g*\(s^*\) - Closed Sets in Topological Spaces

N. Gayathri

1 Department of Mathematics, Sri Krishna Arts and Science College, Coimbatore, Tamilnadu, India.

Abstract: In this paper, a new class of sets, namely \(g^*s^*\)-closed sets was introduced and some of their properties were studied. Further the notion of \(g^*s^*\)-continuous maps, \(g^*s^*\)- irresolute maps, \(T_b^*\)-spaces, \(g_Tb^*\)-spaces, \(\ast gTb^*\)-spaces, \(g^*s^*\)-compactness, \(g^*s^*\)-connectedness were introduced and its properties are investigated.

Keywords: \(g^*s^*\)-closed set, \(g^*s^*\)-continuous maps, \(g^*s^*\)- irresolute maps, \(T_b^*\)-spaces, \(g_Tb^*\)-spaces, \(\ast gTb^*\)-spaces, \(g^*s^*\)-compactness, \(g^*s^*\)-connectedness.

1. Introduction

The concept of generalized closed sets and semi-open sets were introduced and studied by Norman Levine [7] respectively. Arya and Nour [4] defined generalized semi-closed sets for obtaining some characterizations of s-normal spaces. Bhattacharya and Lahiri [5] introduced and investigated semi-generalized closed sets. The concept of generalized semi-pre closed sets was introduced by Dontchev [6]. Palaniappan and Rao [14] introduced \(rg\)-closed sets. Pauline Mary Helen, Ponnuthai and Veronica [15] introduced and studied \(g^*\)-closed sets. Anitha [3] introduced \(g^*s\)-closed sets. \(ga\)-closed sets and \(ag\)-closed sets were introduced by Maki et. al. [10] and some of their properties were investigated. In this paper we introduce a new class of called \(g^*s^*\)-closed sets and study the relationship of \(g^*s^*\)-closed sets with the above mentioned sets. We also obtain basic properties of \(g^*s^*\)-closed sets and introduced \(g^*s^*\)-continuous maps and \(g^*s^*\)- irresolute maps.

2. Preliminaries

Throughout this paper \((X, \tau)\) and \((Y, \sigma)\) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset of a space \((X, \tau)\), \(cl(A)\), \(int(A)\) and \(scl(A)\) denote the closure of A, the interior of A and semi – closure of A respectively. The class of all subsets of a space \((X, \tau)\) is denoted by \(C(X, \tau)\).

**Definition 2.1.** A subset \(A\) of a topological space \((X, \tau)\) is called

(i). a semi – openset [8] if \(A \subseteq cl(int(A))\) and semi-closed set if \(int(cl(A)) \subseteq A\).

(ii). a semi – preopenset [2] \((=\beta – open\ [1])\) if \(A \subseteq cl(int(cl(A)))\) and a semi – preclosed [2] set \((=\beta – closed\ [1])\) if \(int(cl(int(A))) \subseteq A\).

* E-mail: gayupadmagayu@gmail.com
Definition 2.2. A subset $A$ of a topological space $(X, \tau)$ is called

(i). $g$-closed set [7] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

(ii). $gs$-closed set [4] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

(iii). $w$-closed set [18] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.

(iv). $g^*$-closed set [19] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $(X, \tau)$.

(v). $g^{**}$-closed set [15] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $gs$-open in $(X, \tau)$.

(vi). $gsp$-closed set [6] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

(vii). $g^s$-closed set [3] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $gs$-open in $(X, \tau)$.

(viii). rg-closed set [14] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $(X, \tau)$.

Definition 2.3. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i). $g^*$-continuous [19] if the inverse image $f^{-1}(V)$ of every closed set in $(Y, \sigma)$ is $g^*$-closed in $(X, \tau)$.

(ii). $gs$-continuous [4] if the inverse image $f^{-1}(V)$ of every closed set in $(Y, \sigma)$ is $gs$-closed in $(X, \tau)$.

(iii). gsp-continuous [6] if the inverse image $f^{-1}(V)$ of every closed set in $(Y, \sigma)$ is gsp-closed in $(X, \tau)$.

(iv). $g^s$-continuous [3] if the inverse image $f^{-1}(V)$ of every closed set in $(Y, \sigma)$ is $g^s$-closed in $(X, \tau)$.

