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Correction to the Number of Homomorphisms From Quaternion Group into Some Finite Groups*

Research Article

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Abstract: Using only elementary group theory, we determine the number of homomorphisms from quaternion group into some finite groups.

MSC: 20K30.

Keywords: The dihedral group, the quaternion group, the modular group, homomorphisms.

1. Introduction

The number of homomorphisms from quaternion group into some finite groups have been showed by the reference [1]. But, some results are mistakes. For readers' convenience, these theorems are corrected in this paper. We fix some notations used in this paper: the dihedral group $D_n =< x_n,\ y_n \mid x_n^n = e = y_n^2,\ x_n y_n = y_n x_n^{-1} >$ the quaternion group $Q_m =< a_m,\ b_m \mid a_m^{2m} = e = b_n^4,\ a_m b_m = b_m a_m^{-1} >$ the quasi-dihedral group $QD_{2^\alpha} =< s_\alpha,\ t_\alpha \mid s_\alpha^{2^{\alpha-1}} = e = t_\alpha^2,\ t_\alpha s_\alpha = s_\alpha^{2^{\alpha-2}-1} t_\alpha >$ the modular group $M_{p^\beta} =< r_\beta,\ f_\beta \mid r_\beta^{p^{\beta-1}} = e = f_\beta^p,\ f_\beta r_\beta = r_\beta^{p^{\beta-2}+1} f_\beta >$. Write (m,n) for the greatest common divisor of m and n. Denote by $m \mid n$ the m divides n. Denote by $\varphi(n)$ the number of positive integers not exceeding n which are co-prime to n. Other notation used will be mostly standard, refer to [2].

2. Proof of the Theorems

For readers' convenience, Theorem 3.2 in [1] is corrected here as

Theorem 2.1. *Let m be a positive integer and n a positive even integer such that $n \equiv 2 \pmod 4$. Then the number of group homomorphisms from Q_m into D_n is $4 + 4n + n(\sum_{k|(m,n)} \varphi(k))$, if m is even; $2 + n(\sum_{k|(m,n)} \varphi(k))$, if m is odd.*

Proof. Suppose that ρ: $Q_m \longrightarrow D_n$ is a group homomorphism. Since $\rho(b_m^4) = \rho(b_m)^4 = e$, it follows that $|\rho(b_m)| \mid (4,n)$. By $n \equiv 2 \ (mod\ 4)$, we obtain that $|\rho(b_m)| \mid 2$, this implies that $\rho(b_m) \in \{e,\ x_n^{\frac{n}{2}},\ x_n^{\gamma} y_n\}$, where $0 \leq \gamma < n$. Noting that $\rho(a_m b_m)^2 = \rho(b_m)^2 = e$, we have $|\rho(a_m)| \mid m$. This implies either $\rho(a_m) = x_n^{\alpha} y_n$ or $\rho(a_m) = x_n^{\beta}$, where $0 \leq \alpha, \beta < n$. If $\rho(b_m) = e$, then $\rho(a_m b_m)^2 = \rho(b_m)^2 = \rho(a_m)^2 = e$ and $|\rho(a_m)| \mid (2,m)$. When m is even, we have $\rho(a_m) \in \{e,\ x_n^{\alpha} y_n,\ x_n^{\frac{n}{2}}\}$,

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it follows that there are $n+2$ homomorphisms in this case. When m is odd, $\rho(a_m) = e$, thus we have trivial homomorphism in this case.

If $\rho(b_m) = x_n^{\gamma} y_n$ and $\rho(a_m) = x_n^{\beta}$,where $0 \le \gamma, \beta < n$, then $|\rho(a_m)| \mid (m, n)$. Thus there are $n(\sum_{k|(m,n)} \varphi(k))$ such homomorphisms. If $\rho(b_m) = x_n^{\gamma} y_n$ and $\rho(a_m) = x_n^{\alpha} y_n$, then $\rho(a_m b_m) = \rho(a_m)\rho(b_m) = x_n^{\alpha-\gamma}$. On the other hand, $\rho(a_m b_m) = \rho(b_m)\rho(a_m^{-1}) = x_n^{\gamma-\alpha}$, so we obtain that $x_n^{2(\alpha-\gamma)} = e$, $\alpha - \gamma \in \{0, \ \frac{n}{2}\}$. Thus we have $2n$ such homomorphisms.

