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Vertex Removal and the Edge Domination Number of Graphs

Research Article

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Abstract: The paper is about the edge domination of the graph. The change in the edge domination number is observed under the operation of vertex removal. Necessary and sufficient condition is proved under which the edge domination number decreases when a vertex is removed from the graph. Calculation of the number of minimum edge dominating sets of the graph is given.

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1. Introduction

Various graph operations have been considered in graph theory. In particular, the effect of these operations on various parameters like domination number, independence number, vertex covering number etc. have been studied by several authors [3, 4]. A particular operation known as vertex removal has also been considered by several authors. For example, the domination number of the graph may increase, decrease or remains unchange when a vertex is removed from the graph [4]. It is interesting to know what happens to the parameters related to edges of the graph when a vertex is removed from the graph. The concept of edge domination number of the graph. We prove necessary and sufficient condition under which the edge domination number of the avertex is removed from the graph. We prove necessary and sufficient condition under which the edge domination number decreases when a vertex is removed from the graph. We consider only finite, simple and undirected graphs. For the graph G = (V(G), E(G)), the set of vertices $\{w \mid vw \in E(G)\} = N(v)$ is the vertex neighborhood of an edge e = uv, denoted by N[e]. We follow West [5] for graph theoretical notations and Haynes et al. [4] for the terms related to the domination.

For the graph G, a subset F of an edge set E(G) is said to be an edge dominating set of the graph G if for every edge e not in F is adjacent to some edge in F. An edge dominating set F of E is a minimal edge dominating set if F does not have a proper subset which is an edge dominating set. An edge dominating set with minimum cardinality is a minimum edge

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dominating set. The cardinality of a minimum edge dominating set is the edge domination number (denoted by $\gamma'(G)$) of the graph G [4]. The minimum edge dominating set is also known as a γ' -set of the graph G. The characterization of the minimal dominating set of the graph is given in [4]. We give the characterization of the minimal edge dominating set of the graph by following theorem.

Theorem 1.1. The edge dominating set F is a minimal edge dominating set if and only if for each edge $e \in F$, one of the following two conditions holds.

(1) e is an isolated edge of F.

(2) there exists an edge $f \in E - F$ such that f is adjacent to only one edge in F namely e.

Definition 1.2. Let G be a graph. $F \subset E(G)$ and $e \in F$. The edge private neighborhood of e with respect to the set F is the set of edges $\{f \in E(G) | N[f] \cap F = \{e\}\}$, denoted by $pn_e[e, F]$.

The characterization of the minimal edge dominating set can be restated with the following Theorem.

Theorem 1.3. Let G be a graph and F be an edge dominating set of G then F is a minimal edge dominating set if and only if for every edge e in F, $pn_e[e, F] \neq \phi$.

2. Main Results

The following theorem says that the edge domination number of the graph does not increase when a vertex is removed from the graph.

Theorem 2.1. Let G be a graph and $v \in V(G)$ then $\gamma'(G - v) \leq \gamma'(G)$.

Proof. Let F be a minimum edge dominating set of G.

Case 1: There is no edge incident to v which is in F, then |F| is an edge domination number of G - v. Therefore $\gamma'(G - v) \leq |F| = \gamma'(G)$.

Case 2: Suppose that there is an edge uv which is in F. If there is no other edge of the graph which is incident at u then $F - \{uv\}$ is an edge dominating set of G - v. If there is an edge f different from uv, which is incident at u then consider the set $F_0 = (F - \{uv\}) \cup \{f\}$. Then obviously F_0 is an edge dominating set of G - v. Therefore $\gamma'(G - v) \leq |F_0| = |F| = \gamma'(G)$. Thus from both the cases, we have $\gamma'(G - v) \leq \gamma'(G)$.

Let G be a graph and $v \in V(G)$. Suppose that deg(v) = k and vu_1, vu_2, \ldots, vu_k are all the edges incident at v. Let F be a minimum edge dominating set of G. Now consider the subgraph G - v. Let m be the number of edges of G - v which are incident at u_1 or u_2 or \ldots or u_k but they are not members of F. Let these edges be f_1, f_2, \ldots, f_m . Now consider the set $F_1 = (F - \{vu_1, vu_2, \ldots, vu_k\}) \cup \{f_1, f_2, \ldots, f_m\}$ then obviously F_1 is an edge dominating set of G - v and $|F_1| \leq |F|$. If m = k then obviously $|F_1| = |F|$ and if m < k then $|F_1| < |F|$. Thus we have the following theorem.

Theorem 2.2. Let G be a graph and $v \in V(G)$ then $\gamma'(G - v) < \gamma'(G)$ if m < k.