(v). $g^{**}$-continuous [19] if the inverse image $f^{-1}(V)$ of every closed set in $(Y, \sigma)$ is $g^{**}$-closed in $(X, \tau)$.

Definition 2.4. A topological space $(X, \tau)$ is said to be

(i). a $T^*_2$-space [19] if every $g^*$-closed set in $(X, \tau)$ is closed in $(X, \tau)$.

(ii). a $T^{**}_2$-space [15] if every $g^{**}$-closed set in $(X, \tau)$ is closed in $(X, \tau)$.

3. Properties of $g^s$-closed sets

We now introduce the following definition.

Definition 3.1. A subset $I$ of $(X, \tau)$ is said to be a $g^s$-closed set if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g^*$-open in $(X, \tau)$. The class of all $g^s$-closed subset of $(X, \tau)$ is denoted by $G^s \subseteq C(X, \tau)$.

Proposition 3.2. Every closed set is $g^s$-closed.

The converse of the above proposition need not be true and in general it can be seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$. Let $A = \{a, b\}$, then $A$ is $g^s$-closed but not closed. So, the class of $g^s$-closed sets is properly contained in the class of closed sets.

Proposition 3.4. Every $g^*$-closed set is $g^s$-closed set.

The converse of the above proposition need not be true and in general it can be seen from the following example.

Example 3.5. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}\}$. Let $A = \{b, c\}$, then $A$ is $g^s$-closed but not $g^*$-closed.
Proposition 3.6. Every $g^{**}$-closed set is $g^*s^*$-closed set.

The converse of the above proposition need not be true and in general it can be seen from the following example.

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{b\}$, then $A$ is $g^*s^*$-closed but not $g^{**}$-closed.

Proposition 3.8. Every $g^*s^*$-closed set is $g^*s^*$-closed set.

The converse of the above proposition need not be true and in general it can be seen from the following example.

Example 3.9. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a, b\}$, then $A$ is $g^*s^*$-closed but not $g^*s^*$-closed set.

Proposition 3.10. Every $g^*s^*$-closed set is gs-closed set.

The converse of the above proposition need not be true and in general it can be seen from the following example.

Example 3.11. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Let $A = \{b, c\}$, then $A$ is gs-closed but not $g^*s^*$-closed.

Proposition 3.12. Every $g^*s^*$-closed set is gs-p-closed set.

The converse of the above proposition need not be true and in general it can be seen from the following example.

Example 3.13. In Example 3.11, $A = \{c\}$. Then $A$ is gs-closed but not $g^*s^*$-closed.

Remark 3.14. $g^*s^*$-closedness is independent of g-closedness.

Example 3.15. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Let $A = \{c\}$, then $A$ is $g^*s^*$-closed but not g-closed set. In example [3.11], $A = \{b\}$. Then $A$ is g-closed but not $g^*s^*$-closed.

Remark 3.16. $g^*s^*$-closedness is independent of w-closedness (or) s*g-closedness.

Example 3.17. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{c\}\}$. Let $A = \{b, c\}$. In Example 3.11, $A$ is $g^*s^*$-closed but not w-closed set (or) s*g-closed.

Remark 3.18. $g^*s^*$-closedness is independent of rg-closedness.

In Example 3.7, $A = \{a, b\}$. Then $A$ is rg-closed but not $g^*s^*$-closed. In Example 3.7, $A = \{b\}$. Then $A$ is $g^*s^*$-closed but not rg-closed. Thus we have the following diagram.

![Diagram](image-url)

where $A \rightarrow B$ implies $B$ and $A \not\rightarrow B$ represents $A$ does not imply $B$ (resp. $A$ and $B$ are independent).
4. \( g^*s^* \)-Continuous Maps and \( g^*s^* \)-Irresolute Maps.

**Definition 4.1.** A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) from a topological space \( (X, \tau) \) to a topological space \( (Y, \sigma) \) is called \( g^*s^* \)-continuous if the inverse image of every closed set in \( (Y, \sigma) \) is \( g^*s^* \)-closed in \( (X, \tau) \).

**Theorem 4.2.** Every continuous map is \( g^*s^* \)-continuous.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be continuous. Let \( F \) be a closed set in \( (Y, \sigma) \) then \( f^{-1}(F) \) is closed in \( (X, \tau) \). Since every closed set is \( g^*s^* \)-closed, \( f^{-1}(F) \) is \( g^*s^* \)-closed in \( (X, \tau) \). \( f \) is \( g^*s^* \)-continuous in \( (X, \tau) \).