If $\rho(b_m) = x_n^{\frac{n}{2}}$ and $\rho(a_m) = x_n^{\beta}$, then $|\rho(a_m)| \mid (m, n)$ and $\rho(a_m b_m) = \rho(a_m)\rho(b_m) = x_n^{\frac{n}{2}+\beta}$. On the other hand, $\rho(a_m b_m) = \rho(b_m)\rho(a_m^{-1}) = x_n^{\frac{n}{2}-\beta}$, this implies that $x_n^{2\beta} = e$, $\beta \in \{0, \ \frac{n}{2}\}$. When m is odd, we obtain that $\rho(a_m) = e$, thus there is 1 homomorphism in this case. When m is even, we have $\rho(a_m) \in \{e, \ \frac{n}{2}\}$, so there are 2 homomorphisms in this case.

If $\rho(b_m) = x_n^{\frac{n}{2}}$ and $\rho(a_m) = x_n^{\alpha} y_n$, then $\rho(a_m^m b_m) = \rho(a_m)^m \rho(b_m) = (x_n^{\alpha} y_n)^m (x_n^{\frac{n}{2}})$. On the other hand, $\rho(a_m^m b_m) = \rho(b_m)^3 = x_n^{\frac{3n}{2}}$, this implies that $(x_n^{\alpha} y_n)^m = e$. Note that $|x_n^{\alpha} y_n| = 2$ and m is even, thus we have n such homomorphisms. Hence we get the result. □

Theorem 3.3 in [1] is corrected here as

Theorem 2.2. *Let m be a positive integer and n a positive even integer such that $n \equiv 0(\ mod\ 4)$. Then the number of group homomorphisms from Q_m into D_n is $4 + n(\sum_{k|(m,n)} \varphi(k))$, if m is odd; $4 + 4n + n(\sum_{k|(m,n)} \varphi(k))$, if m is even.*

Proof. Suppose that ρ: $Q_m \longrightarrow D_n$ is a group homomorphism. Since $\rho(b_m^4) = \rho(b_m)^4 = e$, it follows that $|\rho(b_m)| \mid (4, 2n)$. Noting that $(4, 2n) = 4$, this implies that $\rho(b_m) \in \{e, \ x_n^{\frac{n}{2}}, \ x_n^{\frac{n}{4}}, \ x_n^{\frac{3n}{4}}, \ x_n^{\gamma} y_n\}$, where $0 \le \gamma < n$. By $\rho(a_m) \in D_n$, we obtain either $\rho(a_m) = x_n^{\alpha} y_n$ or $\rho(a_m) = x_n^{\beta}$, where $0 \le \alpha, \beta < n$.

If $\rho(b_m) \in \{e, \ x_n^{\frac{n}{2}}\}$ and $\rho(a_m) = x_n^{\beta}$, then $|\rho(a_m)| \mid m$ and $\rho(a_m b_m) = x_n^{\beta} \rho(b_m)$. On the other hand, $\rho(a_m b_m) = \rho(b_m)\rho(a_m)^{-1} = \rho(b_m) x_n^{-\beta}$, this implies that $x_n^{\beta} \rho(b_m) = \rho(b_m) x_n^{-\beta}$ and $\beta \in \{0, \ \frac{n}{2}\}$. When m is odd, $\rho(a_m)$ must be e, thus we have 2 homomorphisms in this case. When m is even, $\rho(a_m) = e$ or $x_n^{\frac{n}{2}}$, thus we have 4 homomorphisms in this case.