Theorem 2.3. Let G be a graph and $v \in V(G)$. If $\gamma'(G - v) < \gamma'(G)$ and F_1 is a minimum edge dominating set of G - v and e is an edge of G incident at v then $F_1 \cup \{e\} = F$ is a minimum edge dominating set of G.

Proof. Let f be any edge of G. If v is an end vertex of f and $f \neq e$ then f is adjacent to e. If v is not an end vertex of f then f is an edge of G - v and hence f is adjacent to some edge h of F_1 which is also a member of F. Thus f is adjacent to some member of F. This proves that F is an edge dominating set of G. Obviously, F is a minimum edge dominating set of

Theorem 2.4. Let G be a graph and $v \in V(G)$ then $\gamma'(G - v) < \gamma'(G)$ if and only if for every minimum edge dominating set F_1 of G - v, there is a neighbor u of v in G such that for every edge uw in G (except uv), $uw \notin F_1$.

Proof. Suppose that $\gamma'(G - v) < \gamma'(G)$. Let F_1 be any minimum edge dominating set of G - v then F_1 can not be an edge dominating set of G. Therefore there is an edge e (not dominated by any edge of F_1) such that $e \notin F_1$. If there is an edge uw such that $uw \in F_1$ then it means that e is dominated by an edge of F_1 which is not true. Thus no edge incident to u belongs to F_1 .

Conversely, Suppose the condition is satisfied. Let F be a subset of E(G) such that $|F| \leq \gamma'(G-v)$. Suppose that F is an edge dominating set of G. Suppose that $vu_1, vu_2, ..., vu_k$ are the edges which are in F. Let $F_1 = (F - \{vu_1, vu_2, ..., vu_k\}) \cup \{u_1w_1, u_2w_2, ..., u_kw_k\}$ then F_1 is an edge dominating set of G - v and $|F_1| = |F|$. Suppose that $|F_1| < \gamma'(G-v)$ then we have an obvious contradiction. Suppose that $|F_1| = \gamma'(G-v)$ then for every edge vu_i (i = 1, 2, ..., k) there is an edge u_iw_i which is in F_1 and $|F_1| = \gamma'(G-v)$ means that F_1 is a minimum edge dominating set of G - v. Which violates our hypothesis. Therefore $\gamma'(G-v) < \gamma'(G)$.

Theorem 2.5. Let G be a graph and $v \in V(G)$ then $\gamma'(G - v) < \gamma'(G)$ if and only if there is a minimum edge dominating set F of G which contains an edge vu such that

- (1) $F \{vu\}$ is an edge dominating set of G v.
- (2) For every edge uw in G v, $uw \notin F$.

Proof. If the conditions stated in the theorem are satisfy then obviously $\gamma'(G-v) < \gamma'(G)$.

Conversely, Suppose that $\gamma'(G-v) < \gamma'(G)$. Let F_1 be a minimum edge dominating set of G - v. By the Theorem 2.4, there is an edge vu such that for every edge uw in G - v, uw $\notin F_1$. Let $F = F_1 \cup \{uv\}$. Let f be any edge of G. If f is an edge of G - v then f is adjacent to some member of F_1 . If v is an end vertex of f then f = vu or f is adjacent to vu. Therefore from all the above cases, f is in F or f is adjacent to some member of F. Thus F is an edge dominating set of G. Also $|F| = |F_1| + 1$. Since $\gamma'(G - v) < \gamma'(G)$, F is a minimum edge dominating set of G. It is obvious that conditions (1) and (2) are satisfied by F.

Corollary 2.6. Let G be a graph and $v \in V(G)$ then $\gamma'(G-v) < \gamma'(G)$ if and only if there is a minimum edge dominating set F of G and an edge vu such that $vu \in F$ and vu is an isolated edge in F (that is, vu is not adjacent to any other edge of F) and $F - \{vu\}$ is an edge dominating set of G - v.

Corollary 2.7. Let G be a graph and $v \in V(G)$. If $\gamma'(G - v) < \gamma'(G)$ then $\gamma'(G - v) = \gamma'(G) - 1$.

We consolidate the above theorems and corollary to state the following theorem.

Theorem 2.8. Let G be a graph and $v \in V(G)$ then the following statements are equivalent.

