Perfect domination edge subdivision critical and stable Graphs

Research Article

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Abstract: Let G be a graph. A subset S of vertices in a graph G is a perfect dominating set if every vertex in V\S is adjacent to exactly one vertex in S. A graph is perfect domination edge subdivision critical if the subdivision of an arbitrary edge increases the perfect domination number. On the other hand, a graph is perfect domination edge subdivision stable if the subdivision of an arbitrary edge leaves the perfect domination number unchanged. In this paper, we initiate the study of perfect domination critical and stable graphs upon edge subdivision. We discuss some graphs which are perfect domination critical and stable.

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1. Introduction

Let G = (V, E) be a graph with vertex set V and edge set E, such that |V| = n and |E| = m. A dominating set S is a subset of V such that every vertex in V\S is adjacent to at least one vertex in S. Further, S is a perfect dominating set if every vertex of V\S is adjacent to exactly one vertex in S. The perfect domination number denoted γp(G) is the minimum cardinality of a perfect dominating set in G. Any perfect dominating set of cardinality γp(G) is called a γp(G)—set.

When we study a graphical parameter, it is equally important to study its behavior of the parameter when the graph is modified by applying the graph operations. For instance, the effects of the operations like removing or adding an edge or a vertex have been considered on the parameter domination number. The graphs whose domination number increase or decreases by such operations are named as domination critical graphs. Sumner et.al.[10] initiated the study of domination critical graphs. Favaron et. al.[6] have studied the effect of domination criticalness on the diameter of a graph. Removal of a vertex can increase the domination number by more than one, but can decrease it by at most one. Motivated by this Brigham[1] defined the concept of vertex domination critical graph. Further properties of these graphs were explored in [7]. Van der Merwe [11] initiated the study of those graphs where the total domination number decreases upon the addition of any edge. Chellali et.al. [2] studied double domination stable graphs upon edge removal. Chen et. al. [3] studied connected domination critical graphs i.e. graphs whose connected domination number decreases when an edge in the complement of

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2. Effect of Edge Subdivision

In this section, we consider the effects that subdivision of an edge in a graph has on the perfect domination number. We begin with the remark that the perfect domination number of a path $P_n$ and a cycle $C_n$ on $n$ vertices is easy to compute.

**Observation 2.1.** For an integer $n \geq 1$, $\gamma_p(P_n) = \left\lceil \frac{n}{4} \right\rceil$.

**Observation 2.2.** For an integer $n \geq 1$, we have

$$
\gamma_p(C_n) = \begin{cases} 
\frac{n}{3} & \text{if } n \equiv 0 \pmod{3}; \\
\left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 1 \pmod{3}; \\
\left\lceil \frac{n}{3} \right\rceil + 2 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
$$

Subdivision of an edge in a graph can cause its perfect domination number to increase, to decrease, or to remain the same. This can be illustrated with the following example. First, let $G$ be a star with four vertices. Then $\gamma_p(G) = 1$ and $\gamma_p(G^*_e) = 2$, for any edge $e$ in $G$. To show the decrease in $\gamma_p$, consider a Cycle graph $G$ with 5 vertices, then $\gamma_p(G) = 3$ but $\gamma_p(G^*_e) = 2$. Finally, let $G$ be a path with 4 vertices. Then $\gamma_p(G) = 2$ and $\gamma_p(G^*_e) = 2$, for any edge $e$ in $G$. Therefore, for any graph $G$, we may define the following weak partition of its edge set $E(G)$, where by a weak partition of a set we mean a partition of the set in which some of the subsets may be empty.

**Definition 2.3.** For a graph $G$, we define a weak partition $E(G) = E^0(G) \cup E^+(G) \cup E^-(G)$, where

$$
E^0(G) = \{e \in E(G) | \gamma_p(G^*_e) = \gamma_p(G)\},
$$

$$
E^+(G) = \{e \in E(G) | \gamma_p(G^*_e) > \gamma_p(G)\},
$$

$$
E^-(G) = \{e \in E(G) | \gamma_p(G^*_e) < \gamma_p(G)\}.
$$

**Definition 2.4.** A graph $G$ is defined to be perfect domination edge subdivision critical or $\gamma_p$-critical for short, if $\gamma_p(G^*_e) > \gamma_p(G)$ for every edge $e \in E(G)$.

In symbols, $G$ is $\gamma_p$-critical if $E(G) = E^+(G)$. We too consider those graphs whose perfect domination number changes upon the subdivision of an edge. However, we consider both the types of changes, that is, increase and decrease in perfect domination number on subdividing an edge in $G$. We are now in a position to define our two main concepts in this paper.