The converse of the above theorem need not be true in general and it can be seen from the following example.

**Example 4.3.** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}\} \), \( \sigma = \{\phi, Y, \{b\}, \{a, c\}, \{a, b\}, \{a, b, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. The inverse image of every closed set in \( (Y, \sigma) \) is \( g^*s^* \)-closed, but \( f^{-1}([c]) = \{c\} \) is not closed in \( (X, \tau) \).

**Theorem 4.4.** Every \( g^*s^* \)-continuous map is \( g^*s^* \)-continuous.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( g^*s^* \)-continuous. Let \( F \) be a closed set in \( (Y, \sigma) \) then \( f^{-1}(F) \) is \( g^*s^* \)-closed in \( (X, \tau) \). Since every \( g^*s^* \)-closed set is \( g^*s^* \)-closed, \( f^{-1}(F) \) is \( g^*s^* \)-closed in \( (X, \tau) \). \( f \) is \( g^*s^* \)-continuous in \( (X, \tau) \).

The converse of the above theorem need not be true in general and it can be seen from the following example.

**Example 4.5.** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{\phi, X, \{b\}, \{b, c\}\} \), \( \sigma = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. The inverse image of every closed set in \( (Y, \sigma) \) is \( g^*s^* \)-closed, but \( f^{-1}([b, c]) = \{b, c\} \) is not \( g^*s^* \)-closed in \( (X, \tau) \).

**Theorem 4.6.** Every \( g^* \)-continuous map is \( g^*s^* \)-continuous.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( g^* \)-continuous. Let \( F \) be a closed set in \( (Y, \sigma) \) then \( f^{-1}(F) \) is \( g^* \)-closed in \( (X, \tau) \). Since every \( g^*s^* \)-closed set is \( g^*s^* \)-closed, \( f^{-1}(F) \) is \( g^*s^* \)-closed in \( (X, \tau) \). \( f \) is \( g^*s^* \)-continuous in \( (X, \tau) \).

The converse of the above theorem need not be true in general and it can be seen from the following example.

**Example 4.7.** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{\phi, X, \{b\}\}, \sigma = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. The inverse image of every closed set in \( (Y, \sigma) \) is \( g^*s^* \)-closed, but \( f^{-1}([b, c]) = \{b, c\} \) which is not \( g^*s^* \)-closed in \( (X, \tau) \).

**Theorem 4.8.** Every \( g^** \)-continuous map is \( g^*s^* \)-continuous.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( g^** \)-continuous. Let \( F \) be a closed set in \( (Y, \sigma) \) then \( f^{-1}(F) \) is \( g^** \)-closed in \( (X, \tau) \). Since every \( g^**s^* \)-closed set is \( g^*s^* \)-closed, \( f^{-1}(F) \) is \( g^*s^* \)-closed in \( (X, \tau) \). \( f \) is \( g^*s^* \)-continuous in \( (X, \tau) \).

The converse of the above theorem need not be true in general and it can be seen from the following example.

**Example 4.9.** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}, \{b, a, b\}\}, \sigma = \{\phi, X, \{a\}, \{c\}, \{a, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a mapping defined by \( f(a) = a, f(b) = c, f(c) = b \). The inverse image of every closed set in \( (Y, \sigma) \) is \( g^*s^* \)-closed, but \( f^{-1}([a, b]) = \{a, c\} \) which is not \( g^** \)-closed in \( (X, \tau) \).

**Theorem 4.10.** Every \( g^*s^* \)-continuous map is \( gs \)-continuous.
Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be $g^{*}s^{*}$-continuous. Let $F$ be a closed set in $(Y, \sigma)$ then $f^{-1}(F)$ is $g^{*}s^{*}$-closed in $(X, \tau)$. Since every $g^{*}s^{*}$-closed set is $gs$-closed, $f^{-1}(F)$ is $gs$-closed in $(X, \tau)$. $f$ is $gs$-continuous in $(X, \tau)$. □

The converse of the above theorem need not be true in general and it can be seen from the following example.

**Example 4.13.** Let $X = Y = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}, \sigma = \{\phi, X, \{c\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping defined by $f(a) = a, f(b) = c, f(c) = b$. The inverse image of every closed set in $(Y, \sigma)$ is $gs$-closed, but $f^{-1}([b]) = \{c\}$ which is not $g^{*}s^{*}$-closed in $(X, \tau)$.