(1)
$$\gamma'(G-v) < \gamma'(G)$$

- (2) There is a minimum edge dominating set F of G and an edge uv in F such that for every edge uw in G(except uv), uw $\notin F$ and $F \{uv\}$ is an edge dominating set of G v.
- (3) For every minimum edge dominating set F_1 of G v, there is a neighbor u of v in G such that for every edge uw in G, $uw \notin F_1$.
- (4) there is a minimum edge dominating set F of G and an edge vu such that $vu \in F$ and vu is an isolated edge in F and F - $\{vu\}$ is an edge dominating set of G - v.

3. Counting the Number of Minimum Edge Dominating Sets of the Graph

Let G be a graph and $v \in V(G)$ such that $\gamma'(G-v) < \gamma'(G)$. Let $F_1, F_2, ..., F_k$ be all the minimum edge dominating sets of G - v. Suppose that $u_{11}, u_{12}, ..., u_{1j_1}$ are the neighbors of v such that for each $i = 1, 2, ..., j_1$, for every u_{1i} , every edge incident to vu_{1i} is not in F_1 . Let $E_{1i} = F_1 \cup \{vu_{1i}\}$ ($i = 1, 2, ..., j_1$) then obviously, E_{1i} are minimum edge dominating sets of G. Similarly, if $u_{21}, u_{22}, ..., u_{2j_2}$ are the neighbors of v such that for each $i = 1, 2, ..., j_2$, for every u_{2i} , every edge incident to vu_{2i} is not in F_2 . Let $E_{2i} = F_2 \cup \{vu_{2i}\}$ ($i = 1, 2, ..., j_2$) then the sets E_{2i} are minimum edge dominating sets of G. Continuing in this way, for every l = 1, 2, ..., k, we get minimum edge dominating sets $E_{l1}, E_{l2}, ..., E_{lj_l}$ from the minimum edge dominating sets F_l of G - v. Thus, we have

$$\bigcup_{l=1}^{k} \left(\bigcup_{i=1}^{j_l} E_{li} \right) = \mathcal{A}_v$$

Suppose that $v_1, v_2, ..., v_n$ are vertices of the graph G such that $\gamma'(G - v_i) < \gamma'(G)$ for i = 1, 2, ..., n then as discussed above, if we have all the minimum edge dominating sets of $G - v_i$, we obtain a family of minimum edge dominating sets of G. Let us denote this family of minimum edge dominating sets by \mathcal{A}_{v_i} then the collection of minimum edge dominating sets of G is $\mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} \cup \cdots \cup \mathcal{A}_{v_n}$.

Example 3.1. Consider the cycle graph C_5 with vertices 1, 2, 3, 4, 5.



Figure 1. Cycle graph C_5

Note that $\gamma'(G-i) < \gamma'(G)$ for i = 1, 2, 3, 4, 5. For i = 5, $f_5 = \{23\}$ is the only minimum edge dominating set of G - 5. 1 is a neighbor of 5 in G and there is only one edge incident to 1 namely $\{12\}$ which is not in the minimum edge dominating set f_5 . Therefore $E_5^1 = f_5 \cup \{15\}$ is a minimum edge dominating set of G. Similarly, $E_5^2 = f_5 \cup \{45\}$ is also a minimum edge dominating set of G. Thus, $\mathcal{A}_5 = E_5^1 \cup E_5^2$.

Similarly, for i = 1, since {34} is the minimum edge dominating set of G - 1. The sets $E_1^1 = \{34\} \cup \{12\}$ and $E_1^2 = \{34\} \cup \{15\}$ are the minimum edge dominating sets of G. For i = 2, since {45} is the minimum edge dominating set of G - 2. The sets $E_2^1 = \{45\} \cup \{23\}$ and $E_2^2 = \{45\} \cup \{21\}$ are the minimum edge dominating sets of G. For i = 3, since {51} is the minimum edge dominating set of G - 3. The sets $E_3^1 = \{51\} \cup \{32\}$ and $E_3^2 = \{51\} \cup \{34\}$ are the minimum edge dominating sets of G. For i = 4, since $\{12\}$ is the minimum edge dominating set of G - 4. The sets $E_4^1 = \{12\} \cup \{43\}$ and $E_4^2 = \{12\} \cup \{45\}$ are the minimum edge dominating sets of G.

Therefore the number of minimum edge dominating sets of the graph $G = C_5$ is

 $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5 = (E_1^1 \cup E_1^2) \cup (E_2^1 \cup E_2^2) \cup (E_3^1 \cup E_3^2) \cup (E_4^1 \cup E_4^2) \cup (E_5^1 \cup E_5^2).$

Remark 3.2. From the above discussion, it is clear that if $\gamma'(G-v) < \gamma'(G)$ then the number of minimum edge dominating sets of the graph G is greater than or equal to the number of minimum edge dominating sets of G-v.

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