**Definition 2.5.** A graph $G$ is $\gamma_p$-changing if $\gamma_p(G^*_e) \neq \gamma_p(G)$ for every edge $e \in E(G)$, while a graph $G$ is $\gamma_p$-stable if $\gamma_p(G^*_e) = \gamma_p(G)$ for every edge $e \in E(G)$.
Thus a graph $G$ is $\gamma_p$-changing if the subdivision of any edge from $G$ either increases or decreases the perfect domination number. That is, $E(G) = E^-(G) \cup E^+(G)$. A graph $G$ is $\gamma_p$-stable if $E(G) = E^0(G)$. It follows that $\gamma_p$-critical graphs are a subset of $\gamma_p$-changing graphs.

**Definition 2.6.** The multi-star graph $K_m(a_1, a_2, \ldots, a_m)$ is a graph of order $a_1 + a_2 + \cdots + a_m + m$ formed by joining $a_1, a_2, \ldots, a_m$ end-edges to $m$ vertices of $K_m$.

**Example 2.7.**

1. Let $G$ be a complete graph. Then we have $\gamma_p(G) = 1$. Clearly, subdividing any arbitrary edge of $G$, we observe that $\gamma_p(G_e) = 2$. Therefore, for any positive integer $n$, the complete graph $K_n$ is a $\gamma_p$-changing graph.

2. Let $G$ be a star with $n$ vertices. Then $\gamma_p(G) = 1$. But $\gamma_p(G^*_e) = 2$, for any edge $e \in E(G)$. In particular, the multi-star graph $K(a_1, a_2, \ldots, a_k)$ is $\gamma_p$-changing graph.

3. Let $G \cong K_{m,n}$ be a complete bipartite graph. Then $G$ is $\gamma_p$-changing.

The following observation follows from the definition of the edge subdivision.

**Observation 2.8.** If $G$ is connected then $G^*_e$ is also connected. Further, if $G$ is not a connected graph then both $G$ and $G^*_e$ have same number of components.

## 3. Main Results

Here, we discuss some of the graphs which are $\gamma_p$-changing and $\gamma_p$-stable.

**Lemma 3.1.** For $n \geq 3$, the Cycle graph $C_n$ is $\gamma_p$-changing.

**Proof.** Let $C_n$ be a cycle graph and let $V = \{v_1, v_2, \ldots, v_n\}$ denotes the vertex set of the Cycle with $n \geq 3$. We first note that, the graph obtained by subdividing an arbitrary edge of $C_n$ is the cycle graph $C_{n+1}$. Now, we may consider the following possible cases here:

**Case 1:** Suppose $n \equiv 0 \pmod{3}$. Then, we have $\gamma_p(C_n) = \frac{n}{3}$ and the set $S = \{v_1, v_4, \ldots, v_{n-2}\}$ will be the $\gamma_p$-set of $C_n$. On subdividing an arbitrary edge of $C_n$ we obtain the cycle graph $C_{n+1}$. Further, we have $\gamma_p(C_{n+1}) = \lceil \frac{n}{3} \rceil + 1$. In fact $\gamma_p$-set of $C_{n+1}$ is obtained by adding one more vertex to the $\gamma_p$-set of $C_n$. Thus, the subdivision of an edge, here, in this case increases the perfect domination number of the graph.

**Case 2:** Suppose $n \equiv 1 \pmod{3}$. This case is similar to the above.

**Case 3:** Suppose $n \equiv 2 \pmod{3}$. By Observation 2, we have $\gamma_p(C_n) = \lceil \frac{n}{3} \rceil + 2$. Clearly, the graph obtained by subdividing an arbitrary edge of $C_n$ is $C_{n+1}$. The $\gamma_p$-set of $C_{n+1}$ is obtained by removing exactly one vertex from the $\gamma_p$-set of $C_n$. Therefore, in this case, the subdivision of an arbitrary edge in the graph decreases the perfect domination number of the graph.

**Lemma 3.2.** For $n \geq 1$, the Path graph $P_n$ is $\gamma_p$-stable if and only if $n \equiv 1$ or $2 \pmod{3}$.

**Proof.** Let $P_n$ be a cycle graph and $V = \{v_1, v_2, \ldots, v_n\}$ denotes the vertex set of the Path with $n$ vertices. Here also, We note that, the graph obtained by subdividing an arbitrary edge of $P_n$ is the path graph $P_{n+1}$. We consider two cases here:

**Case 1:** Suppose $n = 3k + 1$, for some integer $k \geq 1$. from the observation 2.1, we have $\gamma_p(P_n) = k + 1$. Since, the edge subdivision graph of $P_n$ is the graph $P_{n+1}$, we have again by the same observation that $\gamma_p(P_{n+1}) = k + 1$. Hence $P_n$ is edge stable whenever $n = 3k + 1$. 