If $\rho(b_m) \in \{e, \ x_n^{\frac{n}{2}}\}$ and $\rho(a_m) = x_n^{\alpha} y_n$, then $\rho(a_m^m b_m) = \rho(a_m)^m \rho(b_m) = (x_n^{\alpha} y_n)^m \rho(b_m)$. On the other hand, $\rho(a_m^m b_m) = \rho(b_m)^3$, it follows that $(x_n^{\alpha} y_n)^m = e$ and ρ is group homomorphism only when m is even. Thus we have $2n$ homomorphisms in this case. If $\rho(b_m) = x_n^{\gamma} y_n$ and $\rho(a_m) = x_n^{\beta}$, where $0 \le \gamma, \beta < n$, this implies that $|\rho(a_m)| \mid (m, n)$. Thus there are $n(\sum_{k|(m,n)} \varphi(k))$ homomorphisms in this case.

If $\rho(b_m) = x_n^{\gamma} y_n$ and $\rho(a_m) = x_n^{\alpha} y_n$, then ρ is well defined only when m is even and ρ is homomorphism when $\alpha - \gamma \in \{0, \ \frac{n}{2}\}$. So we have $2n$ homomorphisms in this case. If $\rho(b_m) \in \{x_n^{\frac{n}{4}}, \ x_n^{\frac{3n}{4}}\}$ and $\rho(a_m) = x_n^{\alpha} y_n$. Noting that $\rho(a_m b_m)^2 = (\rho(a_m b_m))^2 = (x_n^{\alpha-\frac{n}{4}} y_n)^2 = e$. But $\rho(a_m b_m)^2 = \rho(b_m)^2 \ne e$, thus ρ is not well defined.

If $\rho(b_m) \in \{x_n^{\frac{n}{4}}, \ x_n^{\frac{3n}{4}}\}$ and $\rho(a_m) = x_n^{\beta}$, then $\rho(a_m b_m) = x_n^{\beta} \rho(b_m)$. On the other hand, $\rho(a_m b_m) = \rho(a_m) x_n^{-\beta}$, this implies that $x_n^{2\beta} = e$ and $\beta \in \{0, \ \frac{n}{2}\}$. Note that $|\rho(a_m)| \nmid m$, we obtain that $\beta = \frac{n}{2}$, thus $\rho(a_m)$ must be $\frac{n}{2}$ and m is odd. Thus we have 2 homomorphisms in this case. Hence we get the result. □

Theorem 4.2 in [1] is corrected here as

Theorem 2.3. *Suppose m is an even positive integer and $\alpha > 3$ is any integer. Then the number of homomorphisms from Q_m into QD_{2^α} is $4 + 2^{\alpha+1} + 2^{\alpha-2}(\sum_{k|(m,2^{\alpha-2})} \varphi(k) + \sum_{k|(2m,2^{\alpha-2}),k\nmid m} \varphi(k))$.*

Proof. Suppose ρ: $Q_m \longrightarrow QD_{2^\alpha}$ is a group homomorphism. Since $|\rho(b_m)| \mid 4$, we obtain either $\rho(b_m) = s_\alpha^t$ or $\rho(b_m) = s_\alpha^{k_2} t_\alpha$, where $0 \leq t, k_2 < 2^{\alpha-1}$. As $|\rho(a_m)| \mid (2m, 2^\alpha)$, this implies that either $\rho(a_m) = s_\alpha^n$ or $\rho(a_m) = s_\alpha^{k_1} t_\alpha$, where $0 \leq n, k_1 < 2^{\alpha-1}$.

If $\rho(b_m) = s_\alpha^t$ and $\rho(a_m) = s_\alpha^n$, where $t \in \{0,\ 2^{\alpha-2}\}$, then $|\rho(b_m)| = 2$, $|\rho(a_m)| \mid m$ and $\rho(a_m b_m) = s_\alpha^{n+t}$. On the other hand, $\rho(a_m b_m) = \rho(b_m)\rho(a_m)^{-1} = s_\alpha^{t-n}$, it follows that $s_\alpha^{2n} = e$. Noting that $0 \leq n < 2^{\alpha-1}$, we have $n \in \{0,\ 2^{\alpha-2}\}$. Thus we have 4 homomorphisms in this case. If $\rho(b_m) = s_\alpha^t$ and $\rho(a_m) = s_\alpha^n$, where $t \in \{2^{\alpha-3},\ 3\cdot 2^{\alpha-3}\}$, then $|\rho(b_m)| = 4$, $|\rho(a_m)| \nmid m$ and $\rho(a_m b_m) = s_\alpha^{n+t}$. On the other hand, $\rho(a_m b_m) = \rho(b_m)\rho(a_m)^{-1} = s_\alpha^{t-n}$, it follows that $s_\alpha^{2n} = e$ and $|\rho(a_m)| \mid 2$. But $|\rho(a_m)| \nmid m$, thus ρ is not a homomorphism.