**Theorem 4.12.** Every $g^{*}s^{*}$-continuous map is $gsp$-continuous.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be $g^{*}s^{*}$-continuous. Let $F$ be a closed set in $(Y, \sigma)$ then $f^{-1}(F)$ is $g^{*}s^{*}$-closed in $(X, \tau)$. Since every $g^{*}s^{*}$-closed set is $gsp$-closed, $f^{-1}(F)$ is $gsp$-closed in $(X, \tau)$. $f$ is $gsp$-continuous in $(X, \tau)$. □

The converse of the above theorem need not be true and in general it can be seen from the following example.

**Example 4.13.** Let $X = Y = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}, \sigma = \{\phi, X, \{c\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping defined by $f(a) = a, f(b) = c, f(c) = b$. The inverse image of every closed set in $(Y, \sigma)$ is $gs$-closed, but $f^{-1}([b]) = \{c\}$ which is not $g^{*}s^{*}$-closed in $(X, \tau)$. Thus, we have the following diagram.

![Diagram](image)

**Definition 4.14.** A map $f : (X, \tau) \to (Y, \sigma)$ from a topological space $(X, \tau)$ to a topological space $(Y, \sigma)$ is called $g^{*}s^{*}$-irresolute if the inverse image of every $g^{*}s^{*}$-closed set in $(Y, \sigma)$ is $g^{*}s^{*}$-closed in $(X, \tau)$.

**Theorem 4.15.** Every $g^{*}s^{*}$-irresolute map is $g^{*}s^{*}$-continuous.

The converse of the above theorem need not be true and in general it can be seen from the following example.

**Example 4.16.** Let $X = Y = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, X, \{b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping defined by $f(a) = b, f(b) = c, f(c) = a$. Let $\{a, c\}$ be a closed set in $(Y, \sigma)$. But $f^{-1}([a, c]) = \{a, b\}$ which is not $g^{*}s^{*}$-closed set in $(X, \tau)$. $f$ is $g^{*}s^{*}$-continuous. $\{c\}$ is a $g^{*}s^{*}$-closed set in $(Y, \sigma)$. But $f^{-1}([c]) = \{a\}$ which is not $g^{*}s^{*}$-closed in $(X, \tau)$. $f$ is not $g^{*}s^{*}$-irresolute.

**Theorem 4.17.** Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$, then

(i). $g \circ f : (X, \tau) \to (Z, \eta)$ is $g^{*}s^{*}$-continuous if $f$ is $g^{*}s^{*}$-irresolute and $g$ is $g^{*}s^{*}$-continuous.

(ii). $g \circ f : (X, \tau) \to (Z, \eta)$ is $g^{*}s^{*}$-irresolute if $f$ and $g$ are $g^{*}s^{*}$-irresolute.

(iii). $g \circ f : (X, \tau) \to (Z, \eta)$ is $g^{*}s^{*}$-continuous if $f$ is $g^{*}s^{*}$-continuous and $g$ is $g^{*}s^{*}$-irresolute.
5. Applications Of $g^*s^*$-Closed Set.

As application of $g^*s^*$-closed sets, new spaces namely, Tb*-space, gTb*-space and *gTb*-space are introduced. We introduce the following definitions.

**Definition 5.1.** A space $(X,\tau)$ is said to be a Tb* space if every $g^*s^*$-closed set in $(X,\tau)$ is closed in $(X,\tau)$.

**Theorem 5.2.** Every Tb* space is $T_{1/2}s^*$-space.

**Proof.** Let $(X,\tau)$ be a Tb* space. Let $A$ be a $g^*$-closed set in $(X,\tau)$. But by proposition (3.4), every $g^*$-closed set is $g^*s^*$-closed. Since $(X,\tau)$ is a Tb* space, $A$ is closed in $(X,\tau)$. $(X,\tau)$ is a $T_{1/2}s^*$-space.

The converse of the above theorem need not be true and in general it can be seen from the following example.

**Example 5.3.** In example [3.5], Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Here, $(X,\tau)$ is a $T_{1/2}s^*$-space and the set $\{b, c\}$ is $g^*s^*$-closed but not closed. $(X,\tau)$ is not a Tb* space.