Case 2: Suppose \( n = 3k + 2 \), for some integer \( k \geq 1 \). Then, the graph obtained by subdividing an arbitrary edge of \( P_n \) is again a path graph with \( 3(k + 1) \) vertices. Thus by observation 2.1 we have, \( \gamma_p(P_n) = \gamma_p(P_{n+1}) = k + 1 \). Hence \( P_n \) is edge stable whenever \( n \equiv 1 \) or \( 2(\text{mod} \ 3) \).

Conversely, suppose \( n \equiv 0(\text{mod} \ 3) \). Then, \( n = 3k \), for some integer \( k \geq 1 \). From the observation 2.1, we have \( \gamma_p(P_n) = k \). But, the graph obtained by subdividing an arbitrary edge of \( P_n \) is again a path graph with \( 3k + 1 \) vertices. Hence, we have \( \gamma_p(P_{n+1}) = k + 1 > k = \gamma_p(P_n) \).

\[ \square \]

Corollary 3.3. For \( n \geq 1 \), the Path graph \( P_n \) is \( \gamma_p \)–changing if and only if \( n \equiv 0(\text{mod} \ 3) \).

Proposition 3.4. Let \( G \) be any graph with \( \gamma_p(G) = 1 \). Then \( G \) is \( \gamma_p \)–changing.

Proof. Let \( G \) be a graph and let \( S = \{v\} \) be a \( \gamma_p(G) \)–set of \( G \). Clearly, the degree of the vertex \( v \) is \( n - 1 \), where \( n = |V| \). Then, subdividing an arbitrary edge \( e = uv \) of \( G \) and inserting the vertex \( x \) between \( u \) and \( v \) it follows that the vertex \( u \) will be not be dominated by \( v \). Therefore, at least two vertices are required to obtain the perfect dominating set of \( G \) and so we get, \( \gamma_p \geq 2 \). Thus \( G \) is a \( \gamma_p \)–changing graph.

\[ \square \]

Proposition 3.5. Let \( G \) be any graph having an isolated vertex. Then \( \overline{G} \) is a \( \gamma_p \)–changing graph.

Proof. Let \( G \) be any graph with an isolated vertex \( v \). Then \( S = \{v\} \) is a perfect dominating set of \( G \). Hence \( \gamma_p(\overline{G}) = 1 \) and so from Proposition 1, it follows that \( \overline{G} \) is a \( \gamma_p \)–changing graph.

\[ \square \]

Proposition 3.6. Let \( G \) be any graph with \( n \) vertices. Then \( 2 \leq \gamma_p(G^\ast) \leq n \).

Proof. The first inequality follows from the proposition 3.1 and the second inequality follows trivially.

The bounds established in proposition 3 are sharp. This may be seen through the following example. The case of the lower bound follows by proposition 2.1. The sharpness of the upper bound is achieved may be seen by taking a wheel \( G \cong W_n \) with \( n \geq 4 \) vertices. Then, for any value of \( n \), \( \gamma_p(G) = 1 \) but if \( e = uv \) is incident to vertex at the center of \( G \) then \( \gamma_p(G^\ast) = n \).

Otherwise, \( \gamma_p(G^\ast) = 2 \).

As a consequence of above proposition, the graphs \( K_n, K_{n-1}, W_n \) are \( \gamma_p \)–changing graphs.

Proposition 3.7. Let \( G \) be a graph. Then \( G \) is \( \gamma_p \)–changing if the end vertices of a subdividing edge lies in the \( \gamma_p \)–set of \( G \).

Proof. Suppose \( S \) is a \( \gamma_p \)–set of \( G \) and \( e = uv \) is a subdividing edge of \( G \). On edge subdivision, the vertex \( x \) is added to the vertex set of \( G \). Clearly \( S \) will not be a \( \gamma_p \)–set of \( G^\ast \), since \( x \) is adjacent to both \( u \) and \( v \). Hence, \( \gamma_p(G^\ast) \geq |S| + 1 \). On the other hand, \( S \cup \{x\} \) is a perfect dominating set of \( G^\ast \). Hence \( \gamma_p(G) < \gamma_p(G^\ast) \).

\[ \square \]

References


