G)$. Also the set of all vertices of G induces the graph G, we have$\gamma g_e(G) \leq p$. □

Remark 2.4. *The bounds in Theorem 2.3 are sharp. In Figure 1,* $2 < \gamma g(G) < \gamma g_e(G) < p$.

Theorem 2.5. *For any connected graph G of order p,* $2 \leq \gamma g(G) \leq \gamma g_e(G) \leq p$.

Proof. Since a geodetic domination set need at least two vertices, $2 \leq \gamma g\ (G)$. Also every EGD set is a geodetic domination set, $\gamma g(G) \leq \gamma g_e(G)$. Since the vertex set of G is both edge geodetic and domination set, $\gamma g_e(G) \leq p$. □

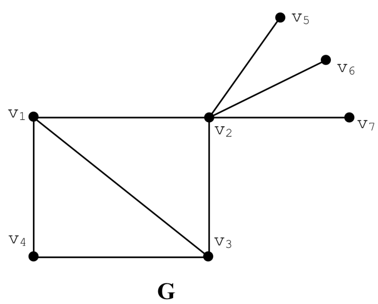

Figure 2:

Remark 2.6. *The bounds in Theorem 2.5 are sharp. In Figure 2,* $\gamma g(G) = 4$, $\gamma g_e(G) = 6$ *and* $p = 7$.

Theorem 2.7. *Each semi-extreme vertex of G belongs to every EGD set of G.*

Proof. Let S be an EGD set of G. Let u be a semi-extreme vertex of G. Take $u \notin S$. Let v be a vertex of $< N(u) >$ such that $\deg_{<N(u)>}(v) = |N(u)| - 1$. Let $v_1,\ v_2, \ldots,\ v_k$ (k $\geq$ 2) be the neighborhood of v in $< N\ (u) >$. Since S is also an edge geodetic set of G, the edge vu lies on the geodetic path $P : w,\ w_1, \ldots, v_i,\ u,\ v,\ v_j, \ldots,\ t$ where w, t $\in$ S. Since u is a semi extreme vertex of G, v and $v_j$are adjacent in G and so P is not a geodetic path of G. This contradicts our assumption. □

Theorem 2.8. *For a semi-extreme vertex G of order p, $\gamma g_e(G) = p$.*

Proof. Since each semi-extreme vertex belongs to every edge geodetic set and $V(G)$ is itself a domination set, the result follows. □

Theorem 2.9. *Each extreme vertex of G belongs to every EGD set of G.*

Proof. Since each extreme vertex of G belongs to every edge geodetic set of G, the result follows. □

Remark 2.10. *The set of all extreme vertices need not form an EGD set. Consider P_n of path graph having more than four vertices.*

Corollary 2.11. *For the complete graph K_p, $\gamma g_e(G) = p$.*

Theorem 2.12. *For a cycle Graph C_n of n vertices, $\gamma g_e(C_n) = 2$, when $n \leq 6$ and it is equal to $[(n-r) \div 3] + 1$ when $n > 6$, where r is the reminder when n is divided by 3.*

Proof. Since G is a cycle, two non-adjacent vertices in G defines an edge geodetic set so that $g_e(C_n) = 2$. Again each vertex dominates three vertices in a cycle, the result follows. □

Theorem 2.13. *For the complete bipartite graph $K_{m,n}$,*

$$\gamma_{g_e}(G) = \begin{cases} 2, & if\ m = n = 1 \\ n, & if\ n \geq 2,\ m = 1 \\ \min\{m, n\}, & if\ m, n \geq 2 \end{cases}$$

Proof. (i) When $m = n = 1$: $K_{m,n} = K_2$, complete graph of two vertices. Hence by Corollary 2.11, $\gamma g_e(G) = 2$. **(ii)** Here each n vertices are extreme vertices and belongs to every EGD set. **(iii)** Without loss of generality assume that $m \leq$ n. Take X = $\{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ be a partition of G. Consider $S = X$. Then S is a minimum edge geodetic set (By Theorem 2.11 of [2]). Also the set S dominate every vertex in G and is the minimum dominating set. Thus S is a minimum EGD set. Therefore $\gamma g_e(G) = |S| = m = \min\{m, n\}$. □

Theorem 2.14. *Let G be a connected graph, u be a cut vertex of G and let S be an EGD set of G. Then every component of $G - u$ contains some vertices of S.*

Proof. Let u be a cut vertex of G and S bean EGD set of G. Let there is some component, say C_1 of G $- u$ such that C_1 have no vertices of S. By Theorem 2.8, S contains all the extreme vertices of G so that C_1 has no extreme vertex of G. Hence C_1 has an edge ab. Since S is an EGD, ab lies on some $v - w$ path P: $v, v_1, v_2, \ldots, u, \ldots, a, b, \ldots, u_1, \ldots u, \ldots, w$ which is geodetic. Since u is a cut vertex of G, every path traverses through u. Then $v - a$ and b $- w$ are sub paths of P both contain u. Therefore P is not a path which is a contradiction. □