**Theorem 5.4.** Let $f : (X,\tau) \to (Y,\sigma)$ be a $g^*s^*$-continuous mapping. If $(X,\tau)$ is Tb* space, then $f$ is continuous.

**Proof.** Let $f : (X,\tau) \to (Y,\sigma)$ be $g^*s^*$-continuous. Let $F$ be a closed set in $(Y,\sigma)$. Then $f^{-1}(F)$ is $g^*s^*$-closed in $(X,\tau)$. Since $(X,\tau)$ is Tb* space, $f^{-1}(F)$ is closed in $(X,\tau)$. $f$ is continuous.

**Theorem 5.5.** Every Tb* space is $T_{1/2} * s^*$-space.

**Proof.** Let $(X,\tau)$ be a Tb* space. Let $A$ be $g^{**}$-closed set in $(X,\tau)$. But by proposition (3.6), every $g^{**}$-closed set is $g^*s^*$-closed. Since $(X,\tau)$ is Tb* space, $A$ is closed in $(X,\tau)$, which implies, $g^*s^*$-closed set is closed. $(X,\tau)$ is a $T_{1/2} * s^*$-space.

The converse of the above theorem need not be true and in general it can be seen from the following example.

**Example 5.6.** In example [3.7], Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Here, $(X,\tau)$ is a $T_{1/2} * s^*$-space and the sets $\{a\}, \{b\}$ are $g^*s^*$-closed but not closed. $(X,\tau)$ is not Tb* space.

**Definition 5.7.** A space $(X,\tau)$ is said to be gTb* space if every $g^*s^*$-closed set in $(X,\tau)$ is $g^*$-closed in $(X,\tau)$.

**Theorem 5.8.** Every gTb* space is Tb* space.

**Proof.** Let $(X,\tau)$ be a Tb* space. Let $A$ be $g^*s^*$-closed set in $(X,\tau)$. Since $(X,\tau)$ is a Tb* space, $A$ is closed in $(X,\tau)$. But we know that, every closed set is $g^*$-closed. Hence, $A$ is $g^*$-closed set in $(X,\tau)$. $(X,\tau)$ is a gTb* space.

**Theorem 5.9.** Let $f : (X,\tau) \to (Y,\sigma)$ be $g^*s^*$-continuous mapping. If $(X,\tau)$ is gTb* space, then $f$ is $g^*$-continuous.

**Proof.** Let $f : (X,\tau) \to (Y,\sigma)$ be $g^*s^*$-continuous. Let $F$ be a closed set in $(Y,\sigma)$. Then $f^{-1}(F)$ is $g^*s^*$-closed in $(X,\tau)$. Since $(X,\tau)$ is gTb* space, $f^{-1}(F)$ is $g^*$-closed in $(X,\tau)$. $f$ is $g^*$-continuous.

**Definition 5.10.** A space $(X,\tau)$ is said to be *gTb* space if every $g^*s^*$-closed set in $(X,\tau)$ is $g^*s$-closed in $(X,\tau)$.

**Theorem 5.11.** Every gTb* space is *gTb* space.

**Proof.** Let $(X,\tau)$ be a Tb* space. Let $A$ be $g^*s^*$-closed set in $(X,\tau)$. Since $(X,\tau)$ is Tb* space, $A$ is closed in $(X,\tau)$. But we know that, every closed set is $g^*s$-closed. Hence, $A$ is $g^*s$-closed set in $(X,\tau)$. $(X,\tau)$ is a *gTb* space.

**Theorem 5.12.** Let $f : (X,\tau) \to (Y,\sigma)$ be $g^*s^*$-continuous mapping. If $(X,\tau)$ is *gTb* space, then $f$ is $g^*s$-continuous.

**Proof.** Let $f : (X,\tau) \to (Y,\sigma)$ be $g^*s^*$-continuous. Let $F$ be a closed set in $(Y,\sigma)$. Then $f^{-1}(F)$ is $g^*s^*$-closed in $(X,\tau)$. Since $(X,\tau)$ is a *gTb* space, $f^{-1}(F)$ is $g^*s$-closed in $(X,\tau)$. $f$ is $g^*s$-continuous.
6. G*S*-Compactness

Definition 6.1. A collection \( \{A_i/i \in A\} \) of \( g^*s^* \)-open sets in a topological space \( X \) is called a \( g^*s^* \)-open cover of a subset \( B \) of \( X \) if \( B \subset \cup_{i \in A} A_i \).