If $\rho(b_m) = s_\alpha^{k_2} t_\alpha$ and $\rho(a_m) = s_\alpha^n$, where k_2 is odd, then $|\rho(b_m)| = 4$ and $|\rho(a_m)| \nmid m$. Noting that $\rho(b_m)^2 = \rho(a_m b_m)^2 = (\rho(a_m)\rho(b_m))^2 = s_\alpha^{(k_2+n)2^{\alpha-2}} \neq e$ and k_2 is odd, it follows that n is even. Thus we have $2^{\alpha-2}(\sum_{k|(2m,2^{\alpha-2}),k\nmid m} \varphi(k))$ homomorphisms in this case. If $\rho(b_m) = s_\alpha^{k_2} t_\alpha$ and $\rho(a_m) = s_\alpha^n$, where k_2 is even, then $|\rho(b_m)| = 2$ and $|\rho(a_m)| \mid m$. Noting that $\rho(b_m)^2 = \rho(a_m b_m)^2 = (\rho(a_m)\rho(b_m))^2 = s_\alpha^{(k_2+n)2^{\alpha-2}} = e$ and k_2 is even, this implies that n is even and $|\rho(a_m)| \mid 2^{\alpha-2}$. Thus we have $2^{\alpha-2}(\sum_{k|(m,2^{\alpha-2})} \varphi(k))$ homomorphisms in this case.

If $\rho(b_m) = s_\alpha^t$ and $\rho(a_m) = s_\alpha^{k_1} t_\alpha$, where $t \in \{2^{\alpha-3},\ 3\cdot 2^{\alpha-3}\}$, $0 \leq k_1 < 2^{\alpha-1}$, then $|\rho(b_m)| = 4$ and $\rho(a_m^m b_m) = (s_\alpha^{k_1} t_\alpha)^m s_\alpha^t$. On the other hand, $\rho(a_m^m b_m) = s_\alpha^{3t}$, this implies that $(s_\alpha^{k_1} t_\alpha)^m \neq e$. When $m \equiv 0(mod\ 4)$, $(s_\alpha^{k_1} t_\alpha)^m = e$, but $(s_\alpha^{k_1} t_\alpha)^m \neq e$, thus ρ is not a homomorphism in this case; when $m \equiv 2(mod\ 4)$, $(s_\alpha^{k_1} t_\alpha)^m = (s_\alpha^{k_1} t_\alpha)^2 \neq e$, implying that $|\rho(a_m)| = 4$ and k_1 is odd, so we have $2^{\alpha-1}$ homomorphisms in this case.

If $\rho(b_m) = s_\alpha^t$ and $\rho(a_m) = s_\alpha^{k_1} t_\alpha$, where $t \in \{0,\ 2^{\alpha-2}\}$, $0 \leq k_1 < 2^{\alpha-1}$, then $|\rho(b_m)| = 2$. Noting that $(s_\alpha^{k_1} t_\alpha)^m = s_\alpha^{2t} = e$, when $m \equiv 0(mod\ 4)$, $(s_\alpha^{k_1} t_\alpha)^m = e$, we have 2^α homomorphisms in this case; when $m \equiv 2(mod\ 4)$, k_1 must be even, we have $2^{\alpha-1}$ homomorphisms in this case.