Theorem 2.15. *Let T be a tree such that N (x) belongs to end vertices for every internal vertex $x \in T$. Then EGD number is equal to the number of end vertices in T.*

Proof. Let S be the set of all end vertices of T. Since each extreme vertex belongs to EGD set of T,S is the subset of every EGD set of T. That is $\gamma g_e(T) \geq |S|$. The converse is trivial. □

Theorem 2.16. *Let G be a connected graph of order p. If there exist a unique vertex $v \in V(G)$ such that v is not a semi extreme vertex of G, $\gamma g_e(G) = p - 1$.*

Proof. If G is a connected graph having a unique non semi-extreme vertex v, then edge geodetic number of G, $g_e(G) = p - 1$ by Theorem 2.19 of [2]. Now every $p - 1$ vertices of a graph is always a domination set, these $p - 1$ vertices form a minimum EGD set. □

Corollary 2.17. *Let G be a connected graph of order $p \geq 3$. If G contains exactly one universal vertex, then $\gamma g_e(G) = p-1$.*

Corollary 2.18. *For the wheel graph $W_{1,p-1}$with $p \geq 4$; γg_e $(W_{1,p-1}) = p - 1$.*

Theorem 2.19. *Let G be a connected graph of order $p \geq 2$, then γg_e $(G) = 2$ if and only if there exist an edge geodetic set $S = \{x_1, x_2\}$ of G such that $d_m(x_1, x_2) \leq 3$.*

Proof. Let $\gamma g_e(G) = 2$. Take $S = \{x_1, x_2\}$ as an EGD set. If $d_m(x_1, x_2) \geq 4$, then the diametrical path contains at least three internal vertices. Then $\gamma g_e(G) \geq 3$ and is a contradiction. Thus $d_m(x_1, x_2) \leq 3$. Conversely, let $d_m(x_1, x_2) \leq 3$. If $S = \{x_1, x_2\}$ is an edge geodetic set, then it is also a dominating set. Therefore $\gamma g_e(G) = 2$. □

3. Realization Results

Theorem 3.1. *For any two integers p, $q \geq 2$, there exist a connected graph G such that $\gamma(G) = p$, $g_e(G) = q$ and $\gamma g_e(G)$ $= p + q$.*

Proof. Consider C_6 with vertex set $\{c_1, c_2, c_3, c_4, c_5, c_6\}$. Let H be the graph obtained by adding $q - 1$ vertices x_1, x_2,...,x_{q-1} with C_6 and join them at vertex c_1. Let G be the graph obtained from H by adding a path of 3 $(p - 2) + 1$ vertices say w$_0$,w_1, w_2,...,$w_{3(p-2)}$ where w_0 is adjacent with c_4 (Figure 3).

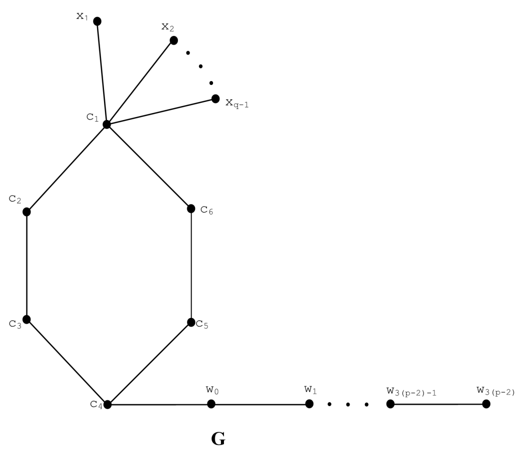

Figure 3:

Let $S_1 = \{c_1, c_4, w_2, w_5,..., w_{3(p-2)-1}\}$. Then S_1 is a minimum dominating set of G. Clearly S_1 Contains p vertices so that γ $(G) = p$. Take $S_2 = \{x_1, x_2,..., x_{q-1},w_{3(p-2)}\}$. Then S_2 is a minimum edge geodetic set of G. Thus $g_e(G) = q$. Now, $S_3 = \{x_1, x_2,...,x_{q-1},w_{3(p-2)}$,w_2,w_5,...,$w_{3(p-2)-1}$,c_1, $c_4\}$ is a minimum EGD set so that $\gamma g_e(G) = p + q$. □

Theorem 3.2. *For any two integers p and q such that $2 \leq p \leq q$ there exists a connected graph G with γg $(G) = p$ and $\gamma g_e(G\ ) = q$.*

Proof. Consider the following cases.

Case 1: Let $p \geq 3$, $q \geq 4$, $q \neq$ p + 1.