Definition 6.2. A topological space \( X \) is \( G^*S^* \)-compact if every \( g^*s^* \)-open cover of \( X \) has a finite sub cover.

Definition 6.3. A subset \( B \) of a topological space \( X \) is said to be \( G^*S^* \)-compact relative to \( X \) if for every collection of \( g^*s^* \)-open subsets of \( X \) such that \( B \subset \cup_{i \in A} A_i \), there exists a finite subset \( A_0 \) of \( A \) such that, \( B \subset \cup_{i \in A_0} A_i \).

Definition 6.4. A subset \( B \) of \( X \) is \( G^*S^* \)-compact if \( B \) is \( G^*S^* \)-compact as a subspace of \( X \).

Proposition 6.5. A \( g^*s^* \)-closed subset of \( G^*S^* \)-compact space is \( G^*S^* \)-compact relative to \( X \).

Proof. Let \( A \) be a \( g^*s^* \)-closed subset of \( G^*S^* \)-compact space \( X \). Then \( A^c \) is \( g^*s^* \)-open in \( X \). Let \( M \) be a cover of \( A \) by \( g^*s^* \)-open sets in \( X \). Then, \( M, A^c \) is a \( g^*s^* \)-open cover of \( X \). Since \( X \) is \( G^*S^* \)-compact, it has a finite sub-cover, namely \( G_1, G_2, \ldots, G_n \). Therefore, we have obtained a finite \( g^*s^* \)-open sub-cover of \( A \). Thus, \( A \) is \( G^*S^* \)-compact relative to \( X \).

Proposition 6.6.

(i). A \( g^*s^* \)-continuous image of a \( G^*S^* \)-compact space is compact.

(ii). If a map \( f : X \to Y \) is \( g^*s^* \)-irresolute and a subset \( B \) of \( X \) is \( G^*S^* \)-compact relative to \( X \), then the image \( f(B) \) is \( G^*S^* \)-compact relative to \( X \).

Proof.

(i). Let \( f : X \to Y \) be a \( g^*s^* \)-continuous map from a \( G^*S^* \)-compact space onto a topological space \( Y \). Let \( A_i : i \in A \) be an open cover of \( Y \). Then \( \{f^{-1}(A_i) : i \in A\} \) is a \( g^*s^* \)-open cover of \( X \). Since \( X \) is \( G^*S^* \)-compact, it has a finite subcover, namely \( \{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\} \). Since \( f \) is onto, \( A_1, A_2, \ldots, A_n \) is an open cover of \( Y \) and so \( Y \) is compact.

(ii). Let \( A_i : i \in A \) be any collection of \( g^*s^* \)-open subsets of \( Y \) such that \( f(B) \subset \cup A_i \). Then \( B \subset \cup f^{-1}(A_i) \). Therefore, we have \( f(B) \subset \cup A_i \). Thus, \( f(B) \) is \( G^*S^* \)-compact relative to \( Y \).

Theorem 6.7. If the product space of two non-empty spaces is \( G^*S^* \)-compact, then each of the factor spaces is \( G^*S^* \)-compact.

Proof. Let \( X \times Y \) be the product space of non-empty spaces \( X \) and \( Y \). Obviously, the projection \( p : X \times Y \to Y \) from \( X \times Y \) onto \( X \) is \( g^*s^* \)-irresolute map. In fact, let \( F \) be any \( g^*s^* \)-closed set of \( X \). Then it follows that, \( F \times \{p^{-1}(F)\} \) is \( g^*s^* \)-closed in \( X \times Y \) and hence \( p \) is \( g^*s^* \)-irresolute. Now, suppose that \( X \times Y \) is \( G^*S^* \)-compact. By using Proposition, we obtain that the \( g^*s^* \)-irresolute image \( p(X \times Y)(= X) \) is \( G^*S^* \)-compact. For \( Y \), the proof is similar to the case of \( X \).

7. G*S*-Connectedness

Definition 7.1. A topological space \( X \) is \( G^*S^* \)-connected if \( X \) cannot be written as a disjoint union of two non-empty \( g^*s^* \)-open sets. A subset \( V \) of \( X \) is \( G^*S^* \)-connected if \( V \) is \( G^*S^* \)-connected as a subspace.