If $\rho(b_m) = s_\alpha^{k_2} t_\alpha$ and $\rho(a_m) = s_\alpha^{k_1} t_\alpha$, then $\rho(a_m b_m) = s_\alpha^{k_1+k_2(2^{\alpha-2}-1)}$. Since $\rho(a_m b_m) = s_\alpha^{k_2-k_1}$, it follows that $s_\alpha^{2(k_1-k_2)+k_2 2^{\alpha-2}} = e$. When k_2 is even, $k_1 - k_2 \in \{0,\ 2^{\alpha-2}\}$, we have $2^{\alpha-1}$ homomorphisms; when k_2 is odd, $k_1 - k_2 \in \{2^{\alpha-3},\ 3\cdot 2^{\alpha-3}\}$, we have $2^{\alpha-1}$ homomorphisms in this case. Hence we get the result. □

Theorem 5.2 in [1] is corrected here as

Theorem 2.4. *Let m is a positive integer and $\alpha > 3$. Then the number of homomorphisms from Q_m into M_{2^α} is 12, if m is odd; 32, if m is even.*

Proof. Suppose ρ: $Q_m \longrightarrow M_{2^\alpha}$ is a group homomorphism, then we may assume that $\rho(a_m) = r_\alpha^{k_1} f_\alpha^{m_1}$ and $\rho(b_m) = r_\alpha^{k_2} f_\alpha^{m_2}$, where $|r_\alpha^{k_1}| \mid (2m, 2^{\alpha-1})$, $|r_\alpha^{k_2}| \mid 4$, $m_1, m_2 = 0,\ 1$. Since $\rho(a_m b_m) = r_\alpha^{k_1+k_2+m_1 k_2 2^{\alpha-2}} f_\alpha^{m_1+m_2}$ and $|\rho(a_m b_m)| \mid 4$, we obtain that $k_1 + k_2 \in \{0,\ 2^{\alpha-3},\ 3\cdot 2^{\alpha-3},\ 5\cdot 2^{\alpha-3},\ 7\cdot 2^{\alpha-3},\ 2^{\alpha-2},\ 3\cdot 2^{\alpha-2},\ 2^{\alpha-1}\}$.

If $k_2 \in \{0, 2^{\alpha-2}\}$ and $m_2 \in \{0,1\}$, then $\rho(a_m b_m)^2 = \rho(b_m)^2 = e$ and $|\rho(a_m)| \mid m$. When m is odd, $\rho(a_m)$ must be e, we have 4 homomorphisms in this case; when $m \equiv 2(mod 4)$, $|\rho(a_m)| \mid (m, 2^\alpha) = 2$, we obtain that $k_1 \in \{0,\ 2^{\alpha-2}\}$, we have 16 such homomorphisms in this case; when $m \equiv 0(mod 4)$, we have $|\rho(a_m)| \mid (m, 2^\alpha) = 4$, it follows that $k_1 \in \{0,\ 2^{\alpha-2},\ 2^{\alpha-3},\ 3 \cdot 2^{\alpha-3}\}$, we have 32 such homomorphisms in this case.

If $k_2 \in \{2^{\alpha-3}, 3 \cdot 2^{\alpha-3}\}$ and $m_2 \in \{0,1\}$, then $\rho(a_m b_m)^2 = \rho(b_m)^2 \neq e$ and $|\rho(a_m)| \nmid m$. When m is odd, we have $|\rho(a_m)| \mid (2m, 2^\alpha) = 2$, this implies that $|\rho(a_m)| = 2$ and $k_1 = 2^{\alpha-2}$. Thus we have 8 such homomorphisms in this case. When $m \equiv 2(mod 4)$, note that $|\rho(a_m)| \mid (2m, 2^\alpha) = 4$, it follows that $|\rho(a_m)| = 4$ and $k_1 \in \{2^{\alpha-3},\ 3 \cdot 2^{\alpha-3}\}$. Thus we have 16 such homomorphisms in this case. When $m \equiv 0(mod 4)$, we have $\rho(a_m)^m = e$, but $|\rho(a_m)| \nmid m$, thus ρ is not a homomorphism. Hence we get the result. □

References

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