Take G_1 as the graph given in Figure 4. Now G_1 is obtained by adding two set of vertices $\{x_1, x_2, \ldots, x_{p-2}\}$ and $\{y_1, y_2, \ldots, y_{q-p}\}$ with the path $P : u, v, w$ in G_1 such a way that each $x_i$join with u and v and each y_i join with u, v, w but not mutually. Let $S_1 = \{x_1, x_2, \ldots, x_{p-2}, u, w\}$. Then S_1 is a minimum geodetic domination set of G_1. Therefore $\gamma g(G_1) = p$. Since v is the unique universal vertex, by Corollary 2.2, $\gamma g_e(G_1) = G_1 - 1 = (q - p + p - 2 + 3) - 1 = q$.

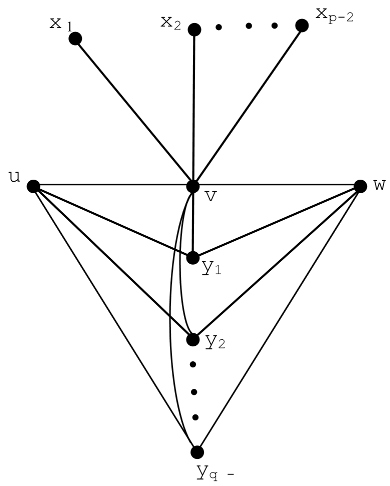

Figure 4: G_1

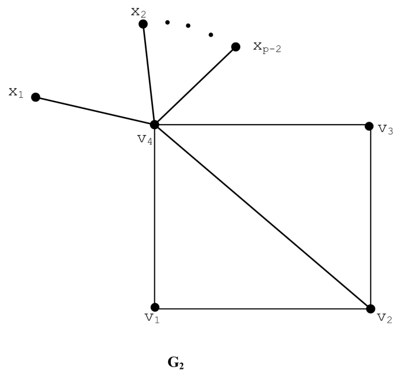

Figure 5: G_2

Case 2: $p \geq 3$, $q \geq 4$, $q = p + 1$.

Consider the following graph G_2 (Figure 5). Take $S_2 = \{x_1, x_2, \ldots, x_{p-2}, v_1, v_3\}$. It is a minimum geodetic domination set of G_2. Therefore $\gamma g(G_2) = p$. Now S_2 is not an EGD set since the edge $v_2v_4$not lies in any edge geodetic path. But $S_2 \cup \{v_2\}$ is an EGD set. Therefore $\gamma g_e(G_2) = (p - 2 + 3) = p + 1 = q$.

Case 3: Let $p = 2$, $q \geq 4$.

Consider the graph G_3 given in Figure 6. G_3 is obtained using the path $P : u, v, w$ of three vertices, by adding $q - 2$ new vertices $x_1, x_2, \ldots, x_{q-2}$ and join these vertices with u, v, w. Here v is a universal vertex. Therefore $\gamma g_e(G_3) = G_3 - 1 = q - 2 + 3 - 1 = q$. But $S_3 = \{u, w\}$ is a geodetic domination set of G_3. Therefore the geodetic domination number $\gamma g_e(G_3) = 2$.

Case 4: Let $p = 2$, $q = 3$.

Consider the graph G_4 given in Figure 7. Here $S_4 = \{x_2, x_4\}$ is a dominating set but not an EGD set. Therefore $\gamma g_e(G_4) = p$. Take $S_5 = \{x_2, x_3, x_4\}$. It is a minimum EGD set. Therefore $\gamma g_e(G_4) = 3$.

Case 5: Let $p = q$. Take T as the bipartite graph $K_{1,p}$. Then $\gamma g(T) = \gamma g_e(T) = p$.

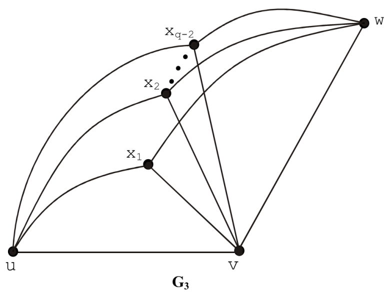

Figure 6: G_3

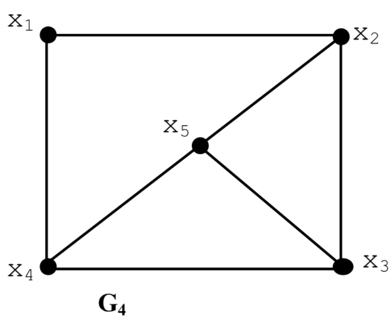

Figure 7: G_4

□

4. Conclusion

The results used in this article can extended to find properties of upper EGD set, forcing EGD set and EGD number of join of graphs, EGD number of composition of graphs and EGD hull number of graphs and so on.

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