Proposition 7.2. For a topological space \( X \), the following conditions are equivalent.
(i). $X$ is $G^*S^*$-connected.

(ii). The only subsets of $X$ which are both $g^*s^*$-open and $g^*s^*$-closed are empty set and $X$.

(iii). Each $g^*s^*$-continuous map of $X$ into a discrete space $Y$ with at least two points is a constant map.

Proof.

(i) $\Rightarrow$ (ii): Let $U$ be a $g^*s^*$-open and $g^*s^*$-closed subset of $X$. Then $X=U$ is both $g^*s^*$-closed and $g^*s^*$-open. Since $X$ is the disjoint union of the $g^*s^*$-open sets $U$ and $X-U$, one of these must be empty, that is $U = \emptyset \cup X = X$.

(ii) $\Rightarrow$ (i): Suppose $X = A \cup B$ where $A$ and $B$ are disjoint non-empty $g^*s^*$-open subsets of $X$. Since $A$ is a $g^*s^*$-open subset of $X$, by condition (ii), it may be $g^*s^*$-closed and $A = \emptyset$ or $A = X$. If $A = \emptyset$, $X=B$. If $A=X$, $B=\emptyset$. Thus, $X$ is $g^*s^*$-connected.

(ii) $\Rightarrow$ (iii): Let $f:X \to Y$ be a $g^*s^*$-continuous map. Then $X$ is covered by $g^*s^*$-open and $g^*s^*$-closed covering $\{f^{-1}(y) / y \in Y\}$. By assumption, $f^{-1}(y) = \emptyset$ or $X$ for each $x \in X$. If $f^{-1}(y) = \emptyset$ for all $y \in Y$ then $f$ fails to be a map. Then, there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \emptyset$ and hence $f^{-1}(y) = X$ which shows that $f$ is a constant map.

(iii) $\Rightarrow$ (ii): Let $U$ be both $g^*s^*$-open and $g^*s^*$-closed in $X$. Suppose $U \neq \emptyset$. Let $f : X \to Y$ be a $g^*s^*$-continuous map defined by $f(U) = \{y\}$ and $f(X-U) = \{w\}$ for some distinct points $y$ and $w$ in $Y$. By assumption, $f$ is a constant. Therefore, $U = X$.

It is obvious that every $G^*S^*$-connected space is connected. The following example shows that the converse is not true.

Example 7.3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then the topological space is $(X, \tau)$ is connected. However, since $\{b\}$ is both $g^*s^*$-closed and $g^*s^*$-open in $X$. By Proposition 7.2, $X$ is not $G^*S^*$-connected.

Proposition 7.4. If $X$ is $T^*_b$-space and connected, then $X$ is $g^*s^*$-connected.

Proof. Let $X$ be $T^*_b$-space and connected. Assume that $X$ can be written in the form $X = A \cup B$ where $A$ and $B$ are nonempty disjoint and $g^*s^*$-open sets in $X$. Since $X$ is $T^*_b$-space, every $g^*s^*$-open set is open and so $X = A \cup B$ where $A$ and $B$ are disjoint nonempty and open sets in $X$. This contradicts the fact that $X$ is connected. Therefore $X$ is $g^*s^*$-connected.

Proposition 7.5. If $f:X \to Y$ is $g^*s^*$-continuous surjection and $X$ is $g^*s^*$-connected then $Y$ is $g^*s^*$-connected.

Proof. Suppose that $Y$ is not connected. Let $Y = A \cup B$ where $A$ and $B$ are disjoint nonempty open sets in $Y$. Since $f$ is $g^*s^*$-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty $g^*s^*$-open in $X$. This contradicts the fact that $X$ is $g^*s^*$-connected. Hence $Y$ is connected.

Proposition 7.6. If $f:X \to Y$ is $g^*s^*$-continuous map from a connected space $X$ into a topological space $Y$, then $Y$ is $g^*s^*$-connected.

Proof. Let $Y$ be not $g^*s^*$-connected. Then $Y$ can be written as $Y = A \cup B$ where $A$ and $B$ are disjoint nonempty $g^*s^*$-open sets in $Y$. Since $f$ is $g^*s^*$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open sets in $X$. Also $X = f^{-1}(A) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. This contradicts the fact that $X$ is connected. Therefore $Y$ is $g^*s^*$-connected.

References

N.Gayathri